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ON INVERSE CATEGORIES AND TRANSFER IN COHOMOLOGY

MARKUS LINCKELMANN*

Department of Mathematical Sciences, Institute of Mathematics,
University of Aberdeen, King's College, Fraser Noble Building,
Aberdeen AB24 3UE, UK

Abstract It follows from methods of B. Steinberg, extended to inverse categories, that finite inverse category algebras are isomorphic to their associated groupoid algebras; in particular, they are symmetric algebras with canonical symmetrizing forms. We deduce the existence of transfer maps in cohomology and Hochschild cohomology from certain inverse subcategories. This is in part motivated by the observation that, for certain categories $\mathcal{C}$, being a Mackey functor on $\mathcal{C}$ is equivalent to being extendible to a suitable inverse category containing $\mathcal{C}$. We further show that extensions of inverse categories by abelian groups are again inverse categories.

Keywords: inverse category; transfer; cohomology

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1. Introduction

For $\mathcal{C}$ a small category, k a commutative ring and A a k-module, we denote by $H^*(\mathcal{C}; A)$ the cohomology of $\mathcal{C}$ with coefficients in the constant functor from $\mathcal{C}$ to $\mathrm{Mod}(k)$ sending every object in $\mathcal{C}$ to A and every morphism in $\mathcal{C}$ to the identity map on A; in other words, $H^n(\mathcal{C}; A)$ is the nth right derived functor of the limit functor over $\mathcal{C}$ evaluated at the constant functor A. For $A = k$ this is a graded k-algebra, the Ext-algebra of the constant functor with value k. See, for instance, [**22**, § 5] for a brief introduction and further references on functor cohomology. Given a small category $\mathcal{C}$, a subcategory $\mathcal{D}$ and a commutative ring k, the restriction from $\mathcal{C}$ to $\mathcal{D}$ induces an algebra homomorphism on cohomology from $H^*(\mathcal{C}; k)$ to $H^*(\mathcal{D}; k)$. It is not known whether there are a transfer map $H^*(\mathcal{D}; k) \to H^*(\mathcal{C}; k)$ with good formal properties or transfer maps between the Hochschild cohomology algebras $HH^*(k\mathcal{C})$ and $HH^*(k\mathcal{D})$ of the category algebras $k\mathcal{C}$, $k\mathcal{D}$ over k, unless the left and right Kan extensions of a functor $\mathcal{D} \to \mathcal{C}$ coincide (cf. [**17**]). This seems to be a rare phenomenon, however. Besides the categories of finite groups and their subgroups, one known example is the canonical functor from a transporter category of p-centric subgroups of a finite group to its centric linking systems in [**2**, 1.2, 1.3]. Finite inverse categories and their subcategories provide further examples for this phenomenon because these can be played back to groupoids. Following [**13**], a small

* Present address: Department of Mathematical Science, City University London, Northampton Square, London EC1V 0HB, UK (markus.linckelmann.1@city.ac.uk).

category $\mathcal{C}$ is called an *inverse category* if for any morphism $s\colon X \to Y$ in $\mathcal{C}$ there is a unique morphism $\hat{s}\colon Y \to X$ such that $s \circ \hat{s} \circ s = s$ and $\hat{s} \circ s \circ \hat{s} = \hat{s}$. Just as in the case of inverse semigroups, the morphism set of $\mathcal{C}$ admits a partial order. We shall review the relevant background in § 2. An algebra A over a commutative ring k is called *symmetric* if A is finitely generated projective as a k-module and if A is isomorphic, as an A–A-bimodule, to its k-dual $\mathrm{Hom}_k(A, k)$. A *symmetrizing form of* A is a linear map $\tau\colon A \to k$ corresponding to 1_A under some bimodule isomorphism $A \cong \mathrm{Hom}_k(A, k)$. A symmetric algebra over a field is, in particular, self-injective. We refer the reader to [**3**] for a more detailed account on properties of symmetric algebras. Essentially, by extending arguments due to Steinberg in the context of inverse semigroups in [**21**], we shall show that the category algebras of finite inverse categories are symmetric and admit canonical choices of symmetrizing forms.

Theorem 1.1. *Let k be a commutative ring and let $\mathcal{C}$ be a finite inverse category. Then $k\mathcal{C}$ is isomorphic to a direct product of matrix algebras over finite group algebras; in particular, $k\mathcal{C}$ is symmetric. Moreover, the linear map $\tau\colon k\mathcal{C} \to k$ sending a morphism s in $\mathcal{C}$ to the number of idempotent morphisms e in $\mathcal{C}$ satisfying $e \leqslant s$ is a symmetrizing form for $k\mathcal{C}$.*

Thus, if k is a field, then the algebra $k\mathcal{C}$ is self-injective. There are examples, due to Xu [**23**], of finite categories for which not even the quotient of the Hochschild cohomology by its nil-radical is finitely generated. In contrast, for finite inverse categories, the above theorem has the following consequence.

Corollary 1.2. *Let $\mathcal{C}$ be a finite inverse category and let k be a commutative Noetherian ring. Then $HH^*(k\mathcal{C})$ and $H^*(\mathcal{C}; k)$ are finitely generated graded commutative k-algebras.*

One can describe the Hochschild and ordinary cohomology of finite inverse categories more precisely in terms of products of the Hochschild cohomology and ordinary cohomology of group algebras; this will be an easy consequence of the explicit description in Theorem 4.1, below, of an isomorphism between $k\mathcal{C}$ and a direct product of matrix algebras over group algebras. For the same reason, standard results on Schur multipliers for finite groups carry over to finite inverse categories, such as the following.

Corollary 1.3. *Let $\mathcal{C}$ be a finite inverse category and let k be an algebraically closed field. The abelian group $H^2(\mathcal{C}; k^\times)$ is finite.*

As before, $H^2(\mathcal{C}; k^\times)$ denotes the second cohomology group of the constant functor from $\mathcal{C}$ to the category of abelian groups sending every object to $k^\times$ and every morphism to the identity map on $k^\times$. The classes in $H^2(\mathcal{C}; k^\times)$ correspond to certain extensions of $\mathcal{C}$, which are also shown to be inverse categories in Theorem 2.7, below. The point of the next result is that it implies that for certain subcategory algebras of finite inverse categories we do indeed have transfer maps in cohomology.

Theorem 1.4. *Let k be a commutative ring, let $\mathcal{C}$ be a finite inverse category and let $\mathcal{D}$ be an inverse subcategory of $\mathcal{C}$ having the property that, for any object Y in $\mathcal{D}$, the*

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endomorphism monoid $\mathrm{End}_{\mathcal{D}}(Y)$ *contains all idempotents in* $\mathrm{End}_{\mathcal{C}}(Y)$*. Set* $i = 1_{k\mathcal{D}} = \sum_{Y \in \mathrm{Ob}(\mathcal{D})} \mathrm{Id}_Y$*, the sum taken in* $k\mathcal{C}$*. Then* $k\mathcal{C}i$ *is finitely generated projective as a left* $k\mathcal{C}$*-module and as a right* $k\mathcal{D}$*-module; similarly,* $ik\mathcal{C}$ *is finitely generated projective as a left* $k\mathcal{D}$*-module and as a right* $k\mathcal{C}$*-module. Moreover, we have* $ik\mathcal{C} \cong \mathrm{Hom}_k(k\mathcal{C}i, k)$ *as* $k\mathcal{C}$*–*$k\mathcal{D}$*-bimodules.*

Proofs of the above results and some further consequences will be given in § 4. In terms of functor categories over $\mathcal{C}$ and $\mathcal{D}$ instead of module categories over $k\mathcal{C}$ and $k\mathcal{D}$, respectively, Theorem 1.4 implies that the left and right Kan extensions associated with the inclusion functor $\mathcal{D} \to \mathcal{C}$ coincide. Indeed, restriction of functors along this inclusion corresponds on modules to the functor $ik\mathcal{C} \otimes_{k\mathcal{C}} -$, and this functor has a left adjoint and a right adjoint that are both isomorphic to the functor $k\mathcal{C}i \otimes_{k\mathcal{D}} -$. One can also show this by directly calculating the relevant left and right Kan extensions. Any A–B-bimodule M for two symmetric algebras A, B which is finitely generated projective as a left and right module gives rise to an array of transfer maps. This is based on the fact that the k-dual $M^* = \mathrm{Hom}_k(M, k)$ is a B–A-bimodule that is again finitely generated projective as a left and right module. This fact implies that the functors $M \otimes_B -$ and $M^* \otimes_A -$ between the module categories of A and B are left *and* right adjoint to each other, and every choice of symmetrizing forms on A and B determines adjunction isomorphisms. Thus, more precisely, the bimodule M together with a choice of symmetrizing forms on A and B induces transfer maps between the Hochschild cohomology rings of the two algebras (cf. [**15**, 2.9]), between the graded centres of their stable or derived categories (cf. [**17**, 4.1]) as well as between Ext-groups of modules and their images under tensoring with the bimodule and its dual, satisfying reciprocity and compatibility properties described in [**17**, 4.3, 4.8] and [**16**, 5.1]. Specialized to the situation of the above theorem, where we have a canonical choice of symmetrizing forms by Theorem 1.1, this means that induction and restriction between the module categories over $k\mathcal{C}$ and $k\mathcal{D}$, truncated by the idempotent i, induce transfer maps

$$\mathrm{htr}_{\mathcal{D}}^{\mathcal{C}} = \mathrm{tr}_{k\mathcal{C}i} \colon HH^*(k\mathcal{D}) \to HH^*(k\mathcal{C}),$$
$$\mathrm{htr}_{\mathcal{C}}^{\mathcal{D}} = \mathrm{tr}_{ik\mathcal{C}} \colon HH^*(k\mathcal{C}) \to HH^*(k\mathcal{D}),$$

as well as transfer maps

$$\mathrm{tr}_{\mathcal{D}}^{\mathcal{C}}(M, N) \colon \mathrm{Ext}_{k\mathcal{C}}^*(\mathrm{Res}_{k\mathcal{D}}^{k\mathcal{C}}(M), \mathrm{Res}_{k\mathcal{D}}^{k\mathcal{C}}(N)) \to \mathrm{Ext}_{k\mathcal{C}}^*(M, N)$$

for any two $k\mathcal{C}$-modules M, N, and

$$\mathrm{tr}_{\mathcal{C}}^{\mathcal{D}}(U, V) \colon \mathrm{Ext}_{k\mathcal{C}}^*(\mathrm{Ind}_{k\mathcal{D}}^{k\mathcal{C}}(U), \mathrm{Ind}_{k\mathcal{D}}^{k\mathcal{C}}(V)) \to \mathrm{Ext}_{k\mathcal{D}}^*(U, V)$$

for any two $k\mathcal{D}$-modules U, V.

We describe, for future reference, the special case with constant coefficients. We write abusively $\mathrm{tr}_{\mathcal{D}}^{\mathcal{C}}$ instead of $\mathrm{tr}_{\mathcal{D}}^{\mathcal{C}}(k, k)$. We identify as usual $H^0(\mathcal{C}; k) = \lim_{\mathcal{C}}(k)$ with the set of families $(\lambda_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ of elements $\lambda_X \in k$ satisfying $\lambda_X = \lambda_Y$ for any two objects X, Y in $\mathcal{C}$ for which the morphism set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is non-empty, and we denote by 1 the unit

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element of $H^0(\mathcal{D};k)$. We need, furthermore, some notation that will be reviewed in some more detail in the background section (§ 2). We denote by μ the Möbius function of the partially ordered set $\mathrm{Mor}(\mathcal{C})$ of morphisms in $\mathcal{C}$. Two idempotent endomorphisms e, f in $\mathcal{C}$ are called *isomorphic* if there is a morphism s in $\mathcal{C}$ such that $s \circ \hat{s} = e$ and $\hat{s} \circ s = f$. The endomorphisms s satisfying $s \circ \hat{s} = e = \hat{s} \circ s$ form a group, with unit element e, which we denote by $\mathcal{C}_e$. Using a similar notation for $\mathcal{D}$, if f is an idempotent endomorphism in $\mathcal{D}$, then $\mathcal{D}_f$ is a subgroup of $\mathcal{C}_f$. For any idempotent endomorphism e in $\mathcal{C}$, we set $n(e) = \sum_f [\mathcal{C}_f \colon \mathcal{D}_f]$, where the sum is taken over a set of representatives of the $\mathcal{D}$-isomorphism classes of idempotents in $\mathrm{Mor}(\mathcal{D})$ contained in the $\mathcal{C}$-isomorphism class of e, with the usual convention $n(e) = 0$ if the sum is empty. We set $\pi^{\mathcal{C}}_{\mathcal{D}} = \sum_{(e,f)} n(e)\mu(f,e)f$, where (e,f) runs over the set of ordered pairs of idempotents e, f in $\mathrm{Mor}(\mathcal{C})$ satisfying $f \leqslant e$. We have $\pi^{\mathcal{C}}_{\mathcal{D}} \in HH^0(k\mathcal{C}) = Z(k\mathcal{C})$. We denote by $\tau^{\mathcal{C}}_{\mathcal{D}}$ the image of $\pi^{\mathcal{C}}_{\mathcal{D}}$ in $H^0(\mathcal{C};k) = \lim_{\mathcal{C}}(k)$ under the canonical algebra homomorphism $HH^*(k\mathcal{C}) \to H^*(\mathcal{C};k)$.

Theorem 1.5. *Let k be a commutative ring, let $\mathcal{C}$ be a finite inverse category and let $\mathcal{D}$ be an inverse subcategory of $\mathcal{C}$. Suppose that for any object Y in $\mathcal{D}$ the endomorphism monoid $\mathrm{End}_{\mathcal{D}}(Y)$ contains all idempotent endomorphisms in $\mathrm{End}_{\mathcal{C}}(Y)$. Denote by $\alpha\colon HH^*(k\mathcal{C}) \to H^*(\mathcal{C};k)$ and $\beta\colon HH^*(k\mathcal{D}) \to H^*(\mathcal{D};k)$ the canonical algebra homomorphisms. There are graded k-linear transfer maps $\mathrm{tr}^{\mathcal{C}}_{\mathcal{D}}\colon H^*(\mathcal{D};k) \to H^*(\mathcal{C};k)$ and $\mathrm{htr}^{\mathcal{C}}_{\mathcal{D}}\colon HH^*(k\mathcal{D}) \to HH^*(k\mathcal{C})$ with the following properties:*

(i) *$\mathrm{tr}^{\mathcal{C}}_{\mathcal{D}}(\theta) \cdot \zeta = \mathrm{tr}^{\mathcal{C}}_{\mathcal{D}}(\theta \cdot \mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}(\zeta))$ for any $\zeta \in H^*(\mathcal{C};k)$ and any $\theta \in H^*(\mathcal{D};k)$;*

(ii) *$\alpha(\mathrm{htr}^{\mathcal{C}}_{\mathcal{D}}(\eta)) = \mathrm{tr}^{\mathcal{C}}_{\mathcal{D}}(\beta(\eta))$ for any $\eta \in HH^*(k\mathcal{D})$;*

(iii) *we have $\mathrm{htr}^{\mathcal{C}}_{\mathcal{D}}(1) = \pi^{\mathcal{C}}_{\mathcal{D}}$, and $\pi^{\mathcal{C}}_{\mathcal{D}}$ is invertible in $Z(k\mathcal{C})$ if and only if $n(e)$ is invertible in k for all idempotent morphisms e in $\mathcal{C}$;*

(iv) *we have $\mathrm{tr}^{\mathcal{C}}_{\mathcal{D}}(1) = \tau^{\mathcal{C}}_{\mathcal{D}}$, and $\tau^{\mathcal{C}}_{\mathcal{D}}$ is invertible in $\lim_{\mathcal{C}}(k)$ if and only if $n(e)$ is invertible in k for all minimal idempotent morphisms e in $\mathcal{C}$;*

(v) *if $\tau^{\mathcal{C}}_{\mathcal{D}}$ is invertible, then $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}$ is injective and $\mathrm{tr}^{\mathcal{C}}_{\mathcal{D}}$ is surjective.*

This will be proved at the end of § 5. We show further in Theorem 6.2 that any small category $\mathcal{C}$ in which all morphisms are monomorphisms and in which pull-backs exist can be embedded canonically into an inverse category, preserving cohomology with constant coefficients.

Theorem 1.6. *Let $\mathcal{C}$ be a small category in which all morphisms are monomorphisms and in which pull-backs exist. There is an inverse category $\hat{\mathcal{C}}$ containing $\mathcal{C}$ such that, for any module A over a commutative ring k, the restriction induces an isomorphism $H^*(\hat{\mathcal{C}};A) \cong H^*(\mathcal{C};A)$.*

In the situation of Theorem 1.6, being a Mackey functor on $\mathcal{C}$ in the sense of Dress [**4**, Part I] and Jackowski and McClure [**10**] is essentially equivalent to being extendible to the inverse category $\hat{\mathcal{C}}$; see Remark 6.4.

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2. Basic properties of inverse categories

We collect in this section basic ideas from the theory of semigroups and categories of partial maps translated to inverse categories; see [**14**, Chapter 1] and [**7**, **8**, **13**]. Let $\mathcal{C}$ be an inverse category; that is, $\mathcal{C}$ is a small category such that for any morphism $s\colon X \to Y$ in $\mathcal{C}$ there is a unique morphism $\hat{s}\colon Y \to X$ satisfying $s \circ \hat{s} \circ s = s$ and $\hat{s} \circ s \circ \hat{s} = \hat{s}$. Then, in particular, for any object X in $\mathcal{C}$, the monoid $\mathrm{End}_{\mathcal{C}}(X)$ is an inverse monoid, and hence any two idempotents in $\mathrm{End}_{\mathcal{C}}(X)$ commute (cf. [**14**, § 1.1, Theorem 3]). For any morphism $s\colon X \to Y$ the morphisms $\hat{s} \circ s$ and $s \circ \hat{s}$ are idempotents in $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{End}_{\mathcal{C}}(Y)$, respectively. For any idempotent e in $\mathrm{End}_{\mathcal{C}}(X)$ we have $\hat{e} = e$, and for any two composable morphisms s, t in $\mathcal{C}$ we have $\widehat{t \circ s} = \hat{s} \circ \hat{t}$; in particular, an inverse category $\mathcal{C}$ is isomorphic to its opposite $\mathcal{C}^{\mathrm{op}}$. If e is an idempotent in $\mathrm{End}_{\mathcal{C}}(X)$ and $s\colon X \to Y$ is a morphism in $\mathcal{C}$, then $f = s \circ e \circ \hat{s}$ is an idempotent in $\mathrm{End}_{\mathcal{C}}(Y)$ satisfying $s \circ e = f \circ s$; indeed, using that the idempotents $\hat{s} \circ s$ and e commute, we have $f \circ f = s \circ e \circ \hat{s} \circ s \circ e \circ \hat{s} = s \circ e \circ e \circ \hat{s} \circ s \circ \hat{s} = s \circ e \circ \hat{s} = f$ and $f \circ s = s \circ e \circ \hat{s} \circ s = s \circ \hat{s} \circ s \circ e = s \circ e$. Similarly, if f is an idempotent in $\mathrm{End}_{\mathcal{C}}(Y)$, then $e = \hat{s} \circ f \circ s$ is an idempotent in $\mathrm{End}_{\mathcal{C}}(X)$ satisfying $s \circ e = f \circ s$. We use these elementary computational rules without further comment. As in the case of inverse semigroups, one can define a partial order on the set of morphisms in $\mathcal{C}$. For any two morphisms $s, t\colon X \to Y$ we write $s \leqslant t$ if $s = t \circ e$ for some idempotent $e \in \mathrm{End}_{\mathcal{C}}(X)$. The proof of the (well-known) fact that this is actually a partial order and the proof, below, of an extension of the Preston–Wagner Representation Theorem to this context follow closely the proofs of corresponding statements for inverse semigroups in [**14**, Chapter 1]; we include details for the convenience of the reader.

Lemma 2.1. *Let $\mathcal{C}$ be an inverse category and let $s, t\colon X \to Y$ be morphisms in $\mathcal{C}$. The following statements are equivalent:*

(i) $s \leqslant t$;

(ii) $s = f \circ t$ *for some idempotent* $f \in \mathrm{End}_{\mathcal{C}}(Y)$;

(iii) $\hat{s} \leqslant \hat{t}$;

(iv) $s = s \circ \hat{s} \circ t$;

(v) $s = t \circ \hat{s} \circ s$.

Proof. The equivalence (i) $\iff$ (ii) follows from the remarks preceding the lemma. The equality $s = t \circ e$ for some idempotent $e \in \mathrm{End}_{\mathcal{C}}(X)$ is equivalent to $\hat{s} = \hat{e} \circ \hat{t} = e \circ \hat{t}$, whence follows the equivalence (i) $\iff$ (iii). If $s = t \circ e$ for some idempotent $e \in \mathrm{End}_{\mathcal{C}}(X)$, then $s = s \circ \hat{s} \circ s = t \circ e \circ \hat{s} \circ s = t \circ \hat{s} \circ s \circ e = t \circ \hat{s} \circ s$ and, since $\hat{s} \circ s$ is an idempotent, we get the equivalence (i) $\iff$ (v). A similar argument yields the equivalence (ii) $\iff$ (iv). □

Proposition 2.2. *Let $\mathcal{C}$ be an inverse category, let X, Y, Z be objects in $\mathcal{C}$ and let $s, t\colon X \to Y$ and $u, v\colon Y \to Z$ be morphisms in $\mathcal{C}$. The relation '$\leqslant$' defines a partial order*

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on the set of morphisms that is compatible with composition of morphisms in $\mathcal{C}$; that is, if $s \leqslant t$ and $u \leqslant v$, then $u \circ s \leqslant v \circ t$.

Proof. Since $s = s \circ \hat{s} \circ s$, we have $s \leqslant s$. If $s \leqslant t$ and $t \leqslant s$, then, using that the idempotents $\hat{s} \circ s$ and $\hat{t} \circ t$ commute, we get that $s = t \circ \hat{s} \circ s = s \circ \hat{t} \circ t \circ \hat{s} \circ s = s \circ \hat{t} \circ t = t$. If $s \leqslant t$ and $t \leqslant w$ for some further morphism $w\colon X \to Y$, then $s = s \circ \hat{s} \circ t = s \circ \hat{s} \circ t \circ \hat{t} \circ w$. Since the idempotents $s \circ \hat{s}$ and $t \circ \hat{t}$ commute, their product is an idempotent as well, and thus $s \leqslant w$. This shows that '$\leqslant$' is a partial order on the set of morphisms in $\mathcal{C}$. If $s \leqslant t$ and $u \leqslant v$, there are idempotents f, g in $\mathrm{End}_{\mathcal{C}}(Y)$ satisfying $s = f \circ t$ and $u = v \circ g$. Thus, $u \circ s = v \circ g \circ f \circ s$. The idempotents f, g commute; hence, $g \circ f$ is an idempotent, and thus by the remarks at the beginning of this section there is an idempotent h in $\mathrm{End}_{\mathcal{C}}(Z)$ such that $v \circ g \circ f = h \circ v$. Together we get that $u \circ f = h \circ v \circ t$; hence, $u \circ s \leqslant v \circ t$. □

Lemma 2.3. *Let $\mathcal{C}$ be an inverse category, let $s\colon X \to Y$ be a morphism in $\mathcal{C}$ and let U be an object in $\mathcal{C}$. We have $s \circ \mathrm{Hom}_{\mathcal{C}}(U, X) = s \circ \hat{s} \circ \mathrm{Hom}_{\mathcal{C}}(U, Y)$.*

Proof. If $u \in \mathrm{Hom}_{\mathcal{C}}(U, X)$, then $s \circ u = s \circ \hat{s} \circ s \circ u \in s \circ \hat{s} \circ \mathrm{Hom}_{\mathcal{C}}(U, Y)$, and if $v \in \mathrm{Hom}_{\mathcal{C}}(U, Y)$, then $s \circ \hat{s} \circ v \in s \circ \mathrm{Hom}_{\mathcal{C}}(U, X)$, whence we obtain the result. □

A category $\mathcal{C}$ is called *idempotent complete* if for any object X and any idempotent $e \in \mathrm{End}_{\mathcal{C}}(X)$ there are an object Y and morphisms $s\colon X \to Y$ and $t\colon Y \to X$ such that $e = t \circ s$ and $s \circ t = \mathrm{Id}_Y$. Any category $\mathcal{C}$ can be embedded into its *idempotent completion* $\hat{\mathcal{C}}$, also called *Karoubienne*, constructed as follows: the objects of $\hat{\mathcal{C}}$ are pairs (X, e) consisting of an object X in $\mathcal{C}$ and an idempotent e in $\mathrm{End}_{\mathcal{C}}(X)$; a morphism in $\hat{\mathcal{C}}$ from (X, e) to (Y, f) is a triple (e, s, f), where $s\colon X \to Y$ is a morphism in $\mathcal{C}$ satisfying $s \circ e = s = f \circ s$. There is an obvious embedding from $\mathcal{C}$ to $\hat{\mathcal{C}}$, sending an object X in $\mathcal{C}$ to (X, Id_X). With the previous notation, two objects (X, e) and (Y, f) are isomorphic in $\hat{\mathcal{C}}$ if there are morphisms $s\colon X \to Y$ and $t\colon Y \to X$ in $\mathcal{C}$ satisfying $s \circ e = s = f \circ s$, $t \circ f = t = e \circ t$, $t \circ s = e$ and $s \circ t = f$. In that case, we also say that *the idempotents e and f are isomorphic*. Note that in that case $s \circ t \circ s = s$ and $t \circ s \circ t = t$, and hence if $\mathcal{C}$ is an inverse category, we have $t = \hat{s}$. In other words, in an inverse category $\mathcal{C}$, two idempotents $e \in \mathrm{End}_{\mathcal{C}}(X)$ and $f \in \mathrm{End}_{\mathcal{C}}(Y)$ are isomorphic if and only if there is a morphism $s\colon X \to Y$ satisfying $\hat{s} \circ s = e$ and $s \circ \hat{s} = f$.

Proposition 2.4. *Let $\mathcal{C}$ be a small category. Then $\mathcal{C}$ is an inverse category if and only if the idempotent completion $\hat{\mathcal{C}}$ of $\mathcal{C}$ is an inverse category.*

Proof. Let X, Y be objects in $\mathcal{C}$ and let $e \in \mathrm{End}_{\mathcal{C}}(X)$ and $f \in \mathrm{End}_{\mathcal{C}}(Y)$ be idempotents. Let $(e, s, f)\colon (X, e) \to (Y, f)$ be a morphism in $\hat{\mathcal{C}}$; that is, $s\colon X \to Y$ is a morphism in $\mathcal{C}$ satisfying $f \circ s = s = s \circ e$. If $\mathcal{C}$ is an inverse category, then $\hat{s} \circ \hat{f} = \hat{s} = \hat{e} \circ \hat{s}$, so $(f, \hat{s}, e)$ is a morphism in $\hat{\mathcal{C}}$ from (Y, f) to (X, e). An easy verification shows that $\hat{\mathcal{C}}$ is indeed an inverse category, and a similar argument shows the converse. □

By the Preston–Wagner Representation Theorem (cf. [**14**, § 1.5]), any inverse semigroup can be embedded into an inverse monoid $\mathcal{I}(M)$ of partial bijections on a non-empty

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set M. A similar statement holds for inverse categories (cf. [**7**]). A partition of a non-empty set M is a set λ of pairwise disjoint non-empty subsets of M whose union is equal to M. Any such partition gives rise to an inverse category $\mathcal{I}(\lambda)$, defined as follows. The object set of $I(\lambda)$ is equal to λ. For any two subsets U, V of M in λ, the morphism set $\mathrm{Hom}_{\mathcal{I}(\lambda)}(U, V)$ is the set of all bijective maps $s\colon U' \cong V'$ between a subset U' of U and a subset V' of V, including the unique bijection between empty subsets; the morphism $\hat{s}$ is defined as $s^{-1}\colon V' \cong U'$. Composition of morphisms in $\mathcal{I}(\lambda)$ is induced by composition in the inverse monoid $\mathcal{I}(M)$; that is, given U, V, W in λ, a subset U' of U, subsets V', V'' of V, a subset W' of W and bijections $s\colon U' \cong V'$, $t\colon V'' \cong W'$, the composition $t \circ s$ in $\mathcal{I}(\lambda)$ is the induced bijection $t|_{V'\cap V''} \circ s|_{s^{-1}(V'\cap V'')}\colon s^{-1}(V' \cap V'') \cong t(V' \cap V'')$.

Theorem 2.5. *Let $\mathcal{C}$ be an inverse category. Let $M = \mathrm{Mor}(\mathcal{C})$ be the set of all morphisms in $\mathcal{C}$. For any object X in $\mathcal{C}$ denote by M_X the set of all morphisms in $\mathcal{C}$ terminating at X. Then the set of subsets $\lambda = \{M_X \mid X \in \mathrm{Ob}(\mathcal{C})\}$ is a partition of M and there is a functor $\Phi\colon \mathcal{C} \to \mathcal{I}(\lambda)$ with the following properties:*

(i) *Φ sends an object X in $\mathcal{C}$ to the set $M_X \in \lambda$;*

(ii) *Φ induces an injective map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{I}(\lambda)}(\Phi(X), \Phi(Y))$ for any two objects X, Y in $\mathcal{C}$;*

(iii) *$\Phi(\hat{s}) = \widehat{\Phi(s)}$ for any morphism s in $\mathcal{C}$;*

(iv) *we have $s \leqslant t$ if and only if $\Phi(s) \leqslant \Phi(t)$, for any two morphisms s, t in $\mathcal{C}$.*

Proof. The sets M_X, with $X \in \mathrm{Ob}(\mathcal{C})$, are clearly pairwise disjoint and their union is the set of all morphisms in $\mathcal{C}$, so this determines a partition λ of M. By Lemma 2.3, for any morphism $s\colon X \to Y$ in $\mathcal{C}$ we have $s \circ M_X = s \circ \hat{s} \circ M_Y$. We define the functor Φ as follows. On objects, we set $\Phi(X) = M_X$ for any $X \in \mathrm{Ob}(\mathcal{C})$. Let $s\colon X \to Y$ be a morphism in $\mathcal{C}$. Then $\hat{s} \circ s \circ M_X$ is a subset of M_X, and $s \circ \hat{s} \circ M_Y$ is a subset of M_Y. The map $\Phi(s)\colon \hat{s} \circ s \circ M_X \to s \circ \hat{s} \circ M_Y$ sending $f \in \hat{s} \circ s \circ M_X$ to $s \circ f$ is a bijection, with inverse $\Phi(\hat{s})$ sending $g \in s \circ \hat{s} \circ M_Y$ to $\hat{s} \circ g$. Thus, $\Phi(s)$ defined in this way is a morphism from M_X to M_Y in $\mathcal{I}(\lambda)$ satisfying $\Phi(\hat{s}) = \widehat{\Phi(s)}$. We need to show that this assignment is functorial. Let $s\colon X \to Y$ and $t\colon Y \to Z$ be morphisms in $\mathcal{C}$. Both morphisms $\Phi(t \circ s)$ and $\Phi(t) \circ \Phi(s)$ are induced by composing morphism sets with $t \circ s$; we only need to check that their domains in M_X are equal. The map $\Phi(t \circ s)$ is defined on the set $\widehat{t \circ s} \circ t \circ s \circ M_X = \hat{s} \circ \hat{t} \circ t \circ s \circ M_X$. Since $s \circ M_X = s \circ \hat{s} \circ M_Y$, this set is equal to $\hat{s} \circ \hat{t} \circ t \circ s \circ \hat{s} \circ M_Y$. Since idempotents commute, this set is in fact equal to $\hat{s} \circ s \circ \hat{s} \circ \hat{t} \circ t \circ M_Y = \hat{s} \circ \hat{t} \circ t \circ M_Y$. Now, $\Phi(s)(\hat{s} \circ \hat{t} \circ t \circ M_Y) = s \circ \hat{s} \circ \hat{t} \circ t \circ M_Y = s \circ \hat{s} \circ M_Y \cap t \circ \hat{t} \circ M_Y$, where we use again that $s \circ \hat{s}$ and $\hat{t} \circ t$ are commuting idempotents. This intersection is precisely the intersection of the image of $\Phi(s)$ and the domain of $\Phi(t)$, whence we obtain the equality $\Phi(t \circ s) = \Phi(t) \circ \Phi(s)$. Thus, Φ is a functor. By construction, Φ is bijective on objects. An equality $\Phi(s) = \Phi(t)$ implies $s = s \circ \hat{s} \circ s = \Phi(s)(\hat{s} \circ s) = \Phi(t)(\hat{s} \circ s) = t \circ \hat{s} \circ s$; hence $s \leqslant t$, and exchanging the roles of s and t also yields $t \leqslant s$, and hence $s = t$, which shows that Φ is injective on morphism sets. The remaining two statements, (iii) and (iv), are trivial verifications. □

Remarks 2.6.

1. The uniqueness of the morphism $\hat{s}$ in the definition of an inverse category implies that statement (iii) in Theorem 2.5 holds automatically; that is, if $\mathcal{C}$, $\mathcal{D}$ are inverse categories and if $\Phi\colon \mathcal{C} \to \mathcal{D}$ is a functor, then $\Phi(\hat{s}) = \widehat{\Phi(s)}$ for any morphism s in $\mathcal{C}$.

2. It is well known that a monoid S in which all idempotents commute and which admits an involution $s \to \hat{s}$ satisfying $s = s\hat{s}s$ is an inverse monoid. A similar statement holds for inverse categories. An *involution of a small category* $\mathcal{C}$ is a bijection on the morphism set of $\mathcal{C}$ sending a morphism $s\colon X \to Y$ in $\mathcal{C}$ to a morphism $\hat{s}\colon Y \to X$ such that $\hat{\hat{s}} = s$ and $\widehat{t \circ s} = \hat{s} \circ \hat{t}$ for any two composable morphisms s, t. In particular, a small category with an involution is isomorphic to its opposite. One can show that if in addition $s \circ \hat{s} \circ s = s$ for any morphism s in $\mathcal{C}$ and if any two idempotents in $\mathrm{End}_{\mathcal{C}}(X)$ commute, for any object X in $\mathcal{C}$, then $\mathcal{C}$ is an inverse category.

3. If $\mathcal{C}$ is a finite inverse category, then every endomorphism monoid $\mathrm{End}_{\mathcal{C}}(X)$ of an object X of $\mathcal{C}$ has a unique minimal idempotent endomorphism e_X, namely the product of all idempotent endomorphisms in $\mathrm{End}_{\mathcal{C}}(X)$. Thus, for any morphism $s\colon X \to Y$, the set $\{t\colon X \to Y \mid t \leqslant s\}$ has a unique minimal element, namely $t = s \circ e_X = e_Y \circ s$.

4. Let $\mathcal{C}$ be a finite inverse category, let $\mathcal{D}$ be an inverse subcategory of $\mathcal{C}$ and let Y be an object in $\mathcal{D}$. Then $\mathrm{End}_{\mathcal{D}}(Y)$ contains all idempotents in $\mathrm{End}_{\mathcal{C}}(Y)$ if and only if $\mathrm{End}_{\mathcal{D}}(Y)$ is a downwardly closed subposet of $\mathrm{End}_{\mathcal{C}}(Y)$; that is, if and only if, for $s, t \in \mathrm{End}_{\mathcal{C}}(Y)$ satisfying $t \leqslant s$ and $s \in \mathrm{End}_{\mathcal{D}}(Y)$, we have $t \in \mathrm{End}_{\mathcal{D}}(Y)$. Indeed, since $\mathrm{End}_{\mathcal{D}}(Y)$ contains Id_Y, if $\mathrm{End}_{\mathcal{D}}(Y)$ is downwardly closed, it contains all idempotents in $\mathrm{End}_{\mathcal{C}}(Y)$, and the converse follows from Lemma 2.1.

Second cohomology classes of a category $\mathcal{C}$ with coefficients in an abelian group, viewed as a constant functor, correspond to category extensions of $\mathcal{C}$ (cf. [**1**]); see [**22**, § 7] for an exposition of this material. Extensions of inverse categories by constant functors are again inverse categories.

Theorem 2.7. *Let $\mathcal{C}$ be an inverse category and let A be an abelian group, viewed as a constant contravariant functor on $\mathcal{C}$. Let $\mathcal{D}$ be an extension of $\mathcal{C}$ by A. Then $\mathcal{D}$ is an inverse category. Moreover, the canonical functor $\mathcal{D} \to \mathcal{C}$ induces an isomorphism $E(\mathcal{D}) \cong E(\mathcal{C})$ between the partially ordered sets $E(\mathcal{D})$, $E(\mathcal{C})$ of idempotent endomorphisms in $\mathcal{D}$ and $\mathcal{C}$, respectively.*

Proof. An extension category $\mathcal{D}$ of $\mathcal{C}$ by A has the same object set as $\mathcal{C}$ and morphism set $\mathrm{Mor}(\mathcal{D}) \times A$; the composition of morphisms in $\mathcal{D}$ is given by

$$(t, b) \circ (s, a) = (t \circ s, ba\alpha(t, s))$$

for any two composable morphisms s, t in $\mathcal{C}$ and $a, b \in A$, where α is a function sending any two composable morphisms s, t in $\mathcal{C}$ to an element $\alpha(t, s)$ in A, such that α satisfies, for any three composable morphisms s, t, u in $\mathcal{C}$, the 2-cocycle identity

$$\alpha(u \circ t, s)\alpha(u, t) = \alpha(u, t \circ s)\alpha(t, s).$$

This identity is equivalent to the associativity of the composition of morphisms in $\mathcal{D}$. Up to isomorphism of extensions, $\mathcal{D}$ depends only on the class of α in $H^2(\mathcal{C}; A)$. Let s be a morphism in $\mathcal{C}$. The 2-cocycle identity applied to the three composable morphisms $\hat{s}$, $s \circ \hat{s}$, s yields the equality

$$\alpha(\hat{s} \circ s \circ \hat{s}, s)\alpha(\hat{s}, s \circ \hat{s}) = \alpha(\hat{s}, s \circ \hat{s} \circ s)\alpha(s \circ \hat{s}, s).$$

Since $\hat{s} \circ s \circ \hat{s} = \hat{s}$ and $s \circ \hat{s} \circ s = s$, this implies

$$\alpha(\hat{s}, s \circ \hat{s}) = \alpha(s \circ \hat{s}, s).$$

Let $\sigma = (s, a)$ be a morphism in $\mathcal{D}$. A morphism $\hat{\sigma}$ satisfying $\sigma \circ \hat{\sigma} \circ \sigma = \sigma$ must necessarily be of the form $(\hat{s}, a^{-1}\gamma(s))$ for some $\gamma(s) \in A$. Then

$$\sigma \circ \hat{\sigma} \circ \sigma = (s \circ \hat{s}, \alpha(s, \hat{s})\gamma(s)) \circ (s, a) = (s, a\alpha(s \circ \hat{s}, s)\alpha(s, \hat{s})\gamma(s)),$$

so this forces $\gamma(s) = \alpha(s \circ \hat{s}, s)^{-1}\alpha(s, \hat{s})^{-1}$. By the equality above we also have $\gamma(s) = \alpha(\hat{s}, s \circ \hat{s})^{-1}\alpha(s, \hat{s})^{-1}$. Thus,

$$\begin{aligned}
\hat{\sigma} \circ \sigma \circ \hat{\sigma} &= (\hat{s}, a^{-1}\gamma(s)) \circ (s \circ \hat{s}, \alpha(s, \hat{s})\gamma(s)) \\
&= (\hat{s}, a^{-1}\alpha(\hat{s}, s \circ \hat{s})\alpha(s, \hat{s})\gamma(s)^2) \\
&= (\hat{s}, a^{-1}\gamma(s)) \\
&= \hat{\sigma}.
\end{aligned}$$

This shows that $\mathcal{D}$ is indeed an inverse category. Any idempotent endomorphism in $\mathcal{D}$ is necessarily of the form (e, a) for some idempotent endomorphism e in $\mathcal{C}$ and some $a \in A$. The equality $(e, a) = (e, a) \circ (e, a) = (e, a^2\alpha(e, e))$ forces $a = \alpha(e, e)^{-1}$. Thus, the canonical functor $\mathcal{D} \to \mathcal{C}$ induces a bijection between the sets of idempotent endomorphisms $E(\mathcal{D})$ and $E(\mathcal{C})$ in $\mathcal{D}$ and $\mathcal{C}$.

To see that this bijection is an isomorphism of posets we need to show that if e, f are idempotent endomorphisms in $\mathcal{C}$ satisfying $e \leqslant f$, then also $(e, \alpha(e, e)^{-1}) \leqslant (f, \alpha(f, f)^{-1})$. The 2-cocycle identity applied to e, f, f yields $\alpha(e, f)\alpha(f, f) = \alpha(ef, f)\alpha(e, f)$; using the equality $ef = e$, this yields $\alpha(e, f) = \alpha(f, f)$. Thus, $(e, \alpha(e, e)^{-1})(f, \alpha(f, f)^{-1}) = (ef, \alpha(e, f)\alpha(e, e)^{-1}\alpha(f, f)^{-1}) = (e, \alpha(e, e)^{-1})$, as required. □

3. The Möbius function for finite inverse categories

The purpose of this section is to verify that results of Steinberg in [**21**, § 4] carry over, with no difficulty, to finite inverse categories. As before, we give detailed proofs for the convenience of the reader. The Möbius function μ of a finite partially ordered set $(\mathcal{P}, \leqslant)$ with coefficients in a commutative ring k is defined on the set of pairs $(x, y) \in \mathcal{P} \times \mathcal{P}$ satisfying $x \leqslant y$, by $\mu(x, x) = 1$ for $x \in \mathcal{P}$ and $\sum_{x \leqslant u \leqslant y} \mu(u, y) = 0$ if $x < y$. The Möbius

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Inversion Theorem, due to Rota, states that if f, g are functions from $\mathcal{P}$ to k such that $g(x) = \sum_{u\leqslant x} f(u)$ for all $x \in \mathcal{P}$, then $f(x) = \sum_{u\leqslant x} \mu(u,x)g(u)$ for all $x \in \mathcal{P}$. Given a finite inverse category $\mathcal{C}$ and a commutative ring k, we consider the set of morphisms in $\mathcal{C}$ endowed with the natural partial order as in the previous section and use the letter μ for the corresponding Möbius function with coefficients in k. For any morphism s in $\mathcal{C}$, we define an element $\underline{s}$ in the subspace $k\operatorname{Hom}_{\mathcal{C}}(X,Y)$ of the category algebra $k\mathcal{C}$ by setting

$$\underline{s} = \sum_{u\leqslant s} \mu(u,s)u.$$

Möbius inversion implies that $s = \sum_{u\leqslant s} \underline{u}$, and hence that the set of $\underline{s}$, with s running over all morphisms in $\mathcal{C}$, is again a k-basis of $k\mathcal{C}$. Since $u \leqslant s$ if and only if $\hat{u} \leqslant \hat{s}$, we have $\mu(\hat{u},\hat{s}) = \mu(u,s)$.

Proposition 3.1. *Let $\mathcal{C}$ be a finite inverse category and let k be a commutative ring. For any two composable morphisms $s\colon X \to Y$ and $t\colon Y \to Z$ in $\mathcal{C}$, the following hold.*

(i) *We have $t \cdot \underline{s} = \underline{t\circ s}$ if $s\circ\hat{s} \leqslant \hat{t}\circ t$, and $t\cdot\underline{s} = 0$ otherwise.*

(ii) *We have $\underline{t}\cdot\underline{s} = \underline{t\circ s}$ if $s\circ\hat{s} = \hat{t}\circ t$, and $\underline{t}\cdot\underline{s} = 0$ otherwise.*

The proof is essentially a variation of the arguments in [**21**, § 4]; we break this up into a series of lemmas.

Lemma 3.2. *Let $\mathcal{C}$ be an inverse category and let $s\colon X \to Y$, $t\colon Y \to Z$ be two composable morphisms in $\mathcal{C}$. The map sending a morphism $u \leqslant s$ to $t\circ u$ induces a bijection of sets*

$$\{u \mid u \leqslant \hat{t}\circ t\circ s\} \cong \{w \mid w \leqslant t\circ s\}$$

with inverse sending $w \leqslant t\circ s$ to $\hat{t}\circ w$.

Proof. If $u \leqslant \hat{t}\circ t\circ s$, then $u = \hat{t}\circ t\circ u$, and if $w \leqslant t\circ s$, then $w = t\circ s\circ\hat{w}\circ w$; hence, $t\circ\hat{t}\circ w = w$. □

Lemma 3.3. *Let $\mathcal{C}$ be an inverse category and let $s\colon X \to Y$, $t\colon Y \to Z$ be two composable morphisms in $\mathcal{C}$. We have $s\circ\hat{s} \leqslant \hat{t}\circ t$ if and only if $s = \hat{t}\circ t\circ s$.*

Proof. We have $s\circ\hat{s} \leqslant \hat{t}\circ t$ if and only if $s\circ\hat{s} = \hat{t}\circ t\circ s\circ\hat{s}$. Precomposing with s shows that this equality implies the equality $s = \hat{t}\circ t\circ s$, and precomposing by $\hat{s}$ shows that the two are indeed equivalent. □

Lemma 3.4. *Let $\mathcal{C}$ be an inverse category and let s be a morphism in $\mathcal{C}$. We have $\underline{s} = s - \sum_{u<s}\underline{u}$.*

Proof. By Möbius inversion we have $s = \sum_{u\leqslant s}\underline{u} = \underline{s} + \sum_{u<s}\underline{u}$, whence the result follows. □

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Proof of Proposition 3.1. We prove (i) by induction, assuming that the result holds for $s' < s$ instead of s. Assume first that $s \circ \hat{s} \leqslant \hat{t} \circ t$, which, by Lemma 3.3, is equivalent to $s = \hat{t} \circ t \circ s$. Then, for $u \leqslant s$, we also have $u \circ \hat{u} \leqslant \hat{t} \circ t$. By Lemma 3.4, we have $\underline{s} = s - \sum_{u<s} \underline{u}$. By induction, we have

$$t \cdot \underline{s} = t \circ s - \sum_{u<s} t \cdot \underline{u} = t \circ s - \sum_{u<s} \underline{t \circ u}$$

Since $s = \hat{t} \circ t \circ s$, it follows from Lemma 3.2 that if u runs over the morphisms satisfying $u < s$, then $t \circ u$ runs over the morphisms w satisfying $w < t \circ s$. Thus,

$$t \cdot \underline{s} = t \circ s - \sum_{w < t \circ s} \underline{w} = \underline{t \circ s}.$$

Assume next that $s \circ \hat{s} \nleqslant \hat{t} \circ t$; again by Lemma 3.3, this is equivalent to $\hat{t} \circ t \circ s < s$. Thus, if $u \leqslant s$ such that $u \circ \hat{u} \leqslant \hat{t} \circ t$, then $u = u \circ \hat{u} \circ s \leqslant \hat{t} \circ t \circ s < s$. By induction we get $t \cdot \underline{s} = t \circ s - \sum_{u \leqslant \hat{t} \circ t \circ s} \underline{t \circ u}$ and hence Lemma 3.2 implies that

$$t \cdot \underline{s} = t \circ s - \sum_{w \leqslant t \circ s} \underline{w} = t \circ s - t \circ s = 0,$$

which completes the proof of (i). Using (i) and Lemma 3.3 we get

$$\underline{t} \cdot \underline{s} = \sum_{v \leqslant t} \mu(v, t) v \cdot \underline{s} = \sum_{v \leqslant t;\, \hat{v} \circ v \circ s = s} \mu(v, t) \underline{v \circ s}.$$

If $\hat{t} \circ t \circ s < s$, then $\hat{v} \circ v \circ s < s$ for all $v \leqslant t$, so $\underline{t} \cdot \underline{s} = 0$ in that case. Assume now that $\hat{t} \circ t \circ s = s$. We show first that if $v \leqslant t$ such that $\hat{v} \circ v \circ s = s$, then $v \circ s = t \circ s$. Indeed, since $v \leqslant t$ we have $v = t \circ \hat{v} \circ v$; hence, $v \circ s = t \circ \hat{v} \circ v \circ s = t \circ s$. This also implies $t \circ s \circ \hat{s} = v \circ s \circ \hat{s} \leqslant v$. Therefore, we have

$$\underline{t} \cdot \underline{s} = \sum_{v \leqslant t;\, \hat{v} \circ v \circ s = s} \mu(v, t) \underline{v \circ s} = \Bigg(\sum_{t \circ s \circ \hat{s} \leqslant v \leqslant t} \mu(v, t) \Bigg) \underline{t \circ s}.$$

This sum is zero if $t \circ s \circ \hat{s} < s$, and equal to $\underline{t \circ s}$ if $t \circ s \circ \hat{s} = t$. Since the two conditions $\hat{t} \circ t \circ s = s$ and $t \circ s \circ \hat{s} = t$ are equivalent to the equality $s \circ \hat{s} = \hat{t} \circ t$, statement (ii) follows. $\square$

For the sake of completeness we point out that one can also prove Proposition 3.1 by making use of the following observation, which is from the argument used in the proof of [**21**, 4.1].

Lemma 3.5. *Let $\mathcal{C}$ be an inverse category and let $s\colon X \to Y$, $t\colon Y \to Z$ be two composable morphisms in $\mathcal{C}$. Let $w\colon X \to Z$ be a morphism in $\mathcal{C}$ such that $w \leqslant t \circ s$. Then $u = s \circ \hat{w} \circ w$ and $v = w \circ \hat{w} \circ t$ are the unique morphisms in $\mathcal{C}$ satisfying $u \leqslant s$, $v \leqslant t$, $\hat{v} \circ v = u \circ \hat{u}$ and $v \circ u = w$. Moreover, we have $u = \hat{t} \circ w$ and $v = w \circ \hat{s}$.*

Proof. Since $w \leqslant t \circ s$, we have $w = w \circ \hat{w} \circ t \circ s$. Suppose first that $u \leqslant s$, $v \leqslant t$ are morphisms such that $v \circ u = w$ and $\hat{v} \circ v = u \circ \hat{u}$. Then $v \circ \hat{v} \circ w = w = w \circ \hat{u} \circ u$ and $v = v \circ \hat{v} \circ v = v \circ u \circ \hat{u} = w \circ \hat{u}$. Similarly, $u = \hat{v} \circ w$, and hence also $\hat{u} = \hat{w} \circ v$. Since $u \leqslant s$, we have $u = s \circ \hat{u} \circ u = s \circ \hat{w} \circ v \circ \hat{v} \circ w = s \circ \hat{w} \circ w$. Similarly, since $v \leqslant t$, we have $v = v \circ \hat{v} \circ t = w \circ \hat{u} \circ u \circ \hat{w} \circ t = w \circ \hat{w} \circ t$. Conversely, set $u = s \circ \hat{w} \circ w$ and $v = w \circ \hat{w} \circ t$. Then clearly $u \leqslant s$ and $v \leqslant t$. Moreover, $v \circ u = w \circ \hat{w} \circ t \circ s \circ \hat{w} \circ w = w$ because $t \circ s = w$ and $w \circ \hat{w} \circ w = w$.

We show next that $\hat{u} = \hat{w} \circ t$. Indeed, we have $u \circ (\hat{w} \circ t) \circ u = s \circ \hat{w} \circ w \circ t \circ s \circ \hat{w} \circ w = s \circ \hat{w} \circ w = u$, and $(\hat{w} \circ t) \circ u \circ (\hat{w} \circ t) = \hat{w} \circ t \circ s \circ \hat{w} \circ w \circ \hat{w} \circ t = \hat{w} \circ t$. This shows $\hat{u} = \hat{w} \circ t$; hence, $u = \hat{t} \circ w$. A similar argument yields $\hat{v} = s \circ \hat{w}$, and hence $v = w \circ \hat{s}$. Finally, we have $u \circ \hat{u} = (s \circ \hat{w} \circ w) \circ (\hat{w} \circ t) = (s \circ \hat{w}) \circ (w \circ \hat{w} \circ t) = \hat{v} \circ v$, which concludes the proof. □

Remark 3.6. Let $\mathcal{C}$ be a finite inverse category. If s, t are morphisms in $\mathcal{C}$ such that $s \leqslant t$ and such that t is an idempotent endomorphism of an object X in $\mathcal{C}$, then Lemma 2.1, together with the fact that idempotents in $\mathrm{End}_{\mathcal{C}}(X)$ commute, implies that s is also an idempotent endomorphism. Thus, the set $E(\mathcal{C})$ of all idempotent endomorphisms in $\mathcal{C}$ is a downwardly closed subposet of the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms in $\mathcal{C}$. In particular, the Möbius function of $E(\mathcal{C})$ is the restriction of that of $\mathrm{Mor}(\mathcal{C})$. Moreover, $\mathrm{End}_{\mathcal{C}}(X)$ has a unique minimal idempotent e_X, namely the product of all idempotents in $\mathrm{End}_{\mathcal{C}}(X)$. Thus, for any idempotent e in $\mathrm{End}_{\mathcal{C}}(X)$, we have $\sum_{f; f \leqslant e} \mu(f, e) = \sum_{e_X \leqslant f \leqslant e} \mu(f, e) = 0$, unless $e = e_X$, in which case this sum is equal to 1.

4. Finite inverse category algebras

Let $\mathcal{C}$ be an inverse category. The *groupoid associated with* $\mathcal{C}$ is the category, denoted $G(\mathcal{C})$, defined as follows. The objects of $G(\mathcal{C})$ are the pairs (X, e), with X an object in $\mathcal{C}$ and e an idempotent in $\mathrm{End}_{\mathcal{C}}(X)$. A morphism in $G(\mathcal{C})$ from (X, e) to (Y, f) is a triple (e, s, f), where $s \colon X \to Y$ is a morphism in $\mathcal{C}$ satisfying $\hat{s} \circ s = e$ and $s \circ \hat{s} = f$. Note that in that case $s = s \circ e = f \circ s$. In other words, $G(\mathcal{C})$ is the subcategory of the idempotent completion $\hat{\mathcal{C}}$ of $\mathcal{C}$ having the same objects as $\hat{\mathcal{C}}$ and all isomorphisms in $\hat{\mathcal{C}}$ as morphisms; in particular, $G(\mathcal{C})$ is indeed a groupoid (and this definition of $G(\mathcal{C})$ makes sense for any small category $\mathcal{C}$). As mentioned earlier, the idempotents e, f are called isomorphic if the objects (X, e), (Y, f) are isomorphic in $G(\mathcal{C})$. By the above, this is equivalent to the existence of a morphism $(X, e) \to (Y, f)$ in $G(\mathcal{C})$ and also equivalent to the existence of a morphism $s \colon X \to Y$ in $\mathcal{C}$ satisfying $\hat{s} \circ s = e$ and $s \circ \hat{s} = f$. The group $\mathcal{C}_e$ considered in the introduction is canonically isomorphic to $\mathrm{Aut}_{G(\mathcal{C})}(X, e)$ via the map sending $s \in \mathrm{End}_{\mathcal{C}}(X)$ satisfying $s \circ \hat{s} = e = \hat{s} \circ s$ to (e, s, e). The category algebra of the groupoid $G(\mathcal{C})$ turns out to be isomorphic to the category algebra of the category $\mathcal{C}$ itself; this extends [**21**, 4.2] to inverse categories.

Theorem 4.1. *Let $\mathcal{C}$ be a finite inverse category and let k be a commutative ring. The map sending a morphism $(e, s, f) \colon (X, e) \to (Y, f)$ in $G(\mathcal{C})$ to the element $\underline{s}$ in $k\mathcal{C}$, where X, Y are objects in $\mathcal{C}$ and $e \in \mathrm{End}_{\mathcal{C}}(X)$, $f \in \mathrm{End}_{\mathcal{C}}(Y)$ are idempotents, is an isomorphism of k-algebras $kG(\mathcal{C}) \cong k\mathcal{C}$.*

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Proof. The map sending (e,s,f) to $\underline{s}$ is multiplicative by Proposition 3.1 (ii). This map is also a k-linear isomorphism because, by Möbius inversion, the set of $\underline{s}$, with $s \in \mathrm{Mor}(\mathcal{C})$, form again a k-basis of $k\mathcal{C}$. □

Finite inverse categories thus provide examples of non-equivalent categories with isomorphic category algebras. Even though there is a canonical forgetful functor $G(\mathcal{C}) \to \mathcal{C}$, the isomorphism $kG(\mathcal{C}) \cong k\mathcal{C}$ is in general not induced by any functor. In particular, a constant functor on $\mathcal{C}$ need not correspond to a constant functor on $G(\mathcal{C})$. A groupoid algebra over a commutative ring k is well known to be isomorphic to a direct product of matrix algebras over the group algebras of automorphism groups of objects, one factor for each isomorphism class of objects, and the size of that isomorphism class is equal to the size of the involved matrices. Translated to the situation of Theorem 4.1, this yields the following statement.

Corollary 4.2. *Let $\mathcal{C}$ be a finite inverse category, let k be a commutative ring and let E be a set of representatives of the isomorphism classes of idempotent endomorphisms in $\mathcal{C}$. For $e \in E$, denote by $n(e)$ the number of idempotents in $\mathrm{Mor}(\mathcal{C})$ isomorphic to e. We have an isomorphism of k-algebras $k\mathcal{C} \cong \prod_{e\in E} M_{n(e)}(k\mathcal{C}_e)$.*

Here $\mathcal{C}_e$ is, as defined earlier, the group consisting of endomorphisms s of an object X satisfying $s \circ \hat{s} = e = \hat{s} \circ s$, where e is an idempotent endomorphism of X. The isomorphism above shows that if k is a field of characteristic 0 or prime characteristic not dividing the order of any of the groups $\mathcal{C}_e$, where e runs over the idempotents in $\mathrm{Mor}(\mathcal{C})$, then $k\mathcal{C}$ is semi-simple. This generalizes well-known facts about inverse semigroups [**18**, **19**]. This isomorphism shows further that the category of functors from $\mathcal{C}$ to $\mathrm{Mod}(k)$ is equivalent to the product of the module categories of the group algebras $k\mathcal{C}_e$; thus, any block of $k\mathcal{C}$ is Morita equivalent to a block of one of the finite group algebras $k\mathcal{C}_e$. In particular, every functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{Mod}(k)$ decomposes naturally as a direct sum of functors $\mathcal{F} = \bigoplus_{e\in E} \mathcal{F}_e$.

Using the isomorphism from Theorem 4.1, one can describe the functors $\mathcal{F}_e$ and their cohomological properties more explicitly. An idempotent endomorphism e of an object X in $\mathcal{C}$, when viewed as the morphism (e,e,e) in $G(\mathcal{C})$, is mapped under the isomorphism $kG(\mathcal{C}) \cong k\mathcal{C}$ from Theorem 4.1 to the idempotent $\underline{e} = \sum_{f\leqslant e} \mu(f,e) f$ in $k\,\mathrm{End}_{\mathcal{C}}(X)$. If $\mathcal{F}\colon \mathcal{C} \to \mathrm{Mod}(k)$ is a functor, setting $\mathcal{F}(\underline{e}) = \sum_{f\leqslant e} \mu(f,e)\mathcal{F}(f)$, we get that $\mathcal{F}(\underline{e})$ is an idempotent endomorphism of the k-module $\mathcal{F}(X)$, and its image $\mathcal{F}(\underline{e})(\mathcal{F}(X))$ is a $k\mathcal{C}_e$-module. If $\mathcal{F}$ is a constant functor on $\mathcal{C}$, then $\mathcal{F}(\underline{e}) = \sum_{f\leqslant e} \mu(f,e)\,\mathrm{Id}_{\mathcal{F}(X)}$, and hence by Remark 3.6 we have $\mathcal{F}(\underline{e}) = 0$ unless e is a minimal idempotent with respect to the canonical partial order on the morphism set of $\mathcal{C}$. Combining these observations with the isomorphism Theorem 4.1 immediately yields the following corollary.

Corollary 4.3. *Let $\mathcal{C}$ be a finite inverse category, let k be a commutative ring and let E be a set of representatives of the isomorphism classes of idempotent endomorphisms in $\mathcal{C}$.*

(i) *Every functor $\mathcal{F}\colon \mathcal{C} \to \mathrm{Mod}(k)$ can be written uniquely as a direct sum $\mathcal{F} = \bigoplus_{e\in E} \mathcal{F}_e$ of functors $\mathcal{F}_e\colon \mathcal{C} \to \mathrm{Mod}(k)$ satisfying $\mathcal{F}_e(\underline{f}) = \mathcal{F}(\underline{f})$ if e, f are isomorphic idempotent endomorphisms in $\mathcal{C}$, and $\mathcal{F}_e(\underline{f}) = 0$, otherwise.*

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(ii) *For any two functors* $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathrm{Mod}(k)$ *and* $e \in E$ *we have*

$$\mathrm{Ext}^*_{k\mathcal{C}}(\mathcal{F}_e, \mathcal{G}_e) \cong \mathrm{Ext}^*_{k\mathcal{C}_e}(\mathcal{F}(\underline{e})(\mathcal{F}(X)), \mathcal{G}(\underline{e})(\mathcal{G}(X)))$$

and $\mathrm{Ext}^*_{k\mathcal{C}}(\mathcal{F}_e, \mathcal{G}_f) = \{0\}$ *if* $e, f \in E$ *are different.*

(iii) *If* $\mathcal{F} \colon \mathcal{C} \to \mathrm{Mod}(k)$ *is a constant functor, then* $\mathcal{F} = \bigoplus_e \mathcal{F}_e$*, where* e *runs over a set of representatives of the isomorphism classes of minimal idempotent endomorphisms in* $\mathcal{C}$*; in particular,* $\mathcal{F}(\underline{e}) = 0$ *if* e *is an idempotent endomorphism that is not minimal.*

(iv) *We have* $H^*(\mathcal{C}; k) \cong \prod_e H^*(\mathcal{C}_e; k)$*, where* e *runs over a set of representatives of the isomorphism classes of minimal idempotent endomorphisms in* $\mathcal{C}$.

(v) *We have* $HH^*(k\mathcal{C}) \cong \prod_{e \in E} HH^*(k\mathcal{C}_e)$.

The following statement on symmetrizing forms of groupoid algebras is well known and easily verified (we leave the proof to the reader).

Lemma 4.4. *Let* k *be a commutative ring and let* $\mathcal{G}$ *be a finite groupoid. The category algebra* $k\mathcal{G}$ *is symmetric. More precisely, there is a bimodule isomorphism* $(k\mathcal{G})^* \cong k\mathcal{G}$ *sending* $\mu \in (k\mathcal{G})^*$ *to the element* $\sum_{s \in \mathrm{Mor}(\mathcal{G})} \mu(s^{-1})s$ *in* $k\mathcal{G}$*. The symmetrizing form* $\tau \in (k\mathcal{G})^*$ *corresponding to* $1_{k\mathcal{G}}$ *under this isomorphism satisfies* $\tau(s) = 1$ *if* $s = \mathrm{Id}_X$ *for some* $X \in \mathrm{Ob}(\mathcal{G})$*, and* $\tau(s) = 0$ *otherwise.*

Proof of Theorem 1.1. We have already noted in Corollary 4.2 that $k\mathcal{C}$ is a direct product of matrix algebras over finite group algebras. In order to prove the statement on the symmetrizing form, we need to follow the canonical symmetrizing forms on the finite groupoid algebra $kG(\mathcal{C})$ from Lemma 4.4 through the isomorphism in Theorem 4.1. The identity morphism of an object (X, e) in the groupoid $G(\mathcal{C})$ is of the form (e, e, e), which, under the isomorphism from Theorem 4.1, is sent to $\underline{e}$. Thus, $k\mathcal{C}$ inherits a canonical symmetrizing form sending $\underline{s}$ to 1 if s is an idempotent and to zero otherwise. Since $s = \sum_{u \leqslant s} \underline{u}$ by Möbius inversion, the formula for τ follows. □

Proof of Corollary 1.3. By a result of Gerstenhaber [**6**], the Hochschild cohomology of an algebra is graded commutative. The Hochschild cohomology of a finite group algebra over a commutative Noetherian ring is finitely generated. Since Hochschild cohomology is invariant under Morita equivalences and compatible with direct products of algebras, the finite generation of $HH^*(k\mathcal{C})$ follows. By [**23**, Theorem A], $H^*(\mathcal{C}; \underline{k})$ is a quotient of $HH^*(k\mathcal{C})$, whence the result follows. □

Proof of Corollary 1.3. This follows from Corollary 4.3 (iv) and the standard fact [**9**, (11.15)]. □

Lemma 4.5. *Let* k *be a commutative ring, let* $\mathcal{G}$ *be a finite groupoid and let* $\mathcal{H}$ *be a subgroupoid of* $\mathcal{G}$*. Set* $j = 1_{k\mathcal{H}} = \sum_{Y \in \mathrm{Ob}(\mathcal{H})} \mathrm{Id}_Y$*. Then* $k\mathcal{G}j$ *is finitely generated projective as a right* $k\mathcal{H}$*-module and* $jk\mathcal{G}$ *is finitely generated as a left* $k\mathcal{H}$*-module.*

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Proof. Projectivity of a module is invariant under Morita equivalences, and hence we may assume that the objects of $\mathcal{H}$ are pairwise non-isomorphic. Thus, $\mathcal{H}$ is a direct product of groups. The result follows from the fact that the action of $\mathrm{Aut}_{\mathcal{H}}(Y)$ on $\mathrm{Hom}_{\mathcal{G}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{G}}(Y,X)$ is free, for any object Y in $\mathcal{H}$ and any object X in $\mathcal{G}$. □

Under certain circumstances, the map in Theorem 4.1 is functorial, proving, in particular, Theorem 1.4.

Theorem 4.6. *Let k be a commutative ring, let $\mathcal{C}$ be a finite inverse category and let $\mathcal{D}$ be an inverse subcategory of $\mathcal{C}$ such that for any object Y in $\mathcal{D}$ the endomorphism monoid $\mathrm{End}_{\mathcal{D}}(Y)$ contains all idempotent endomorphisms in $\mathrm{End}_{\mathcal{C}}(Y)$. Then the following diagram of non-unitary algebra homomorphisms is commutative:*

$$\begin{tikzcd} kG(\mathcal{D}) \arrow[r, "\cong"] \arrow[d] & k\mathcal{D} \arrow[d] \\ kG(\mathcal{C}) \arrow[r, "\cong"'] & k\mathcal{C} \end{tikzcd}$$

where the horizontal maps are the isomorphisms from Theorem 4.1 and the vertical maps are the injective non-unitary algebra homomorphisms induced by the corresponding inclusions of categories. In particular, setting $j = 1_{k\mathcal{D}} = \sum_{Y\in\mathrm{Ob}(\mathcal{D})} \mathrm{Id}_Y$, the $k\mathcal{C}$–$k\mathcal{D}$-bimodule $k\mathcal{C}j$ and the $k\mathcal{D}$–$k\mathcal{C}$-bimodule $jk\mathcal{C}$ are both finitely generated projective as left and right modules. Moreover, the restriction to $k\mathcal{D}$ of the canonical symmetrizing form of $k\mathcal{C}$ is the canonical symmetrizing form of $k\mathcal{D}$.

Proof. Since $\mathrm{Mor}(\mathcal{D})$ contains all idempotents in endomorphism monoids in $\mathcal{C}$ of objects in $\mathcal{D}$, it follows from Lemma 2.1 (ii) that if s, t are morphisms in $\mathcal{C}$ satisfying $s \leqslant t$ and if t is a morphism in $\mathcal{D}$, then s is a morphism in $\mathcal{D}$ as well. In other words, $\mathrm{Mor}(\mathcal{D})$ is a downwardly closed subposet of $\mathrm{Mor}(\mathcal{C})$, and hence the Möbius function on $\mathrm{Mor}(\mathcal{D})$ is the restriction of the Möbius function on $\mathrm{Mor}(\mathcal{C})$. This means that for a morphism s in $\mathcal{D}$ the expression $\underline{s} = \sum_{u\leqslant s} \mu(u,s)u$ does not depend on whether we regard s as a morphism in $\mathcal{C}$ or in $\mathcal{D}$. This shows the commutativity of the diagram in the statement. The statement on projectivity follows from Lemma 4.5. The description of the canonical symmetrizing forms in Theorem 1.1 implies the last statement. □

Example 4.7. Let λ be a partition of a finite set M and let k be a commutative ring. Any idempotent endomorphism in the inverse category $I(\lambda)$ defined in § 2 is of the form Id_V for a subset V of a set $U \in \lambda$. Given subsets $V \subseteq U$, $V' \subseteq U'$, where $U, U' \in \lambda$, the idempotents Id_V and $\mathrm{Id}_{V'}$ are isomorphic if and only if V and V' have the same cardinality. Thus, setting $m = |M|$ and

$$i(n) = \sum_{U\in\lambda} \binom{|U|}{n}$$

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for any non-negative integer n, we have an isomorphism of k-algebras

$$kI(\lambda) \cong \prod_{0\leqslant n\leqslant m} M_{i(n)}(kS_n),$$

where S_n denotes as usual the symmetric group of degree n, with the convention $i_0 = |\lambda|$ and $S_0 = \{1\}$; the factor for $n = 0$ corresponds to the isomorphism class of idempotents associated with the empty subsets of the subsets of M in λ.

5. Adjunction morphisms for groupoid algebras

Throughout this section, k is a commutative ring. If A is a k-algebra, i is an idempotent in A and B is a unitary subalgebra of iAi, then the 'truncated restriction' functor sending an A-module U to the B-module iU is isomorphic to the functor $iA \otimes_A -$, where iA is considered as a B–A-bimodule. This functor has as left adjoint the 'truncated induction' functor $Ai \otimes_B -$. There is a canonical choice of an adjunction isomorphism (this is a special case of the standard adjunction between Hom-functors and the tensor product; the proof is left to the reader).

Lemma 5.1. *Let A be a k-algebra, let i be an idempotent in A and let B be a unitary subalgebra of iAi. The functor $Ai \otimes_B -$ is left adjoint to the functor $iA \otimes_A -$, and there is a canonical adjunction isomorphism whose unit and counit are as follows:*

(i) *the adjunction unit is represented by the inclusion $B \to iAi \cong iA \otimes_A Ai$, viewed as a B–B-bimodule homomorphism;*

(ii) *the adjunction counit is represented by the A–A-bimodule homomorphism $Ai \otimes_B iA \to A$ induced by multiplication in A.*

If A, B are symmetric and Ai is finitely generated projective as a right B-module, then $Ai \otimes_B -$ is also right adjoint to $iA \otimes_A -$. More precisely, any choice of symmetrizing forms on A, B determines an adjunction isomorphism whose adjunction unit and counit are obtained from dualizing the adjunction unit and counit from the left adjunction of $Ai \otimes_B -$ to $iA - \otimes_A -$; see, for example, [**3**] for some background material. In that case we get *relatively projective elements* $\pi_{Ai} \in Z(A)$ and $\pi_{iA} \in Z(B)$ defined as follows (cf. [**15**, 3.1]): the composition $A \to Ai \otimes_B iA \to A$ of the appropriate adjunction unit and counit yields an endomorphism of A as an A–A-bimodule, which is hence induced by left (or right) multiplication by an element $\pi_{Ai} \in Z(A)$; similarly, the composition $B \to iA \otimes_A Ai \to B$ of the appropriate adjunction unit and counit is an endomorphism of B as a B–B-bimodule, which is hence induced by left or right multiplication by an element $\pi_{iA} \in Z(B)$. The purpose of this section is to calculate these elements when A is a finite groupoid algebra and B is a subgroupoid algebra; for finite group algebras these calculations are well known.

For the remainder of this section, let $\mathcal{G}$ be a finite groupoid, let $\mathcal{H}$ be a subgroupoid of $\mathcal{G}$ and let $j = 1_{k\mathcal{H}} = \sum_{Y\in\mathrm{Ob}(\mathcal{H})} \mathrm{Id}_Y$, the sum taken in $k\mathcal{G}$. As mentioned in Lemma 4.4, the algebra $k\mathcal{G}$ is symmetric with a canonical symmetrizing form $\tau \in (k\mathcal{G})^*$ satisfying

$\tau(s) = 1$ if $s = \mathrm{Id}_X$ for some $X \in \mathrm{Ob}(\mathcal{G})$ and $\tau(s) = 0$ otherwise. The restriction of τ to $k\mathcal{H}$ yields the corresponding symmetrizing form for $k\mathcal{H}$. We consider $k\mathcal{G}$ and $k\mathcal{H}$ to be endowed with their canonical symmetrizing forms. Denote by $\mathrm{Mor}(\mathcal{G}/\mathcal{H})$ the set of all morphisms s in $\mathcal{G}$ that start at an object in $\mathcal{H}$ or, equivalently, that satisfy $s \cdot j = s$ in $k\mathcal{G}$. Two morphisms s, s' in $\mathrm{Mor}(\mathcal{G}/\mathcal{H})$ are called *$\mathcal{H}$-equivalent* if there is a morphism t in $\mathcal{H}$ such that $s' = s \circ t$. Note that $\mathcal{H}$-equivalent morphisms have necessarily the same target. We denote by $\mathcal{G}/\mathcal{H}$ a set of representatives of the $\mathcal{H}$-equivalence classes of morphisms in $\mathrm{Mor}(\mathcal{G}/\mathcal{H})$. We denote by $\pi_{\mathcal{G}/\mathcal{H}}$ the element in $k\mathcal{G}j \otimes_{k\mathcal{H}} jk\mathcal{G}$ defined by

$$\pi_{\mathcal{G}/\mathcal{H}} = \sum_{s \in \mathcal{G}/\mathcal{H}} s \otimes s^{-1}.$$

This element does not depend on the choice of $\mathcal{G}/\mathcal{H}$, and we have $s \cdot \pi_{\mathcal{G}/\mathcal{H}} = \pi_{\mathcal{G}/\mathcal{H}} \cdot s$ for any morphism s in $\mathcal{G}$.

Lemma 5.2. *With the notation above, the functor $k\mathcal{G}j \otimes_{k\mathcal{H}} -$ is right adjoint to the functor $jk\mathcal{G} \otimes_{k\mathcal{G}} -$ and there is a canonical choice of an adjunction isomorphism whose unit and counit are as follows:*

(i) *the adjunction unit is represented by the $k\mathcal{G}$–$k\mathcal{G}$-bimodule homomorphism*

$$k\mathcal{G} \to k\mathcal{G}j \otimes_{k\mathcal{H}} jk\mathcal{G}$$

sending $1_{k\mathcal{G}}$ to $\pi_{\mathcal{G}/\mathcal{H}}$;

(ii) *the adjunction counit is represented by the $k\mathcal{H}$–$k\mathcal{H}$-bimodule homomorphism*

$$jk\mathcal{G} \otimes_{k\mathcal{G}} k\mathcal{G}j \cong jk\mathcal{G}j \to k\mathcal{H}$$

induced by the map sending $s \in \mathrm{Mor}(\mathcal{H})$ to s and $s \in \mathrm{Mor}(\mathcal{G}) \setminus \mathrm{Mor}(\mathcal{H})$ to 0.

Proof. One way to prove this is by dualizing the adjunction unit and counit of the left adjunction of $k\mathcal{G}e \otimes_{k\mathcal{H}} -$ to $ek\mathcal{G} \otimes_{k\mathcal{G}} -$; this is a standard argument (see, for example, [**15**, § 6]). Alternatively, an easy verification shows that the appropriate compositions $k\mathcal{G}j \to k\mathcal{G}j \otimes_{k\mathcal{H}} jk\mathcal{G} \otimes_{k\mathcal{G}} k\mathcal{G}j \to k\mathcal{G}j$ and $jk\mathcal{G} \to jk\mathcal{G} \otimes_{k\mathcal{G}} k\mathcal{G}j \otimes_{k\mathcal{H}} jk\mathcal{G} \to jk\mathcal{G}$ of the given maps in the statement are the identity bimodule endomorphisms on $k\mathcal{G}j$ and $jk\mathcal{G}$, respectively. □

For any object X in $\mathcal{G}$ define an integer $n(X)$ by

$$n(X) = \sum_{Y} [\mathrm{Aut}_{\mathcal{G}}(Y) : \mathrm{Aut}_{\mathcal{H}}(Y)],$$

where Y runs over a set of representatives of the $\mathcal{H}$-isomorphism classes of objects contained in the $\mathcal{G}$-isomorphism class of X, with the usual convention that this is zero if the sum is empty.

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Lemma 5.3. *With the notation above, for any object X in $\mathcal{G}$, the integer $n(X)$ is equal to the number of morphisms in $\mathcal{G}/\mathcal{H}$ ending at X; in particular, we have $|\mathcal{G}/\mathcal{H}| = \sum_X n(x)$, where X runs over a set of representatives of the isomorphism classes of objects in $\mathcal{G}$.*

Proof. If $s\colon Y \to X$ and $s'\colon Y' \to X$ are two morphisms in $\mathcal{G}$ ending at X, then Y and Y' belong to the same $\mathcal{G}$-isomorphism class as X. Moreover, s, s' are $\mathcal{H}$-equivalent if and only if there is a morphism $t\colon Y \to Y'$ in $\mathcal{H}$ satisfying $s' = s \circ t$, which in particular is only possible if Y, Y' are isomorphic objects in $\mathcal{H}$. The number of pairwise inequivalent morphisms from Y to X is clearly equal to the number of pairwise inequivalent automorphisms of Y, and hence equal to $[\mathrm{Aut}_{\mathcal{G}}(Y) : \mathrm{Aut}_{\mathcal{H}}(Y)]$. The result follows. $\square$

We use this to calculate the relatively projective elements $\pi_{k\mathcal{G}j} \in Z(k\mathcal{G})$ and $\pi_{jk\mathcal{G}} \in Z(k\mathcal{H})$.

Lemma 5.4. *With the notation above, we have*

$$\pi_{jk\mathcal{G}} = j = 1_{k\mathcal{H}},$$
$$\pi_{k\mathcal{G}j} = \sum_{X \in \mathrm{Ob}(\mathcal{G})} n(X) \cdot \mathrm{Id}_X .$$

Proof. The composition $k\mathcal{H} \to jk\mathcal{G} \otimes_{k\mathcal{G}} k\mathcal{G}j \to k\mathcal{H}$ of the appropriate adjunction unit and counit sends $1_{k\mathcal{H}}$ to $1_{k\mathcal{H}}$, where we use the first statement of Lemma 5.1 and the second statement of Lemma 5.2. The composition $k\mathcal{G} \to k\mathcal{G}j \otimes_{k\mathcal{H}} jk\mathcal{G} \to k\mathcal{G}$ of the remaining adjunction unit and counit sends $1_{k\mathcal{G}}$ to $\sum_{s \in \mathcal{G}/\mathcal{H}} s \circ s^{-1}$. The lemma follows from Lemma 5.3. $\square$

Proof of Theorem 1.5. The existence of a transfer map $\mathrm{tr}_{\mathcal{D}}^{\mathcal{C}}$ with the property stated in (i) is a purely formal consequence of Theorem 1.4 and the general reciprocity result in [**17**, 4.8]. Similarly, (ii) follows from [**16**, 5.1]. We need to calculate $\mathrm{htr}_{\mathcal{D}}^{\mathcal{C}}(1)$. This element is equal to the relative projective element $\pi_{k\mathcal{C}i}$ in $Z(k\mathcal{C})$, where $i = 1_{k\mathcal{D}}$. We calculate this using the results of this section applied to the groupoids associated with $\mathcal{C}$, $\mathcal{D}$ and then applying Theorem 4.6. By Theorem 4.6, the isomorphism $kG(\mathcal{C}) \cong k\mathcal{C}$ from Theorem 4.1 maps $j = 1_{kG(\mathcal{D})}$ to $i = 1_{k\mathcal{D}}$. Thus, $\pi_{\mathcal{D}}^{\mathcal{C}}$ is the image under this isomorphism of the relatively projective element $\pi_{kG(\mathcal{C})j}$. Note that $\mathrm{Aut}_{G(\mathcal{C})}(X, e) = \mathcal{C}_e$ for any object X in $\mathcal{C}$ and any idempotent endomorphism e of X in $\mathcal{C}$. Thus, by Lemma 5.4, we have $\pi_{kG(\mathcal{C})j} = \sum_e n(e)e$, the sum taken in $kG(\mathcal{C})$, where $n(e) = \sum_f [\mathcal{C}_f\colon \mathcal{D}_f]$, with f running over a set of representatives of the $\mathcal{D}$-isomorphism classes of idempotent endomorphisms in $\mathcal{D}$ contained in the $\mathcal{C}$-isomorphism class of e. The isomorphism $kG(\mathcal{C}) \cong k\mathcal{C}$ from Theorem 4.1 maps this element to the element $\pi_{\mathcal{D}}^{\mathcal{C}} = \sum_e n(e)\underline{e}$. Using $\underline{e} = \sum_{f \leqslant e} \mu(f, e)f$ yields the formula for $\pi_{\mathcal{D}}^{\mathcal{C}}$ as in the paragraph preceding Theorem 1.5. Clearly, $\pi_{kG(\mathcal{C})j}$ is invertible if and only if all $n(e)$ are invertible in k; hence, the same is true for $\pi_{\mathcal{D}}^{\mathcal{C}}$. This proves (iii). In order to prove (iv) we need to calculate $\tau_{\mathcal{D}}^{\mathcal{C}}$ more explicitly. It follows from statement (ii) applied to $\eta = 1 \in HH^0(k\mathcal{D})$ that $\mathrm{tr}_{\mathcal{D}}^{\mathcal{C}}(1)$ is indeed equal to the image $\tau_{\mathcal{D}}^{\mathcal{C}}$ of $\pi_{\mathcal{D}}^{\mathcal{C}}$ under the canonical map $HH^0(k\mathcal{C}) = Z(k\mathcal{C}) \to H^0(\mathcal{C}; k) = \lim_{\mathcal{C}}(k)$. This map sends $\pi_{\mathcal{D}}^{\mathcal{C}}$ to the family $\tau_{\mathcal{D}}^{\mathcal{C}} = (\tau_X)_{X \in \mathrm{Ob}(\mathcal{C})}$ in $\lim_{\mathcal{C}}(k)$, where $\tau_X = \sum_{(e,f)} n(e)\mu(f,e)$, with (e,f) running over the pairs of idempotents in $\mathrm{End}_{\mathcal{C}}(X)$ satisfying $f \leqslant e$. If e is not a minimal idempotent, then by Remark 3.6 we have

$$\sum_{f;\, f \leqslant e} n(e)\mu(f,e) = 0.$$

Thus, $\tau_X = n(e_X)$ for all objects X in $\mathcal{C}$. This shows (iv), and (v) follows from applying (i) with $\theta = 1$. □

6. Embeddings into inverse categories

Example 6.1. Let n be a non-negative integer, and view the totally ordered set $\boldsymbol{n} = \{0, 1, 2, \dots, n\}$ as a category. Then $\boldsymbol{n}$ can be embedded into an inverse category $\hat{\boldsymbol{n}}$ defined as follows: the objects of $\hat{\boldsymbol{n}}$ are those of $\boldsymbol{n}$, and for $0 \leqslant i, j \leqslant n$ the morphism set $\mathrm{Hom}_{\hat{\boldsymbol{n}}}(i,j)$ consists of all triples (i,j,a) such that $0 \leqslant a \leqslant \min\{i,j\}$. The canonical involution on the morphism set is given by $\widehat{(i,j,a)} = (j,i,a)$, and the composition in $\hat{\boldsymbol{n}}$ is defined by $(k,j,b) \circ (i,j,a) = (i,k,\min\{a,b\})$. There is a canonical functor $\boldsymbol{n} \to \hat{\boldsymbol{n}}$, which is the identity on objects and which sends a morphism $i \to j$ in $\boldsymbol{n}$ to the morphism (i,j,i) in $\hat{\boldsymbol{n}}$, where $0 \leqslant i \leqslant j \leqslant n$. Idempotents in $\hat{n}$ are of the form (i,i,a). Two idempotents (i,i,a), (j,j,b) are isomorphic if and only if $a = b$, via the morphisms (i,j,a) and (j,i,a). Thus, the isomorphism class of the idempotent (a,a,a) contains the $n - a + 1$ idempotents (i,i,a), $a \leqslant i \leqslant n$. If a runs from 0 to n, then $b = n - a + 1$ runs from $n+1$ to 1. Moreover, the automorphism group associated with any idempotent is trivial. It follows from Theorem 4.1 that if k is a commutative ring, the algebra $k\hat{\boldsymbol{n}}$ is a direct product of matrix algebras $\prod_{b=1}^{n+1} M_b(k)$.

The above example is a special case of more general embeddings of categories into inverse categories. This is based on a well-known construction principle for categories, described, for instance, in [**20**, § 1]. Given a small category $\mathcal{C}$ in which pull-backs exist, we define a category $\hat{\mathcal{C}}$ as follows. We set $\mathrm{Ob}(\hat{\mathcal{C}}) = \mathrm{Ob}(\mathcal{C})$. A morphism in $\hat{\mathcal{C}}$ from X to Y is an equivalence class $[U; \sigma, \varphi]$ of triples (U, σ, φ) consisting of an object U in $\mathcal{C}$ and two morphisms $\sigma \colon U \to X$, $\varphi \colon U \to Y$, where two such triples (U, σ, φ), (U', σ', φ') are equivalent if there is an isomorphism $\alpha \colon U \cong U'$ in $\mathcal{C}$ such that $\sigma = \sigma' \circ \alpha$ and $\varphi = \varphi' \circ \alpha$. The composition of a morphism $[U; \sigma, \varphi]$ from X to Y and a morphism $[V; \tau, \psi]$ from Y to Z is the morphism $[W; \sigma \circ \mu, \psi \circ \nu]$ such that the square in the following diagram is a pull-back:

$$\begin{tikzcd}
X & & Y & & Z \\
& U \arrow[ul, "\sigma"] \arrow[ur, "\varphi"'] & & V \arrow[ul, "\tau"'] \arrow[ur, "\psi"'] & \\
& & W \arrow[ul, "\mu"] \arrow[ur, "\nu"'] & &
\end{tikzcd}$$

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Using the universal property of pull-backs, one checks that this composition is associative. If $s = [U; \sigma, \varphi]$ is a morphism in $\hat{\mathcal{C}}$, we set $\hat{s} = [U; \varphi, \sigma]$. Clearly, $\hat{\hat{s}} = s$ and $\widehat{t \circ s} = \hat{s} \circ \hat{t}$ for two composable morphisms s and t in $\hat{\mathcal{C}}$. In other words, the map sending s to $\hat{s}$ is an involution on $\hat{\mathcal{C}}$; in particular, $\hat{\mathcal{C}}$ is isomorphic to its opposite category. There is a canonical covariant functor from $\mathcal{C}$ to $\hat{\mathcal{C}}$ defined as the identity on objects and sending a morphism $\varphi \colon X \to Y$ in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to the morphism $[X; \mathrm{Id}_X, \varphi]$ in $\mathrm{Hom}_{\hat{\mathcal{C}}}(X, Y)$. The canonical functor $\mathcal{C} \to \hat{\mathcal{C}}$ is an isomorphism of categories if and only if $\mathcal{C}$ is a groupoid. There is also a canonical contravariant functor sending φ to $[X; \varphi, \mathrm{Id}_X]$; the two embeddings 'differ' by the involution of $\hat{\mathcal{C}}$. The following theorem contains a restatement of Theorem 1.6.

Theorem 6.2. *Let $\mathcal{C}$ be a small category in which any morphism is a monomorphism and in which pull-backs exist. Let k be a commutative ring and let A be a k-module.*

(i) *The category $\hat{\mathcal{C}}$ as defined above is an inverse category, the canonical functor $\mathcal{C} \to \hat{\mathcal{C}}$ is injective on morphisms and $\hat{\mathcal{C}}$ is idempotent complete.*

(ii) *If $\mathcal{C}$ is finite, then $\hat{\mathcal{C}}$ is finite and the category algebra $k\hat{\mathcal{C}}$ is Morita equivalent to $\prod_U k\,\mathrm{Aut}_{\mathcal{C}}(U)$, where U runs over a set of representatives of the isomorphism classes of objects in $\mathcal{C}$.*

(iii) *Restriction induces an isomorphism $H^*(\hat{\mathcal{C}}; A) \cong H^*(\mathcal{C}; A)$.*

(iv) *Let Φ be a covariant functor from $\mathcal{C}$ to an inverse category $\mathcal{D}$ with the property that if*

$$\begin{tikzcd}
 & Y & \\
U \arrow[ur, "\varphi"] & & V \arrow[ul, "\tau"'] \\
 & W \arrow[ul, "\mu"] \arrow[ur, "\nu"'] &
\end{tikzcd}$$

is a pull-back square in $\mathcal{C}$, then the diagram

$$\begin{tikzcd}
 & \Phi(Y) \arrow[dl, "\widehat{\Phi(\varphi)}"'] & \\
\Phi(U) & & \Phi(V) \arrow[ul, "\Phi(\tau)"'] \arrow[dl, "\widehat{\Phi(\mu)}"] \\
 & \Phi(W) \arrow[ul, "\Phi(\mu)"] &
\end{tikzcd}$$

in $\mathcal{D}$ is commutative. Then Φ extends uniquely to a functor $\hat{\Phi}$ from $\hat{\mathcal{C}}$ to $\mathcal{D}$.

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Proof. Let $s = [U; \sigma, \varphi]$ and $t = [V; \tau, \psi]$ be morphisms in $\hat{\mathcal{C}}$ satisfying $s \circ t \circ s = s$ and $t \circ s \circ t = t$. The morphism $s \circ t \circ s$ is represented by a commutative diagram of the form

$$\begin{tikzcd}
X & & Y & & X & & Y \\
& U \arrow[ul, "\sigma"] \arrow[ur, "\varphi"'] & & V \arrow[ul, "\tau"] \arrow[ur, "\psi"'] & & U \arrow[ul, "\sigma"] \arrow[ur, "\varphi"'] & \\
& & W \arrow[ul, "\mu"] \arrow[ur, "\nu"'] & & & & \\
& & & T \arrow[ul, "\delta"] \arrow[uurr, "\eta"'] & & &
\end{tikzcd}$$

Since this also represents s there is an isomorphism $\alpha\colon T \cong U$ such that $\sigma \circ \alpha = \sigma \circ \mu \circ \delta$ and $\varphi \circ \alpha = \varphi \circ \eta$. Using the fact that σ and φ are monomorphisms, this implies

$$\alpha = \eta = \mu \circ \delta.$$

Thus, we may assume $T = U$ and $\eta = \mu \circ \delta = \mathrm{Id}_U$. Precomposing the last identity with μ yields $\mu \circ \delta \circ \mu = \mu$; hence, $\delta \circ \mu = \mathrm{Id}_W$, again because μ is a monomorphism. Thus, μ, δ are inverse isomorphisms, and we hence may assume $W = U$ and $\mu = \delta = \mathrm{Id}_U$. Note that the morphism $\nu\colon U \to V$ therefore satisfies $\varphi = \tau \circ \nu$ and $\sigma = \psi \circ \nu$. Similarly, the equality $t \circ s \circ t \circ t = t$ yields the existence of a morphism $\lambda\colon V \to U$ satisfying $\psi = \sigma \circ \lambda$ and $\tau = \varphi \circ \lambda$. Thus, $\psi = \psi \circ \nu \circ \lambda$, and since ψ is a monomorphism we get $\nu \circ \lambda = \mathrm{Id}_V$. Similarly, since $\varphi = \tau \circ \nu = \varphi \circ \lambda \circ \nu$, we get $\lambda \circ \nu = \mathrm{Id}_U$. Thus, we may assume $V = U$ and $\lambda = \nu = \mathrm{Id}_U$, which implies that $\psi = \sigma$ and $\tau = \varphi$. This shows that $\hat{\mathcal{C}}$ is an inverse category. The functor $\mathcal{C} \to \hat{\mathcal{C}}$ in the statement is clearly injective on morphisms. Using the fact that every morphism in $\mathcal{C}$ is a monomorphism, one verifies that any idempotent in $\hat{\mathcal{C}}$ is of the form $[U; \varphi, \varphi]$ for some morphism $\varphi\colon U \to X$. We have

$$[U; \varphi, \varphi] = [U; \mathrm{Id}_U, \varphi] \circ [U; \varphi, \mathrm{Id}_U] \quad \text{and} \quad [U; \mathrm{Id}_U, \mathrm{Id}_U] = [U; \varphi, \mathrm{Id}_U] \circ [U; \mathrm{Id}_U, \varphi];$$

hence, $[U; \varphi, \varphi]$ splits. This proves (i).

Given morphisms $\varphi\colon U \to X$ and $\psi\colon V \to Y$, the idempotents $e = [U; \varphi, \varphi]$ and $f = [V; \psi, \psi]$ are isomorphic if and only if $U \cong V$. More precisely, if $\mu\colon U \cong V$ is an isomorphism in $\mathcal{C}$ then the morphism $s = [U; \varphi, \psi \circ \mu]$ in $\hat{\mathcal{C}}$ satisfies $\hat{s} \circ s = e$ and $s \circ \hat{s} = f$. We have an automorphism of groups $\mathrm{Aut}_{\mathcal{C}}(U) \cong \hat{\mathcal{C}}_e$ sending $\mu \in \mathrm{Aut}_{\mathcal{C}}(U)$ to $[U; \varphi, \varphi \circ \mu]$, and now (ii) follows from Theorem 4.1.

In order to prove (iii), we need to calculate the right Kan extensions of the canonical functor $\mathcal{C} \to \hat{\mathcal{C}}$. For Y an object in $\hat{\mathcal{C}}$, the undercategory, denoted $\mathcal{C}^Y$, has as objects the pairs (X, s) consisting of an object X in $\mathcal{C}$ and a morphism $s\colon X \to Y$ in $\hat{\mathcal{C}}$. A morphism, in $\mathcal{C}^Y$, from (X, s) to (X', s') is a morphism $\alpha\colon X \to X'$ in $\mathcal{C}$ satisfying $s = s' \circ \alpha$, where we abusively denote the image of α in $\hat{\mathcal{C}}$ by α again. Write $s = [U; \sigma, \varphi]$ and $s' = [U'; \sigma', \varphi']$. The equality $s = s' \circ \alpha$ means that $s' \circ \alpha$ is represented by a diagram of

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the form

$$\begin{tikzcd}
X & & X' & & Y \\
& X \arrow[ul, "\mathrm{Id}_X"] \arrow[ur, "\alpha"'] & & U' \arrow[ul, "\sigma'"'] \arrow[ur, "\varphi'"'] & \\
& & U \arrow[ul, "\sigma"] \arrow[ur, "\nu"'] & &
\end{tikzcd}$$

satisfying $\varphi' \circ \nu = \varphi$; this determines ν uniquely, as φ' is a monomorphism. This shows that if s is in fact a morphism in $\mathcal{C}$ (that is, if $U = X$ and $\sigma = \mathrm{Id}_X$), then there is at most one morphism, in $\mathcal{C}^Y$, from (X, s) to (X', s'). Denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}^Y$ consisting of those (X, s) for which s is a morphism in $\mathcal{C}$. By the previous paragraph, $\mathcal{I}$ is in fact a partially ordered set, and it has a terminal object, namely (Y, Id_Y). The assignment sending an object (X, s) in $\mathcal{C}^Y$ as in the previous diagram to (U, φ) and the morphism $\alpha\colon (X, s) \to (X', s')$ to $\nu\colon (U, \varphi) \to (U', \varphi')$ determines a functor $\Psi\colon \mathcal{C}^Y \to \mathcal{I}$. This functor is easily seen to be right adjoint to the inclusion functor $\mathcal{I} \to \mathcal{C}^Y$. By standard properties of functor cohomology, we get that $H^*(\mathcal{C}^Y; A) \cong H^*(\mathcal{I}; A) \cong A$, concentrated in degree 0 (cf. [**12**, 5.1] or [**10**, 3.1] for the first isomorphism and [**10**, 3.4] for the second). Thus, the base change spectral sequence (cf. [**11**, 5.3] or [**5**, Appendix 2, Theorem 3.6] for the homology version) associated with the functor $\mathcal{C} \to \hat{\mathcal{C}}$ collapses to the isomorphism as stated in (iii).

Finally, if $\Phi\colon \mathcal{C} \to \mathcal{D}$ is a functor as in (iii), where $\mathcal{D}$ is an inverse category, then the unique extension of Φ to a functor $\hat{\Phi}$ of inverse categories from $\hat{\mathcal{C}}$ to $\mathcal{D}$ is the functor sending a morphism $[U; \sigma, \varphi]$ in $\hat{\mathcal{C}}$ to the morphism $\Phi(\varphi) \circ \widehat{\Phi(\sigma)}$; this is functorial thanks to the assumptions on Φ. $\square$

Example 6.3. Let $\mathcal{C}$ be an EI-category; that is, $\mathcal{C}$ is a small category such that any endomorphism of an object X in $\mathcal{C}$ is an automorphism of X. Let $S(\mathcal{C})$ be the subdivision category of $\mathcal{C}$; that is, the objects of $S(\mathcal{C})$ are faithful functors $\sigma\colon \boldsymbol{m} \to \mathcal{C}$, with $m \geqslant 0$; a morphism in $S(\mathcal{C})$ from $\sigma\colon \boldsymbol{m} \to \mathcal{C}$ to $\tau\colon \boldsymbol{n} \to \mathcal{C}$ is a pair (α, φ) consisting of an injective order-preserving map $\alpha\colon \boldsymbol{m} \to \boldsymbol{n}$ and an isomorphism of functors $\varphi\colon \sigma \cong \tau \circ \alpha$. Denote by $S(\mathcal{C})_+$ the category obtained from adding to $S(\mathcal{C})$ the empty chain $\emptyset$ as initial object. All morphisms in $S(\mathcal{C})_+$ are monomorphisms and pull-backs exist in $S(\mathcal{C})_+$; indeed, the pull-back of two morphisms (α, φ) and (α', φ') from $\sigma\colon \boldsymbol{m} \to \mathcal{C}$ and $\sigma'\colon \boldsymbol{m'} \to \mathcal{C}$ to $\tau\colon \boldsymbol{n} \to \mathcal{C}$, respectively, is $\emptyset$ if $\mathrm{Im}(\alpha)$, $\mathrm{Im}(\alpha')$ are disjoint, and otherwise equal to the obvious object $\boldsymbol{k} \cong \mathrm{Im}(\alpha) \cap \mathrm{Im}(\alpha') \to \mathcal{C}$ obtained from restricting τ, where $k + 1$ is the cardinality of this intersection.

Remark 6.4. Let $\mathcal{C}$ be a small category in which pull-backs exist. One of the crucial properties of a Mackey functor M from $\mathcal{C}$ to an abelian category $\mathcal{A}$ in the sense of [**4**, Part I], [**10**, § 5] is that $M = M^*$ is the contravariant part of a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from $\mathcal{C}$ to $\mathcal{A}$ which coincide on objects and which send a pull-back diagram

$$\begin{tikzcd}
& Y & \\
U \arrow[ur, "\varphi"] & & V \arrow[ul, "\tau"'] \\
& W \arrow[ul, "\mu"] \arrow[ur, "\nu"'] &
\end{tikzcd}$$

in $\mathcal{C}$ to a commutative diagram

$$\begin{tikzcd}
& M(Y) \arrow[dl, "M^*(\varphi)"'] & \\
M(U) & & M(V) \arrow[ul, "M_*(\tau)"'] \arrow[dl, "M^*(\mu)"] \\
& M(W) \arrow[ul, "M_*(\mu)"] &
\end{tikzcd}$$

in $\mathcal{A}$. This means exactly that M can be viewed as the restriction to $\mathcal{C}$ of a contravariant functor $\hat{M}\colon \hat{\mathcal{C}} \to \mathcal{A}$ sending X to $M(X)$ and a morphism $[U;\sigma,\varphi]\colon X \to Y$ in $\hat{\mathcal{C}}$ to the morphism $M_*(\sigma) \circ M^*(\varphi)\colon Y \to X$ in $\mathcal{A}$.

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