Determination of the characteristic equation of a matrix

The standard approach for finding the characteristic equation of a matrix $A$ is to expand the left hand side of the equation

$$\det(A - \lambda I) = 0$$

If $A$ is a $3 \times 3$ matrix this exercise can be carried out without extending most students. However, unless $A$ has some simple structures e.g. $A$ is symmetric or tridiagonal, it can be quite a formidable challenge to evaluate the characteristic equation of $A$ when $A$ is a $6 \times 6$ (or higher order) matrix. In this note I describe a simple method of finding the characteristic equation which is known in control theory as Leverrier's method, but which seems little known to mathematicians. The method is based on the following standard results.

**Theorem 1.**

$$\text{Trace } (A) = \text{sum of diagonal elements of } A \ (n \times n)$$

$$= \sum_{i=1}^{n} \lambda_i, \text{ the eigenvalues of } A.$$

and so

$$\text{Trace } (A^r) = \sum_{i=1}^{n} \lambda_i^r.$$

**Theorem 2.** If $S_r = \sum_{i=1}^{n} \lambda_i^r, \ r = 1, 2, \ldots, n$, where the $\lambda_i$ satisfy the polynomial equation

$$\lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n = 0$$

then the following identities hold (the Newton-Girard formulae):

$$S_1 + a_1 = 0$$

$$S_2 + a_1 S_1 + 2a_2 = 0$$

$$S_3 + a_1 S_2 + a_2 S_1 + 3a_3 = 0$$

$$\vdots$$

$$S_n + a_1 S_{n-1} + \ldots + na_n = 0.$$ 

It follows that if we evaluate $A^2, \ldots, A^n$ we can compute $S_r = \text{Trace } (A^r), \ r = 1, \ldots, n$ and then, from the above relations, we can systematically determine $a_1, a_2, \ldots, a_n$ and hence the characteristic equation of the matrix.
EXAMPLE. Find the characteristic equation of the matrix

\[
A = \begin{pmatrix}
4 & 2 & -3 & 1 \\
-1 & 5 & 0 & -2 \\
-2 & 0 & 5 & -1 \\
1 & -3 & 2 & 4
\end{pmatrix}.
\]

We have

\[
A^2 = \begin{pmatrix}
21 & 15 & -25 & 7 \\
-11 & 29 & -1 & -19 \\
-19 & -1 & 29 & -11 \\
7 & -25 & 15 & 21
\end{pmatrix},
\]

\[
A^3 = \begin{pmatrix}
126 & 180 & 180 & 126 \\
800 & 1152 & 1152 & 800
\end{pmatrix},
\]

\[
A^4 = \begin{pmatrix}
800 & 1152 & 1152 & 800
\end{pmatrix}.
\]

Note that we only require the diagonal elements of \( A^3 \) and \( A^4 \) and these can be obtained from \( A \) and \( A^2 \). With the notation as above, it follows that

\[
S_1 = \text{trace } A = 18
\]

\[
S_2 = \text{trace } A^2 = 100
\]

\[
S_3 = \text{trace } A^3 = 612
\]

\[
S_4 = \text{trace } A^4 = 3904
\]

and so, from the Newton-Girard formulae,

\[
a_1 = -18, \quad a_2 = 112, \quad a_3 = -276, \quad a_4 = 220.
\]

Thus the characteristic equation of \( A \) is

\[
\lambda^4 - 18\lambda^3 + 112\lambda^2 - 276\lambda + 220 = 0.
\]

Since every matrix satisfies its characteristic equation we have a simple (but not completely watertight) check to test the validity of the above result, namely to test whether the top left-hand entries of the matrices satisfy

\[
A^4 - 18A^3 + 112A^2 - 276A + 220I = 0
\]

In our case we obtain \( 800 - 18 \cdot 126 \times 112 \cdot 21 - 276 \cdot 4 + 220 = 0 \), which supports the contention that we have found the characteristic equation of \( A \).

I hope to comment elsewhere on the steps which are necessary for the practical implementation of this method for finding the eigenvalues of an \( n \times n \) matrix (\( n \) large) using a high speed computer.

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