Reciprocity sheaves and logarithmic motives

Shuji Saito

Abstract

We connect two developments that aim to extend Voevodsky’s theory of motives over a field in such a way as to encompass non-$\mathbf{A}^1$-invariant phenomena. One is theory of reciprocity sheaves introduced by Kahn, Saito and Yamazaki. The other is theory of the triangulated category $\logDM_{\text{eff}}$ of logarithmic motives launched by Binda, Park and Østvær. We prove that the Nisnevich cohomology of reciprocity sheaves is representable in $\logDM_{\text{eff}}$.

Contents

Introduction 355
1 Preliminaries 358
2 Purity with reduced modulus 361
3 Review of higher local symbols 365
4 Logarithmic cohomology of reciprocity sheaves 367
5 Invariance of logarithmic cohomology under blowups 372
6 Relation with logarithmic sheaves with transfers 376
Acknowledgements 378
References 379

Introduction

We fix once and for all a perfect base field $k$. The main purpose of this paper is to connect two developments that aim to extend Voevodsky’s theory of motives over $k$ in such a way as to encompass non-$\mathbf{A}^1$-invariant phenomena. One is the theory of reciprocity sheaves introduced by Kahn, Saito and Yamazaki [KSY16, KSY22] and developed in [Sai20, BRS22]. Voevodsky’s theory is based on the category $\PST$ of presheaves with transfers, defined as the category of additive presheaves of abelian groups on the category $\Cor$ of finite correspondences: $\Cor$ has the same objects as the category $\Sm$ of separated smooth schemes of finite type over $k$, and morphisms in $\Cor$ are finite correspondences. Let $\NST \subset \PST$ be the full subcategory of Nisnevich sheaves, that is, those objects $F \in \PST$ whose restrictions $F_X$ to the small étale site $X_{\text{ét}}$ over $X$ are Nisnevich sheaves for all $X \in \Sm$. Voevodsky proved that $\NST$ is a Grothendieck abelian
category and defined the triangulated category $\text{DM}^{\text{eff}}$ of effective motives as the localization of the derived category $D(\text{NST})$ of complexes in $\text{NST}$ with respect to an $\mathbf{A}^1$-weak equivalence (see [MVW06, Definition 14.1]). It is equipped with a funtor $M : \text{Sm} \to \text{DM}^{\text{eff}}$ associating the motive $M(X)$ of $X \in \text{Sm}$.

Let $\text{HI}_{\text{Nis}} \subset \text{NST}$ be the full subcategory consisting of $\mathbf{A}^1$-invariant objects, namely such $F \in \text{NST}$ that the projection $\pi_X : X \times \mathbf{A}^1 \to X$ induces an isomorphism $\pi_X^* : F(X) \simeq F(X \times \mathbf{A}^1)$ for any $X \in \text{Sm}$. We say that $F \in \text{HI}_{\text{Nis}}$ is strictly $\mathbf{A}^1$-invariant if $\pi_X$ induces isomorphisms

$$\pi_X^* : H^i_{\text{Nis}}(X, F_X) \simeq H^i_{\text{Nis}}(X \times \mathbf{A}^1, F_{X \times \mathbf{A}^1}) \quad \text{for all } i \geq 0.$$  

The following theorem plays a fundamental role in Voevodsky’s theory.

**Theorem 0.1** (Voevodsky [Voe00]). Any $F \in \text{HI}_{\text{Nis}}$ is strictly $\mathbf{A}^1$-invariant and we have a natural isomorphism

$$H^i_{\text{Nis}}(X, F_X) \simeq \text{Hom}_{\text{DM}^{\text{eff}}}(M(X), L^{\mathbf{A}^1}F[i]) \quad \text{for } X \in \text{Sm},$$

where $L^{\mathbf{A}^1} : D(\text{NST}) \to \text{DM}^{\text{eff}}$ is the localization functor.

Notice that there are interesting and important objects of $\text{NST}$ which do not belong to $\text{HI}_{\text{Nis}}$. Such examples are given by the sheaves $\Omega^i$ of (absolute or relative) differential forms; the $p$-typical de Rham–Witt sheaves $W_m \Omega^i$ of Bloch, Deligne and Illusie; smooth commutative $k$-group schemes with a unipotent part (seen as objects of $\text{NST}$); and the complexes $\mathcal{R}z_*\mathbb{Z}/p^n(n)$ with $\text{ch}(k) = p > 0$, where $\mathbb{Z}/p^n(n)$ is the étale motivic complex of weight $n$ with $\mathbb{Z}/p^n$ coefficients and $\varepsilon$ is the change of site functor from the étale to the Nisnevich topology. For such examples, (0.1.1) fails to hold since $\pi_X : X \times \mathbf{A}^1 \to X$ induces an isomorphism $M(X \times \mathbf{A}^1) \simeq M(X)$ in $\text{DM}^{\text{eff}}$ but the maps induced on the cohomology of those sheaves are not isomorphisms.

The category $\text{RSC}_{\text{Nis}}$ of reciprocity sheaves is a full abelian subcategory of $\text{NST}$ that contains $\text{HI}_{\text{Nis}}$ as well as the non-$\mathbf{A}^1$-invariant objects mentioned above. Heuristically, its objects satisfy the property that for any $X \in \text{Sm}$, each section $a \in F(X)$ ‘has bounded ramification at infinity’ and the objects of $\text{HI}_{\text{Nis}}$ are special reciprocity sheaves with the property that every section $a \in F(X)$ has ‘tame’ ramification at infinity. Slightly more exotic examples of reciprocity sheaves are given by the sheaves $\text{Conn}^1$ (for $\text{ch}(k) = 0$), whose sections over $X$ are rank 1-connections, or $\text{Lisse}_c^1$ (in case $\text{ch}(k) = p > 0$), whose sections on $X$ are the lisse $\mathbb{Q}_p$-sheaves of rank 1. Since $\text{RSC}_{\text{Nis}}$ is an abelian category equipped with a lax symmetric monoidal structure by [RSY22], many more interesting examples can be manufactured by taking kernels, quotients and tensor products (see [BRS22, §11.1] for more examples).

The main purpose of this paper is to establish formula (0.1.1) for all $F \in \text{RSC}_{\text{Nis}}$ in a new category which enlarges $\text{DM}^{\text{eff}}$ (see (0.2)). It is the triangulated category $\text{logDM}^{\text{eff}}$ of logarithmic motives introduced by Binda, Park and Østvær in [BPO22]. Let $\text{lSm}$ be the category of log smooth and separated fs log schemes of finite type over $k$, and $\text{lCor}$ be the category with the same objects as $\text{lSm}$ and whose morphisms are log finite correspondences (see [BPO22, Definition 2.1.1]). Let $\text{PSH}^{\text{ltr}}$ be the category of additive presheaves of abelian groups on $\text{lCor}$ and $\text{Shv}_{\text{dNis}}^{\text{ltr}} \subset \text{PSH}^{\text{ltr}}$ be the full subcategory consisting of those $F$ whose restrictions to $\text{lSm}$ are dividing Nisnevich sheaves (see [BPO22, Definition 3.1.4]). It is shown in [BPO22, Theorem 1.2.1] that $\text{Shv}_{\text{dNis}}^{\text{ltr}}$ is a Grothendieck abelian category, and $\text{logDM}^{\text{eff}}$ is defined as the localization of the derived category $D(\text{Shv}_{\text{dNis}}^{\text{ltr}})$ of complexes in $\text{Shv}_{\text{dNis}}^{\text{ltr}}$ with respect to a $\square$-weak equivalence, where $\square$ is $\mathbb{P}^1$ with the log structure associated to the effective divisor $\infty \hookrightarrow \mathbb{P}^1$.

---

1 This heuristic viewpoint is manifested in [RS21a, Theorem 2].
Reciprocity sheaves and logarithmic motives

(see [BP022, Definition 5.2.1]). It is equipped with a functor \( M : \mathcal{I} \text{Sm} \to \logDM^{\text{eff}} \) associating the logarithmic motive \( M(\mathcal{X}) \) of \( \mathcal{X} \in \mathcal{I} \text{Sm} \). Thanks to [BM12, Theorem 1.1], the standard \( t \)-structure on \( D(\Shv_{\dNis}^{\text{tr}}) \) induces a \( t \)-structure on \( \logDM^{\text{eff}} \) called the homotopy \( t \)-structure, and its heart is identified with the abelian full subcategory \( \text{CI}_{\dNis}^{\text{tr}} \subset \Shv_{\dNis}^{\text{tr}} \) consisting of strictly \( \Box \)-invariant objects in the sense of [BP022, Definition 5.2.2].

We can now state the main result of this paper.

**Theorem 0.2 (Theorems 6.1 and 6.3).** There exists an exact and fully faithful functor

\[
\Log : \text{RSC}_{\text{Nis}} \to \text{CI}_{\dNis}^{\text{tr}} : F \mapsto F^{\log} = \Log(F). \tag{0.2.1}
\]

For \( X \in \text{Sm} \) we have a natural isomorphism

\[
H^i_{\text{Nis}}(X, F_X) \simeq \text{Hom}_{\logDM^{\text{eff}}}(M(X, \text{triv}), L^iF^{\log}[i]), \tag{0.2.2}
\]

where \( L^i : D(\Shv_{\dNis}^{\text{tr}}) \to \logDM^{\text{eff}} \) is the localization functor and \((X, \text{triv})\) is the log scheme with the trivial log structure.

We remark (see Remark 5.6) that, for \( F = \Omega^i \), \( F^{\log}(\mathcal{X}) \) for \( \mathcal{X} \in \mathcal{I} \text{Sm} \) whose underlying scheme is smooth agrees with the sheaf of logarithmic differential forms of \( \mathcal{X} \) at least assuming \( \text{ch}(k) = 0 \).

We now explain the organization of the paper.

In § 1 we discuss some preliminaries and fix notation. We recall the definitions and basic properties of *modulus (pre)sheaves with transfers* from [KMSY21a, KMSY21b, KSY22, Sai20]. These are a generalization of Voevodsky’s (pre)sheaves with transfers to a version with modulus. The category \( \text{MCor} \) of *modulus correspondences* is introduced. Its objects are pairs \( \mathcal{X} = (\overline{X}, D) \), where \( \overline{X} \) is a separated scheme of finite type over \( k \) equipped with an effective Cartier divisor \( D \) such that the interior \( \overline{X} - D = X \) is smooth. The morphisms are finite correspondences on the interiors satisfying admissibility and a properness condition. Let \( \text{MPST} \) be the category of additive presheaves of abelian groups on \( \text{MCor} \). A full subcategory \( \text{MNST} \subset \text{MPST} \) of *Nisnevich sheaves* is defined and there is a functor (see §1(20))

\[
\omega_{\text{CI}} : \text{RSC}_{\text{Nis}} \to \text{MNST}.
\]

For every \( F \in \text{RSC}_{\text{Nis}} \) and \( X \in \text{Sm} \), it provides an exhaustive filtration on the group \( F(X) \) of sections over \( X \) which measures the depth of ramification along a boundary of a partial compactification of \( X \); for \( (\overline{X}, D) \in \text{MCor} \) with \( \overline{X} - D = X \), we get the subgroups \( \bar{F}(\overline{X}, D) \subset F(X) \) with \( \bar{F} = \omega_{\text{CI}}F \) such that \( \bar{F}(\overline{X}, D_1) \subset \bar{F}(\overline{X}, D_2) \) if \( D_1 \leq D_2 \).

In § 2 we prove as a key technical input an analogue of the Zariski–Nagata purity theorem [SGA2, X 3.4] for \( \bar{F}(\overline{X}, D) \) as above. This asserts the exactness of the sequence

\[
0 \to \bar{F}(\overline{X}, D) \to F(X) \to \bigoplus_{\underline{\xi} \in D^{(0)}} \frac{F(\overline{X}_{\underline{\xi}})h_{\underline{\xi}}}{\bar{F}(\overline{X}_{\underline{\xi}}^{\hbar}/\underline{\xi})},
\]

where \( \overline{X} \in \text{Sm} \) and \( D \) is a reduced simple normal crossing divisor, and where \( D^{(0)} \) is the set of the irreducible components of \( D \) and \( \overline{X}^{\hbar}_{\underline{\xi}} \) is the henselization of \( \overline{X} \) at \( \xi \). In [RS21b] this result is generalized to the case where \( D \) may not be reduced under the assumption that \( \overline{X} \) admits a smooth compactification.

---

2 In fact it is defined in [BP022, Definition 5.2.1] as the localization of the homotopy category of complexes in \( \Shv_{\dNis}^{\text{tr}} \) with respect to a \( \Box \)-local descent model structure.

3 It is a logarithmic analogue of Voevodsky’s strict \( \Lambda^2 \)-invariance.

4 The assumption is necessary to use [RS21a, Corollary 6.8] proved in the case \( \text{ch}(k) = 0 \). We expect that it can be dispensed with by using a forthcoming work of K. Rülling extending [RS21a, Corollary 6.8] to the case \( \text{ch}(k) > 0 \).
S. Saito

In § 3 we review higher local symbols for reciprocity sheaves constructed in [RS21e]. These are an effective tool with which one can decide when a given element of \( F(X) \) with \( F \in RSC_{Nis} \) and \( X \in Sm \) belongs to \( \tilde{F}(X, D) \) as above. The construction of the pairing depends on pushforward maps for the cohomology of reciprocity sheaves constructed in [BRS22] (which means that Theorem 0.2 depends on the result of [BRS22]).

In § 4 we prove the following result. Let \( MCor_{ls}^{\text{fin}} \) be the subcategory of \( MCor_{ls} \) whose objects are pairs \((X, D)\) such that \( X \in Sm \) and the reduced divisor \( D_{\text{red}} \) underlying \( D \) is a simple normal crossing divisor on \( X \) and whose morphisms are modulus correspondences satisfying a finiteness conditions instead of the properness condition (see § 1(5)). Then, for \( F \in RSC_{Nis} \), the association

\[
\tilde{F}^{\log} : (X, D) \rightarrow \omega_{i}^{\text{CI}} F(X, D_{\text{red}})
\]

gives a presheaf on \( MCor_{ls}^{\text{fin}} \), which gives rise to a cohomology theory \( H^{i}_{\text{log}}(\cdot, \tilde{F}^{\log}) \) on \( MCor_{ls}^{\text{fin}} \), called the \( i \)th logarithmic cohomology with coefficient \( F \) (see Definition 4.4). The higher local symbols for \( F \) play a fundamental role in the proof of the result.

In § 5 we prove the invariance of logarithmic cohomology under blowups. Let \( A_{ls}^{\text{fin}} \) be the subcategory of \( MCor_{ls}^{\text{fin}} \) whose objects are the same as \( MCor_{ls}^{\text{fin}} \) and whose morphisms are those \( \rho : (Y, E) \rightarrow (X, D) \) where \( E = \rho^{*}D \) and \( \rho \) are induced by blowups of \( X \) in smooth centers \( Z \subset D \) which are normal crossing to \( D \) see the beginning of the section). Then, for \( F \in RSC_{Nis} \) and \( \rho : Y \rightarrow X \) in \( A_{ls}^{\text{fin}} \), we have

\[
\rho^{*} : H^{i}_{\text{log}}(X, F) \cong H^{i}_{\text{log}}(Y, F) \quad \forall i \geq 0.
\]

In § 6 we prove Theorem 0.2, which is a formal consequence of the theorems in §§ 4 and 5.

1. Preliminaries

We fix once and for all a perfect base field \( k \). In this section we recall the definitions and basic properties of modulus sheaves with transfers from [KMSY21a, Sai20].

(1) Denote by \( \text{Sch} \) the category of separated schemes of finite type over \( k \) and by \( Sm \) the full subcategory of smooth schemes. For \( X, Y \in Sm \) an integral closed subscheme of \( X \times Y \) that is finite and surjective over a connected component of \( X \) is called a prime correspondence from \( X \) to \( Y \). The category \( \text{Cor} \) of finite correspondences has the same objects as \( Sm \), and for \( X, Y \in Sm \), \( \text{Cor}(X, Y) \) is the free abelian group on the set of all prime correspondences from \( X \) to \( Y \) (see [Voe00]). We consider \( Sm \) as a subcategory of \( \text{Cor} \) by regarding a morphism in \( Sm \) as its graph in \( \text{Cor} \).

Let \( PST \) be the category of additive presheaves of abelian groups on \( \text{Cor} \) whose objects are called presheaves with transfers. Let \( NST \subseteq PST \) be the category of Nisnevich sheaves with transfers and let

\[
a^{V}_{\text{Nis}} : PST \rightarrow NST
\]

be Voevodsky’s Nisnevich sheafification functor, which is an exact left adjoint to the inclusion \( NST \rightarrow PST \). Let \( HI \subseteq PST \) be the category of \( A_{1} \)-invariant presheaves and put \( HI_{\text{Nis}} = HI \cap NST \subseteq NST \).

(2) Let \( Sm^{\text{pro}} \) be the category of \( k \)-schemes \( X \) which are essentially smooth over \( k \), that is, \( X \) is a limit \( \lim_{\longrightarrow i \in I} X_{i} \) over a filtered set \( I \), where \( X_{i} \) is smooth over \( k \) and all transition maps are étale. Note that \( \text{Spec} K \in Sm^{\text{pro}} \) for a function field \( K \) over \( k \) thanks to the assumption that \( k \) is perfect. We define \( \text{Cor}^{\text{pro}} \) whose objects are the same as \( Sm^{\text{pro}} \) and whose morphisms are defined as [RS21a, Definition 2.2]. We extend \( F \in PST \) to a presheaf on \( \text{Cor}^{\text{pro}} \) by \( F(X) := \lim_{\longrightarrow i \in I} F(X_{i}) \) for \( X \) as above.
Reciprocity sheaves and logarithmic motives

(3) We recall the definition of the category $\mathbf{MCor}$ from [KMSY21a, Definition 1.3.1]. A pair $\mathcal{M} = (X, D)$ consisting of $X \in \text{Sch}$ and an effective Cartier divisor $D$ on $X$ is called a modulus pair if $X - D \in \text{Sm}$. Let $\mathcal{M} = (X, D_X)$, $\mathcal{N} = (Y, D_Y)$ be modulus pairs and $\Gamma \in \text{Cor}(X - D_X, Y - D_Y)$ be a prime correspondence. Let $\Gamma \subseteq X \times Y$ be the closure of $\Gamma$, and let $\Gamma^N \to X \times Y$ be the normalization. We say that $\Gamma$ is admissible (respectively, left proper) if $(D_X)^N \geq (D_Y)^N$ (respectively, if $\Gamma$ is proper over $X$). Let $\text{MCor}(\mathcal{M}, \mathcal{N})$ be the subgroup of $\text{Cor}(X - D_X, Y - D_Y)$ generated by all admissible left proper prime correspondences. The category $\text{MCor}$ has modulus pairs as objects and $\text{MCor}(\mathcal{M}, \mathcal{N})$ as the group of morphisms from $\mathcal{M}$ to $\mathcal{N}$.

(4) Let $\text{MCor}_{\text{is}} \subseteq \text{MCor}$ be the full subcategory of $(X, D) \in \text{MCor}$ with $X \in \text{Sm}$ and $|D|$ a normal crossing divisor on $X$.

(5) Let $\text{MCor}^{\text{fin}} \subseteq \text{MCor}$ be the full subcategory of the same objects such that $\text{MCor}^{\text{fin}}(\mathcal{M}, \mathcal{N})$ are generated by all admissible finite prime correspondences, where finite prime correspondences are defined by replacing the left properness in (3) by finiteness. We also define $\text{MCor}_{\text{is}}^{\text{fin}} = \text{MCor}^{\text{fin}} \cap \text{MCor}_{\text{is}}$.

(6) There is a canonical pair of adjoint functors $\lambda : \text{Cor} \to \text{MCor}$ $X \mapsto (X, \emptyset)$, $\omega : \text{MCor} \to \text{Cor} (X, D) \mapsto X - D$.

(7) There is a full subcategory $\text{MCor} \subseteq \text{MCor}$ consisting of proper modulus pairs, where a modulus pair $(X, D)$ is proper if $X$ is proper. Let $\tau : \text{MCor} \hookrightarrow \text{MCor}$ be the inclusion functor and $\omega = \omega^\tau$.

(8) Let $\text{MPST}$ (respectively, $\text{MPST}^\tau$) be the category of additive presheaves of abelian groups on $\text{MCor}$ (respectively, $\text{MCor}$) whose objects are called modulus presheaves with transfers. For $\mathcal{M} \in \text{MCor}$, let $\mathcal{Z}_{\text{tr}}(\mathcal{M}) = \text{MCor}(\cdot, \mathcal{M})$ be the representable object of $\text{MPST}$. We sometimes write $\mathcal{M}$ for $\mathcal{Z}_{\text{tr}}(\mathcal{M})$ for simplicity.

(9) In the same manner as (2), the category $\text{MCor}_{\text{pro}}$ is defined and $F \in \text{MPST}$ is extended to a presheaf on $\text{MCor}_{\text{pro}}$ (see [RS21a, §3.7]).

(10) The adjunction $\lambda \dashv \omega$ induces a string of four adjoint functors $(\lambda_! = \omega^!, \lambda^* = \omega^!, \lambda_* = \omega^!, \omega_*)$ (see [KMSY21a, Proposition 2.3.1]):

\[
\begin{array}{ccc}
\text{MPST} & \xrightarrow{\omega^!} & \text{PST} \\
\omega^* & \xleftarrow[\omega_!]{\text{PST}} & \text{MCor}^\tau
\end{array}
\]

where $\omega_!, \omega_*$ are localizations and $\omega^!$ and $\omega^*$ are fully faithful.

(11) The functor $\tau$ yields a string of three adjoint functors $(\tau_!, \tau^*, \tau_*):$

\[
\begin{array}{ccc}
\text{MPST} & \xrightarrow{\tau^*} & \text{MPST} \\
\tau_! & \xleftarrow[\tau_*]{\text{MPST}} & \text{MCor}^\tau
\end{array}
\]

where $\tau_!, \tau_*$ are fully faithful and $\tau^*$ is a localization; $\tau_!$ has a pro-left adjoint $\tau^!$, hence is exact (see [KMSY21a, Proposition 2.4.1]). We will denote by $\text{MPST}^\tau$ the essential image of $\tau_!$ in $\text{MPST}$.

(12) The modulus pair $\square := (\mathbb{P}^1, \infty)$ has an interval structure induced by that of $\mathbb{A}^1$ (see [KSY22, Lemma 2.1.3]). We say that $F \in \text{MPST}$ is $\square$-invariant if $p^* : F(\mathcal{M}) \to F(\mathcal{M} \otimes \square)$ is an isomorphism for any $\mathcal{M} \in \text{MCor}$, where $p : \mathcal{M} \otimes \square \to \mathcal{M}$ is the projection. Let $\text{CI}$ be
the full subcategory of \( \text{MPST} \) consisting of all \( \square \)-invariant objects and \( \text{CI}^\tau \subset \text{MPST} \) be the essential image of \( \text{CI} \) under \( \tau \).

(13) Recall from [KSY22, Theorem 2.1.8] that \( \text{CI} \) is a Serre subcategory of \( \text{MPST} \), and that the inclusion functor \( i^\square : \text{CI} \to \text{MPST} \) has a left adjoint \( h_0^\square \) and a right adjoint \( h_0^\square \) given for \( F \in \text{MPST} \) and \( X \in \text{MCor} \) by

\[
h_0^\square(F)(X) = \text{Coker}(i_0^\square - i_1^\square : F(X \otimes \square) \to F(X)),
\]

\[
h_0^\square(F)(X) = \text{Hom}(h_0^\square(X), F).
\]

For \( X \in \text{MCor} \), we write \( h_0^\square(X) = h_0^\square(\mathbb{Z}_{tr}(X)) \in \text{CI} \), and by abuse of notation we also write \( h_0^\square(X) \) for \( \tau h_0^\square(X) \in \text{CI}^\tau \).

(14) For \( F \in \text{MPST} \) and \( X = (X, D) \in \text{MCor} \), write \( F_X \) for the presheaf on the small étale site \( X_{\text{ét}} \) over \( X \) given by \( U \to F(X_U) \) for \( U \to X \) étale, where \( X_U = (U, D_U) \in \text{MCor} \).

We say that \( F \) is a Nisnevich sheaf if \( F_X \) is also one for all \( X \in \text{MCor} \) (see [KMSY21a, §3]). We write \( \text{MNST} \subset \text{MPST} \) for the full subcategory of Nisnevich sheaves and put

\[
\text{MNST}^\tau = \text{MNST} \cap \text{MPST}^\tau, \quad \text{CI}^\text{Nis} = \text{CI}^\tau \cap \text{MNST}^\tau.
\]

By [KMSY21a, Proposition 3.5.3] and [KMSY21b, Theorem 2], the inclusion functor \( i_{\text{Nis}} : \text{MNST} \to \text{MPST} \) has an exact left adjoint \( a_{\text{Nis}} \) such that \( a_{\text{Nis}}(\text{MPST}^\tau) \subset \text{MNST}^\tau \). The functor \( a_{\text{Nis}} \) has the following description. For \( F \in \text{MPST} \) and \( Y \in \text{MCor} \), let \( F_Y, \text{Nis} \) be the usual Nisnevich sheafification of \( F_Y \). Then, for \( (X, D) \in \text{MCor} \), we have

\[
a_{\text{Nis}}F(X, D) = \lim_{f : Y \to X} F((Y, f^*D), \text{Nis})(Y)
\]

where the colimit is taken over all proper maps \( f : Y \to X \) that induce isomorphisms \( Y - |f^*D| \cong X - |D| \).

(15) By [KMSY21b, Proposition 6.2.1], \( \omega^* \) and \( \omega_! \) from (10) respect \( \text{MNST} \) and \( \text{NST} \) and induce a pair of adjoint functors (which for simplicity we write \( \omega_! \) and \( \omega^* \)). Moreover, we have

\[
\omega_! a_{\text{Nis}} = a_{\text{Nis}}^V \omega_!.
\]

By [KSY22, Lemma 2.3.1] and [KMSY21b, Proposition 6.2.1a], for \( F \in \text{PST} \), we have \( F \in \text{HI} \) (respectively, \( F \in \text{HI}_{\text{Nis}} \)) for \( F \in \text{PST} \), we have \( F \in \text{HI} \) (respectively, \( F \in \text{HI}_{\text{Nis}} \)) for \( F \in \text{PST} \) if and only if \( \omega^*F \in \text{CI}^\tau \) (respectively, \( \omega^*F \in \text{CI}^\text{Nis} \)).

We say that \( F \in \text{MPST} \) is semipure if the unit map

\[
u : F \to \omega^* \omega_! F
\]

is injective. For \( F \in \text{MPST} \) (respectively, \( F \in \text{MNST} \)), let \( F^\text{sp} \in \text{MPST} \) (respectively, \( F^\text{sp} \in \text{MNST} \)) be the image of \( F \to \omega^* \omega_! F \) (called the semipurification of \( F \). See [Sai20, Lemma 1.30]). For \( F \in \text{MPST} \) we have

\[
a_{\text{Nis}}(F^\text{sp}) \simeq (a_{\text{Nis}}F)^\text{sp}.
\]

This follows from the fact that \( a_{\text{Nis}} \) is exact and commutes with \( \omega^* \omega_! \). For \( F \in \text{MPST}^\tau \) we have \( F^\text{sp} \in \text{MPST}^\tau \) since \( \tau \) is exact and \( \omega^* \omega_! \tau = \tau \omega^* \omega_! \).

(17) Let \( \text{CI}^{\tau, \text{sp}} \subset \text{CI}^\tau \) be the full subcategory of semipure objects and consider the full subcategory

\[
\text{CI}^{\tau, \text{sp}} = \text{CI}^{\text{Nis}} \subset \text{CI}^{\text{Nis}}.
\]

By [Sai20, Theorems 0.1 and 0.4], we have \( a_{\text{Nis}}(\text{CI}^{\tau, \text{sp}}) \subset \text{CI}^{\tau, \text{sp}} \).
Reciprocity sheaves and logarithmic motives

(18) We write \( \text{RSC} \subseteq \text{PST} \) for the essential image of \( \text{CI} \) under \( \omega^! \) (which is the same as the essential image of \( \text{CI}^{\text{r.sp}} \) under \( \omega^! \) since \( \omega^! = \omega^!_1 \) and \( \omega^! F = \omega^!_1 F^{\text{sp}} \)). Put \( \text{RSC}_{\text{Nis}} = \text{RSC} \cap \text{NST} \). The objects of \( \text{RSC} \) (respectively, \( \text{RSC}_{\text{Nis}} \)) are called reciprocity presheaves (respectively, sheaves). By [Sai20, Theorem 0.1], we have

\[
\alpha^V_{\text{Nis}}(\text{RSC}) \subseteq \text{RSC}_{\text{Nis}}. 
\]

We have \( \text{HI} \subseteq \text{RSC} \) which also contains smooth commutative group schemes (which may have non-trivial unipotent part), the sheaf \( \Omega^i \) of Kähler differentials, and the de Rham–Witt sheaves \( \Omega^i \) (respectively, sheaves). By [Voe00, Lemma 3.1.6] and we can make \( \text{RSC}_{\text{Nis}} \) its full subabelian category as follows. We define the kernel (respectively, cokernel) of a map \( \phi : F \to G \) in \( \text{RSC}_{\text{Nis}} \) to be that of \( \phi \) as a map in \( \text{NST} \). Here we need (1.0.1) to ensure that the cokernel of \( \phi \) in \( \text{NST} \) stays in \( \text{RSC}_{\text{Nis}} \). By definition, a sequence \( 0 \to F \to G \to H \to 0 \) is exact in \( \text{RSC}_{\text{Nis}} \) if and only if it is exact in \( \text{NST} \).

(19) \( \text{NST} \) is a Grothendieck abelian category by [Voe00, Lemma 3.1.6] and we can make \( \text{RSC}_{\text{Nis}} \) its full subabelian category as follows. We define the kernel (respectively, cokernel) of a map \( \phi : F \to G \) in \( \text{RSC}_{\text{Nis}} \) to be that of \( \phi \) as a map in \( \text{NST} \). Here we need (1.0.1) to ensure that the cokernel of \( \phi \) in \( \text{NST} \) stays in \( \text{RSC}_{\text{Nis}} \). By definition, a sequence \( 0 \to F \to G \to H \to 0 \) is exact in \( \text{RSC}_{\text{Nis}} \) if and only if it is exact in \( \text{NST} \).

(20) By [KSY22, Proposition 2.3.7] we have a pair of adjoint functors

\[
\text{CI} \xleftarrow{\omega^!_1} \text{RSC},
\]

where \( \omega^! = h^0_{\text{dR}} \omega^* \) and is fully faithful. It induces a pair of adjoint functors

\[
\text{CI}^{\text{r.sp}} \xleftarrow{\omega^!_1} \text{RSC},
\]

where \( \omega^!_1 = \tau h^0_{\text{dR}} \omega^* \) and is fully faithful. Indeed, let \( F = \tau \hat{F} \) for \( \hat{F} \in \text{CI} \) and \( G \in \text{RSC} \). In view of (13) and the exactness and full faithfulness of \( \tau \), we have

\[
\text{Hom}_{\text{CI}}(F, \tau h^0_{\text{dR}} \omega^* G) \simeq \text{Hom}_{\text{CI}}(\hat{F}, h^0_{\text{dR}} \omega^* G)
\simeq \text{Hom}_{\text{MPST}}(\hat{F}, \omega^* G) \simeq \text{Hom}_{\text{RSC}}(\omega^! F, G).
\]

In view of (15), (1.0.3) induces a pair of adjoint functors

\[
\text{CI}^{\text{r.sp}} \xleftarrow{\omega^!_{1,\text{Nis}}} \text{RSC}_{\text{Nis}}.
\]

2. Purity with reduced modulus

For \( F \in \text{MPST} \), we put

\[
F_{-1} = \text{Ker} \left( \text{Hom}_{\text{MPST}}((\mathbb{P}^1, 0, \infty), F) \xrightarrow{i_1^*} F \right),
F_{-1}^{(1)} = \text{Ker} \left( \text{Hom}_{\text{MPST}}((\mathbb{P}^1, 0 + \infty), F) \xrightarrow{i_1^*} F \right)
\]

Note that if \( F \in \text{CI}^{\text{r.sp}}_{\text{Nis}} \), then \( F_{-1}, F_{-1}^{(1)} \in \text{CI}^{\text{r.sp}}_{\text{Nis}} \) and

\[
F_{-1}^{(1)}(\mathcal{X}) = \text{Hom}_{\text{MPST}}(h^0_{\text{sp}}(\mathbb{P}^1, 0 + \infty)^0, \text{Hom}_{\text{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), F)),
F_{-1}(\mathcal{X}) = \lim_n \text{Hom}_{\text{MPST}}(h_{0,\text{Nis}}^n(\mathbb{P}^1, n \cdot 0 + \infty)^0, \text{Hom}_{\text{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), F))
\]

for \( \mathcal{X} \in \text{MCor}, \) where

\[
h_{0,\text{Nis}}^n(\mathbb{P}^1, n \cdot 0 + \infty)^0 = \text{Coker} \left( \mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec} \ k, \emptyset) \xrightarrow{i_1} h_{0,\text{Nis}}^n(\mathbb{P}^1, n \cdot 0 + \infty) \right).
\]
The existence of retractions in the following lemma was suggested by A. Merici. It implies \( \tau^{(e_1, \ldots, e_r)} F \in \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \) if \( F \in \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \).

**Lemma 2.2.** For \( F \in \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \), the inclusion \( F_{-1}^{(1)} \to F_{-1} \) admits a retraction \( s_F : F_{-1} \to F_{-1}^{(1)} \) such that for any map \( \phi : F \to G \) in \( \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \), the following diagram is commutative:

\[
\begin{array}{ccc}
F_{-1} & \xrightarrow{s_F} & F_{-1}^{(1)} \\
\downarrow{\phi} & & \downarrow{\phi} \\
G_{-1} & \xrightarrow{s_F} & G_{-1}^{(1)}
\end{array}
\]

In particular, \( \tau^{(1)} F \in \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \) if \( F \in \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \).

**Proof.** In view of (2.0.1), this follows from [BRS22, Lemma 2.4]. \( \square \)

**Theorem 2.3.** Let \( F \in \text{CI}_{\text{Nis}}^{\tau_{\text{sp}}} \). Let \( K[t_1, \ldots, t_n] \) be the henselization of \( K[t_1, \ldots, t_n] \) at \( (t_1, \ldots, t_n) \) and \( X = \text{Spec } K[t_1, \ldots, t_n] \) and \( D = \{ t_1^{e_1} \cdots t_n^{e_n} = 0 \} \subset X \) with \( e_1, \ldots, e_n \in \{0, 1\} \).

For a subset \( I \subset [1, n] \) let \( t_I : \mathcal{H} \to X \) be the closed immersion defined by \( \{ t_i = 0 \}_{i \in I} \) and \( D_{\mathcal{H}} = \{ \prod_{j \in [1, n] - I} t_j^{e_j} = 0 \} \subset \mathcal{H} \). Then

\[
R^e t_I^* t^*_\mathcal{H} (X, D) = 0 \quad \text{for } \nu \neq q := |I|,
\]

and there is an isomorphism

\[
(\tau^{(e_i)} F)_{(\mathcal{H}, D_{\mathcal{H}})} \simeq R^q t_I^* t^*_\mathcal{H} F(X, D) \quad \text{with } e_I = (e_i)_{i \in I} \in \mathbb{Z}_{\geq 0}.
\]

**Proof.** The proof is divided into two steps.

**Step 1:** we prove (2.3.1) and (2.3.2) for \( q = |I| = 1 \). For \( \nu = 0 \), (2.3.1) follows from the semipurity of \( F \) and [Sai20, Theorem 3.1]. Thus, it suffices to show (2.3.1) only for \( \nu > 1 \). Let \( J = \{ j \in [1, n] | e_j \neq 0 \} \) and \( r = |J| \). If \( \dim(X) = 0 \), the assertion is trivial. If \( r = 0 \), the assertion follows from [Sai20, Corollary 8.6(3)]. Assume \( r > 0 \) and \( \dim(X) \geq 1 \), and proceed by the double induction on \( r \) and \( \dim(X) \). Without loss of generality, we may assume

\[
(\clubsuit) \quad e_1 \neq 0, \quad \text{and } \mathcal{H} = \{ t_1 = 0 \} \text{ if } \mathcal{H} \subset |D|.
\]

Let \( t : Z \to X \) be the closed immersion defined by \( \{ t_1 = 0 \} \) and \( D_Z = \{ t_2^{e_2} \cdots t_r^{e_r} = 0 \} \subset Z \) and \( D' = \{ t_2^{e_2} \cdots t_r^{e_r} = 0 \} \subset X \). By [Sai20, Lemma 7.1], we have an exact sequence sheaves on \( X_{\text{Nis}} \),

\[
0 \to F(X, D') \to F(X, D) \to t_\ast (F_{-1}^{(e_1)})(Z, D_Z) \to 0,
\]

which gives rise to a long exact sequence of sheaves on \( \mathcal{H}_{\text{Nis}} \),

\[
\cdots \to R^e t_I^* t^*_\mathcal{H} F(X, D') \to R^e t_I^* t^*_\mathcal{H} F(X, D) \to R^e t_I^* t_\ast (F_{-1}^{(e_1)})(Z, D_Z) \to \cdots
\]
Reciprocity sheaves and logarithmic motives

By the induction hypothesis, $R^\nu i_\mathcal{H}^! F(X,D') = 0$ for $\nu > 1$. If $\mathcal{H} \neq \mathcal{Z}$, we have a Cartesian diagram of closed immersions

$$
\begin{array}{ccc}
\mathcal{H} \cap \mathcal{Z} & \xrightarrow{i} & \mathcal{H} \\
\mathcal{H} \cap \mathcal{Z} & \xrightarrow{i_\mathcal{H}} & \mathcal{Z} \\
\xrightarrow{i_{\mathcal{H}\cap\mathcal{Z}}} & & \xrightarrow{i} \\
\mathcal{Z} & \xrightarrow{i} & \mathcal{X}
\end{array}
$$

and we have an isomorphism

$$
R^\nu i_\mathcal{H}^! t_* (F^{(e_1)}_{-1}(\mathcal{Z},D_\mathcal{Z})) \simeq i_*' R^\nu i_{\mathcal{H}\cap\mathcal{Z}}^! (F^{(e_1)}_{-1}(\mathcal{Z},D_\mathcal{Z})).
$$

By the induction hypothesis, $R^\nu i_\mathcal{H}^! t_* (F^{(e_1)}_{-1}(\mathcal{Z},D_\mathcal{Z})) = 0$ for $\nu > 1$, noting that $F^{(e_1)}_{-1} \in \mathbf{C}^{\text{sup}}_{\text{Nis}}$ by Lemma 2.2. So the desired vanishing follows from (2.3.3). Moreover, the assumptions ($\heartsuit$) and $\mathcal{H} \neq \mathcal{Z}$ imply that $\mathcal{H} \not\subset [D]$. Then (2.3.2) (with $q = 1$) follows from [Sai20, Lemma 7.1(2)].

If $\mathcal{Z} = \mathcal{H}$, we have

$$
R^\nu i_\mathcal{H}^! t_* (F^{(e_1)}_{-1}(\mathcal{Z},D_\mathcal{Z})) = R^\nu i_\mathcal{H}^! t_* (F^{(e_1)}_{-1}(\mathcal{Z},D_\mathcal{Z})),
$$

which vanishes for $\nu > 0$. Hence, (2.3.3) gives the desired vanishing together with an exact sequence

$$
0 \to (F^{(e_1)}_{-1})_{(\mathcal{H},D_\mathcal{H})} \to R^1 i_\mathcal{H}^! F(X,D') \to R^1 i_\mathcal{H}^! F(X,D) \to 0.
$$

By [Sai20, Lemma 7.1(2)] we have an isomorphism

$$(F^{(e_1)}_{-1})_{(\mathcal{H},D_\mathcal{H})} \simeq R^1 i_\mathcal{H}^! F(X,D')$$

through which $\delta$ is identified with the map induced by the canonical map $F^{(e_1)}_{-1} \to F_{-1}$. This proves the desired isomorphism (2.3.2) for $\mathcal{Z} = \mathcal{H}$ and completes step 1.

Step 2: we prove the theorem by induction on $q$ assuming $q > 0$. Let $I = \{i_1, \ldots, i_q\} \subset [1, n]$ and $\mathcal{Y} \subset \mathcal{X}$ be the closed subscheme defined by $\{i_1 = 0\}$. Let $i_\mathcal{Y} : \mathcal{Y} \to \mathcal{X}$ and $i_{\mathcal{H},\mathcal{Y}} : \mathcal{H} \to \mathcal{Y}$ be the induced closed immersions. By step 1 we have $R^q i\mathcal{Y}_* F(X,D) = 0$ for $\nu \neq 1$ and we have an isomorphism

$$(\tau^{(e_1)} F)_{(\mathcal{Y},D_\mathcal{Y})} \simeq R^{q+1} i_{\mathcal{H}}^! F(X,D) \quad \text{with} \quad D_\mathcal{Y} = \{\tilde{t}_{i_1}^1 \cdots \tilde{t}_{i_q}^1 \cdots \tilde{t}_n^1 = 0\} \subset \mathcal{Y}.$$  

Note $\tau^{(e_1)} F \in \mathbf{C}^{\text{sup}}_{\text{Nis}}$ by Lemma 2.2. Thus, by the induction hypothesis, we have $R^{\nu} i_{\mathcal{H},\mathcal{Y}}^! \tau^{(e_1)} F_{(\mathcal{Y},D_\mathcal{Y})} = 0$ for $\nu \neq q - 1$. By the spectral sequence

$$E_2^{a,b} = R^a i_{\mathcal{H},\mathcal{Y}}^! R^b i_\mathcal{Y}^! F(X,D) \Rightarrow R^a+b i_{\mathcal{H}}^! F(X,D),$$

we get the desired vanishing (2.3.1) and an isomorphism

$$R^{q+1} i_{\mathcal{H}}^! F(X,D) \simeq R^{q-1} i_{\mathcal{H},\mathcal{Y}}^! R^{q+1} i_\mathcal{Y}^! F(X,D) \simeq R^{q-1} i_{\mathcal{H},\mathcal{Y}}^! (\tau^{(e_1)} F)_{(\mathcal{Y},D_\mathcal{Y})} \simeq (\tau^{(e_1_1, \ldots, e_q)} (\tau^{(e_1)} F))_{(\mathcal{H},D_\mathcal{H})} \simeq (\tau^{(e_1_1, \ldots, e_q)} F)_{(\mathcal{H},D_\mathcal{H})},$$

where the third isomorphism holds by the induction hypothesis. This completes the proof of the theorem.

We say that $\mathcal{X} = (X, D) \in \mathbf{M} \mathbf{C} \mathbf{o} \mathbf{r}$ is reduced if so is $D$. The following Corollaries 2.4 and 2.5 are immediate consequences of Theorem 2.3.
Corollary 2.4. Take $F \in \text{CI}_{\text{Nis}}^{r,sp}$ and $(X, D) \in \text{MCor}_{\text{is}}$ reduced. Let $x \in X^{(n)}$ with $K = k(x)$ and let $X = X_{x}^{h}$ be the henselization of $X$ at $x$. Then

$$H_{i}^{x}(X_{\text{Nis}}, F_{(X, D)}) = 0 \quad \text{for } i \neq n.$$  

Choosing an isomorphism

$$\varepsilon : X \simeq \text{Spec} K\{t_{1}, \ldots, t_{n}\}$$

such that $D_{x} = \{t_{1}^{e_{1}} \cdots t_{n}^{e_{n}} = 0\} \subset X$ with $e_{1}, \ldots, e_{n} \in \{0, 1\}$, there exists an isomorphism depending on $\varepsilon$:

$$\theta_{\varepsilon} : t^{(e_{1}, e_{2}, \ldots, e_{n})} F(x) \simeq H_{x}^{n}(X_{\text{Nis}}, F_{(X, D)}).$$

Corollary 2.5. For $F \in \text{CI}_{\text{Nis}}^{r,sp}$ and $X = (X, D) \in \text{MCor}_{\text{is}}$ reduced, the following sequence is exact:

$$0 \to F(X, D) \to F(X - D, 0) \to \bigoplus_{\xi \in D(0)} F(X_{\xi}^{h}, \xi, \theta).$$

The idea of deducing the following corollary from the above is due to A. Merici.

Corollary 2.6. Let $X = (X, D) \in \text{MCor}_{\text{is}}$ be reduced.

1. Assume given an exact sequence in $\text{MNST}$,

$$0 \to H \xrightarrow{\phi} G \xrightarrow{\psi} F,$$

such that $F, G, H \in \text{CI}_{\text{Nis}}^{r,sp}$ and that $\omega_{1}\psi$ is surjective in $\text{NST}$. If $X$ is henselian local, then

$$0 \to H(X) \to G(X) \to F(X) \to 0$$

is exact.

2. Let $\gamma : F \to G$ be a map in $\text{CI}_{\text{Nis}}^{r,sp}$ such that $\omega_{1}\gamma$ is an isomorphism. Then $F(X) \to G(X)$ is an isomorphism.

3. For $F \in \text{CI}_{\text{Nis}}^{r,sp}$, the unit map $u : F \to \omega_{1}\omega_{1}F$ induces an isomorphism $F(X) \cong \omega_{1}\omega_{1}F(X)$.

Proof. To show (1), it suffices to show the surjectivity of $G(X) \to F(X)$. Let $x \in X$ be the generic point and consider the following commutative diagram of the Cousin complexes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H(X) & \longrightarrow & H(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H_{x}^{1}(X, H_{x}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H_{y}^{2}(X, H_{X}) \\
& & & \downarrow \phi(\eta) & & & \downarrow H_{x}^{1}(\phi) & & \downarrow H_{y}^{2}(\phi) \\
0 & \longrightarrow & G(X) & \longrightarrow & G(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H_{x}^{1}(X, G_{x}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H_{y}^{2}(X, G_{X}) \\
& & & \downarrow \psi(\eta) & & & \downarrow H_{x}^{1}(\psi) & & \downarrow H_{y}^{2}(\psi) \\
0 & \longrightarrow & F(X) & \longrightarrow & F(\eta) & \longrightarrow & \bigoplus_{x \in X^{(1)}} H_{x}^{1}(X, F_{X}) & \longrightarrow & \bigoplus_{y \in X^{(2)}} H_{y}^{2}(X, F_{X}) 
\end{array}
$$

By Corollary 2.4, the horizontal sequences are exact. By the assumption, $\psi(\eta)$ is surjective. By a diagram chase we are reduced to showing the following claim.
Reciprocity sheaves and logarithmic motives

Claim 2.6.1.

(i) For \( x \in X^{(1)} \), the sequence
\[
H^1_X(X, H^1_X) \rightarrow H^1_X(X, G_X) \rightarrow H^1_X(X, F_X)
\]
is exact.

(ii) For \( y \in X^{(2)} \), \( H^2_y(\phi) \) is injective.

To show (i), by Corollary 2.4, it suffices to show the exactness of \( \tau^{(e)} H \rightarrow \tau^{(e)} G \rightarrow \tau^{(e)} F \) for \( e \in \{0, 1\} \). The case \( e = 0 \) follows from the left exactness of the endofunctor \( \text{Hom}_{\text{MPST}}(\mathcal{X}, -) \) on \( \text{MNST} \) for any \( \mathcal{X} \in \text{MCor} \). We have a commutative diagram
\[
\begin{array}{ccc}
\tau^{(1)} H & \xrightarrow{\phi} & \tau^{(1)} G \\
p_H & \downarrow{s_H} & \downarrow{s_G} \\
\tau^{(0)} H & \xrightarrow{\phi} & \tau^{(0)} G
\end{array}
\]
where \( p_* \) are the projections and \( s_* \) is a right inverse of \( p_* \) coming from the retractions from Lemma 2.2. We have
\[
\phi \circ p_H = p_G \circ \phi, \quad \psi \circ p_G = p_F \circ \psi, \quad \phi \circ s_H = s_G \circ \phi, \quad \psi \circ s_G = s_F \circ \psi.
\]
By a diagram chase, the case \( e = 1 \) follows from the case \( e = 0 \).

To show (ii), by Corollary 2.4, it suffices to show the injectivity of \( \tau^{(e)} H \rightarrow \tau^{(e)} G \) for \( e \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \). The case \( e = (0, 0) \) follows from the same left exactness as above, and the other cases from this case thanks to Lemma 2.2.

To show (2), we may assume \( \mathcal{X} \) is henselian local. Then it follows from (1).

Finally, (3) follows from (2) since \( \omega \) is an isomorphism. This completes the proof of the corollary.

\[ \square \]

3. Review of higher local symbols

In this section we recall from [RS21c] the higher local symbols for reciprocity sheaves, which are a fundamental tool to prove Theorem 4.2, one of the main theorems of this paper. First we introduce some basic notation. In this section \( X \) is a reduced noetherian separated scheme of dimension \( d < \infty \) such that \( X^{(0)} = X^{(d)} \).

Let \( K \) be a field. For an integer \( r \geq 0 \), let \( K^M_r(K) \) be the Milnor \( K \)-group of \( K \). Let \( A \) be a local domain with the function field \( K \). For an ideal \( I \subset A \), let \( \overline{K}^M_r(A, I) \subset K^M_r(K) \) denote the subgroup generated by symbols
\[
\{1 + a, b_1, \ldots, b_{r-1}\} \quad \text{with} \quad a \in I, \quad b_i \in A^X.
\]
Let \( A \) be a noetherian excellent one-dimensional local domain with function field \( K \) and residue field \( F \). Let \( \hat{A} \) be the normalization of \( A \) and \( S \) be the set of the maximal ideals of \( \hat{A} \). For \( \mathfrak{m} \in S \), denote \( \kappa(\mathfrak{m}) = \hat{A}/\mathfrak{m} \). Then we define
\[
\partial_A := \sum_{\mathfrak{m} \in S} \text{Nm}_{\kappa(\mathfrak{m})/F} \circ \partial_{\mathfrak{m}} : K^M_r(K) \rightarrow K^M_{r-1}(F),
\]
where \( \partial_{\mathfrak{m}} : K^M_r(K) \rightarrow K^M_{r-1}(\kappa(\mathfrak{m})) \) denotes the tame symbol for the discrete valuation ring \( \hat{A}_{\mathfrak{m}} \), the localization of \( \hat{A} \) at \( \mathfrak{m} \), and \( \text{Nm}_{\kappa(\mathfrak{m})/F} \) is the norm map.

365
For $x, y \in X$, we write
\[ y < x : \iff \overline{\{y\}} \subsetneq \overline{\{x\}}, \quad \text{that is, } y \in \overline{\{x\}} \text{ and } y \neq x. \]

A chain on $X$ is a sequence
\[ \underline{x} = (x_0, \ldots, x_n) \quad \text{with } x_0 < x_1 < \cdots < x_n. \quad (3.0.2) \]
The chain $\underline{x}$ is a maximal Paršin chain (or maximal chain) if $n = d$ and $x_i \in X(i)$. Note that the assumptions on $X$ imply $x_i \in \overline{\{x_{i+1}\}}^{(1)}$. We denote
\[ \text{mc}(X) = \{ \text{maximal chains on } X \}. \]

A maximal chain with break at $r \in \{0, \ldots, d\}$ is a chain $\underline{x}$ with $n = d - 1$ and $x_i \in X(i)$, for $i < r$, and $x_i \in X_{i+1}$, for $i \geq r$. We denote
\[ \text{mc}_r(X) = \{ \text{maximal chain with break at } r \text{ on } X \}. \]

For $\underline{x} = (x_0, \ldots, x_{d-1}) \in \text{mc}_r(X)$, we denote by $b(\underline{x})$ the set of $y \in X(r)$ such that
\[ \underline{x}(y) := (x_0, \ldots, x_{r-1}, y, x_r, \ldots, x_{d-1}) \in \text{mc}(X). \quad (3.0.3) \]

The following properties hold for all $a \in F(K(X))$ (see Remark 3.1 below).

(HS1) Let $X \rightarrow X'$ be an open immersion where $X'$ is an integral $K$-scheme of dimension $d$. Then, for all $\beta \in K_d^M(K(X))$,
\[ (a, \beta)_{X/K, \underline{x}} = (a, \beta)_{X'/K, \underline{x}}. \]

(HS2) Let $\underline{x} = (x_0, \ldots, x_{d-1}, x_d) \in \text{mc}(X)$ and $Z \subset X$ be the closure of $z = x_{d-1}$, and set $\underline{x}' = (x_0, \ldots, x_{d-1}) \in \text{mc}(Z)$. Assume $a \in F(O_{X,z})$ and let $a(z) \in F(K(Z))$ be the restriction of $a$. Then
\[ (a, \beta)_{X/K, \underline{x}} = (a(z), \partial_z \beta)_{Z/K, \underline{x}'} \quad \text{for } \beta \in K_d^M(K(X)), \]
where $\partial_z : K_d^M(K(X)) \rightarrow K_{d-1}^M(K(Z))$ is the map (3.0.1) for $A = O_{X,z}$.

(HS3) Let $D \subset X$ be an effective Cartier divisor with $I_D \subset O_X$ its ideal sheaf. Assume that $X \setminus D$ is regular so that $(X, D) \in \text{MCor}^{\text{pro}}$ and that $a \in F(X, D)$. For $\underline{x} = (x_0, \ldots, x_{d-1}, x_d) \in \text{mc}(X)$, we have
\[ (a, \beta)_{X/K, \underline{x}} = 0 \quad \text{for } \beta \in K_d^M(O_{X,x_{d-1}}, I_D O_X, x_{d-1}). \]

(HS4) Let $\underline{x}' \in \text{mc}_r(X)$ with $0 \leq r \leq d - 1$. For $\beta \in K_d^M(K(X))$,
\[ (a, \beta)_{X/K, \underline{x}'}(y) = 0 \quad \text{for almost all } y \in \overline{\{x'\}}. \]
Assume that either $r \geq 1$ or $r = 0$, $X$ is quasi-projective, and the closure of $x_1$ in $X$ is projective over $K$, where $\underline{x}' = (x_1, \ldots, x_d)$. Then
\[ \sum_{y \in b(\underline{x}')} (a, \beta)_{X/K, \underline{x}'}(y) = 0. \]
Reciprocity sheaves and logarithmic motives

Remark 3.1. Properties (HS1)–(HS4) are slight variants of the (stronger) properties (HS1)–(HS4) in [RS21c, Proposition 5.3], where the Milnor $K$-group $K^M_d(K^h_{X,x})$ of the iterated henselization $K^h_{X,x}$ of $K(X)$ along the chain $x$ is used instead of $K^M_d(K(X))$. The version stated here follows easily using the natural maps $ι_x : K(X) → K^h_{X,x}$ and the commutative diagram in the situation of (HS2),

$$
\begin{array}{c}
K^M_d(K^h_{X,x}) \\
\downarrow ι_x \\
K^M_d(K(X))
\end{array}
\xrightarrow{\partial_x} 
\begin{array}{c}
K^M_{d-1}(K^h_{Z,x'}) \\
\downarrow ι_{x'} \\
K^M_{d-1}(K(Z))
\end{array}
$$

and the commutative diagram in the situation of (HS4),

$$
\begin{array}{c}
K^M_{d-1}(K^h_{X,x'}) \\
\downarrow ι_{x'} \\
K^M_d(K(X))
\end{array}
\xrightarrow{\partial_x} 
\begin{array}{c}
K^M_{d-1}(K^h_{X,x'(y)})
\end{array}
$$

where $\partial_x$ (respectively, $ι_y$) is defined in [RS21c, (4.1.1)] (respectively, [RS21c, (3.2.3)]). We also note that $K^M_d(O_{X,d-1}, I_D O_{X,x-1})$ in (HS2) coincides with the Zariski stalk at $x_{d-1}$ of the sheaf $\overline{V}_{d,X/D}$ defined in [RS21c, 4.4].

For a scheme $Z$ over $k$, write $Z_K = Z ⊗_k K$. If $Z_K$ is integral, we denote by $K(Z)$ the function field of $Z_K$. We quote the following result from [RS21c, Proposition 7.3]. It is a key tool in the proof of Theorem 4.2.

Proposition 3.2. Let $X ∈ Sm$ and assume $D$ is a reduced simple normal crossing divisor on $X$ with $I_D ⊂ O_X$ its ideal sheaf. Let $U ⊂ X$ be an open subset containing all the generic points of $D$. Let $a ∈ F(X\setminus D)$. Assume that, for all function fields $K/k$ and for all $x = (x_0, \ldots, x_{d-1}, x_d) ∈ mc(U_K)$ with $x_{d-1} ∈ D^{(0)}_K$, we have

$$(a, β)_{x_K/k,x} = 0 \quad \text{for all } β ∈ K^M(O_{X,x-1}, I_D O_{X,x-1}).$$

Then $a ∈ F(X, D)$.

4. Logarithmic cohomology of reciprocity sheaves

For $X ∈ (X, D) ∈ MCor_{ls}$, we write $X_{red} = (X, D_{red}) ∈ MCor_{ls}$. We say that $X = (X, D) ∈ MCor_{ls}$ is reduced if $X = X_{red}$.

Definition 4.1. Let $F ∈ MPST$.

(1) We say that $F$ is log-semipure if for any $X ∈ MCor_{ls}$, the map $F(X_{red}) → F(X)$ is injective. Note that if $F$ is semipure, $F$ is log-semipure (cf. §1(16)).

(2) We say that $F$ is logarithmic if it is log-semipure and satisfies the condition that for $X, Y ∈ MCor_{ls}$ with $X$ reduced and $α ∈ MCor_{fin}(Y, X)$, the image of $α^* : F(X) → F(Y)$ is contained in $F(Y_{red}) ⊂ F(Y)$.

Let $MPST_{log}$ be the full subcategory of $MPST$ consisting of logarithmic objects and put $MNST_{log} = MNST ∩ MPST_{log}$.
Theorem 4.2. Any $F \in \text{CI}^{\text{sp}}_{\text{Nis}}$ is logarithmic, that is, $\text{CI}^{\text{sp}}_{\text{Nis}} \subset \text{MNST}_{\text{log}}$.

We need a preliminary lemma for the proof of the theorem.

Lemma 4.3. Let $F \in \text{CI}^{\text{sp}}_{\text{Nis}}$. Let $A^n_K = \text{Spec } K[x_1, \ldots, x_n]$ be the affine space over a function field $K$ over $k$ and $V = \text{Spec } K\{x_1, \ldots, x_n\}$ be the henselization of $A^n_K$ at the origin and $L_i = \{x_i = 0\} \subset V$ for $i \in [1, n]$. For an integer $0 < r \leq n$, the natural map

$$K\{x_{r+1}, \ldots, x_n\}[x_1, \ldots, x_r] \to K\{x_1, \ldots, x_n\}$$

induces a map in $\text{MCor}^{\text{pro}}$ (cf. §1(9)):

$$\rho_r : (V, L_1 + \cdots + L_r) \to (A^n_S, \{x_1 \cdots x_r = 0\}) \simeq (A^1, 0)^{\otimes r} \otimes (S, \emptyset),$$

where $S = \text{Spec } K\{x_{r+1}, \ldots, x_n\}$. It induces

$$\rho^* r : F(A^n_S, \{x_1 \cdots x_r = 0\}) \to F(V, L_1 + \cdots + L_r). \quad (4.3.1)$$

Then $F(V, L_1 + \cdots + L_r)$ is generated by the image of $\rho^* r$ and

$$F(V, L_1 + \cdots + \hat{L}_i + \cdots + L_r) \text{ for } i = 1, \ldots, r.$$

Proof. For $\mathcal{Y} \in \text{MCor}$, let $F^\mathcal{Y}$ be defined by $F^\mathcal{Y}(Z) = F(\mathcal{Y} \times Z)$. Clearly, we have $F^\mathcal{Y} \in \text{CI}^{\text{sp}}_{\text{Nis}}$ for $F \in \text{CI}^{\text{sp}}_{\text{Nis}}$. We prove the lemma by induction on $r$. The case $r = 1$ holds since by [Sai20, Lemmas 7.1 and 5.9], $\rho_1$ induces an isomorphism

$$F(A^1, 0)(S)/F(A^1, 0)(S) \cong F(V, L_1)/F(V).$$

By definition $L_1 = \text{Spec } K\{x_2, \ldots, x_n\}$ and we have a map in $\text{MCor}^{\text{pro}}$,

$$(V, L_1 + \cdots + L_r) \to (A^1, 0) \otimes (L_1, L_1 \cap (L_2 + \cdots + L_r)),$$

induced by the natural map $K\{x_2, \ldots, x_n\}[x_1] \to K\{x_1, \ldots, x_n\}$. By [Sai20, Lemmas 7.1 and 5.9], it induces an isomorphism

$$F(A^1, 0)(L_1, E)/F(A^1, 0)(L_1, E) \cong F(V, L_1 + \cdots + L_r)/F(V, L_2 + \cdots + L_r)$$

with $E = L_1 \cap (L_2 + \cdots + L_r)$. By the induction hypothesis, $F(A^1, 0)(L_1, E)$ is generated by $F(A^1, 0)(L_1, E_j)$ with $E_j = L_1 \cap (L_2 + \cdots + L_j)$ for $j = 2, \ldots, r$ together with the image of the map

$$(F(A^1, 0)^{\otimes r-1}(S) = F(A^1, 0)^{\otimes r}(S) \to F(A^1, 0)(L_1, E)$$

induced by

$$(L_1, E) \to (A^n_S^{-1}, \{x_2 \cdots x_r = 0\}) \simeq (A^1, 0)^{\otimes r-1} \otimes (S, \emptyset)$$

coming from the map $K\{x_{r+1}, \ldots, x_n\}[x_2, \ldots, x_r] \to K\{x_2, \ldots, x_n\}$. This proves the lemma. \qed

Proof of Theorem 4.2. By Corollary 2.6(3), we may assume $F = \varphi_{\text{CG}}$ for $G \in \text{RSC}_{\text{Nis}}$. Take $\mathcal{X} = (X, D), \mathcal{Y} = (Y, E) \in \text{MCor}_{\text{Is}}$ with $\mathcal{X}$ reduced, and let $\alpha \in \text{MCor}^{\text{fin}}(\mathcal{Y}, \mathcal{X})$ be an elementary correspondence. We need to show that $\alpha^*(F(\mathcal{X})) \subset F(\mathcal{Y}_{\text{red}})$. The question is Nisnevich local over $X$ and $Y$. Hence, we may assume $(X, D) = (V, L_1 + \cdots + L_r) \in \text{MCor}^{\text{pro}}$ in the notation of Lemma 4.3. If $r = 0$, we have $\alpha \in \text{MCor}^{\text{fin}}(\mathcal{Y}, \mathcal{X})$ so that

$$\alpha^*(F(\mathcal{X})) = \alpha^*(F(X, \emptyset)) \subset F(Y, \emptyset) \subset F(\mathcal{Y}_{\text{red}}).$$

368
Reciprocity sheaves and logarithmic motives

Assume $r > 0$ and proceed by induction on $r$. By Lemma 4.3, we may then assume

$$(X, D) = \mathcal{M} := (\mathbb{A}^1, 0)^{\otimes r} \otimes (S, \emptyset) \quad \text{for } S \in \text{Sm}^\text{pro}.$$ 

On the other hand, by Corollary 2.5, we have an exact sequence

$$0 \to F(Y, E_{\text{red}}) \to F(Y - E_{\text{red}}, \emptyset) \to \bigoplus_{\xi \in E^{(0)}} F(Y^h_\xi - \xi, \emptyset).$$

Hence, we may replace $Y$ with its Nisnevich neighborhood of a generic point $\xi$ of $E$. Using the assumption that $k$ is perfect, we may then assume the following condition (♠). Recall that $\alpha$ is by definition an integral closed subscheme of $(Y - E) \times (X - D)$ finite surjective over $Y - E$, and its closure $\overline{\alpha}$ in $Y \times X$ is finite surjective over $Y$.

(♠) Let $Y'$ be the normalization of $\overline{\alpha}$ and $E' := E \times_Y Y'$. Then $X, Y, E$ and $E'$ are irreducible, and $\alpha, Y', E_{\text{red}}$ and $E'_{\text{red}}$ are essentially smooth over $k$.

Let $g : Y' \to Y$ and $f : Y' \to X$ be the induced maps. We have $E' = g^* E \geq f^* D$ as Cartier divisors on $Y'$ by the modulus condition for $\alpha$. Hence, these maps induce

$$F(X, D) \xrightarrow{f^*} F(Y', E') \xrightarrow{g_*} F(Y, E).$$

We claim that $\alpha^* : F(X, D) \to F(Y, E)$ agrees with this map. Indeed, this follows from the equality

$$\Gamma_f \circ \Gamma_g = \alpha \in \text{Cor}(Y - E, X - D),$$

where $\Gamma_g \in \text{Cor}(Y - E, Y' - E')$ is the transpose of the graph of $g$ and $\Gamma_f \in \text{Cor}(Y' - E', X - D)$ is the graph of $f$. By definition this follows from the equality

$$\Gamma_g \times_{Y', -E'} \Gamma_f = \alpha \subset (Y - E) \times (X - D)$$

which one can check easily, noting that $Y' \to \overline{\alpha}$ is an isomorphism over $\alpha$ since $\alpha$ is regular by (♠). Then we get a commutative diagram

$$
\begin{array}{cccc}
F(Y', E'_{\text{red}}) & \xrightarrow{g_*} & F(Y, E_{\text{red}}) & \\
\downarrow & & \downarrow & \\
& F(Y', E_{\text{red}} \times_Y Y') & \xrightarrow{g_*} & F(Y, E_{\text{red}})
\end{array}
$$

where the top inclusion comes from the inequality $E_{\text{red}} \times_Y Y' \geq E'_{\text{red}}$ as Cartier divisors on $Y'$ thanks to the semipurity of $F$ (cf. § 1(16)). Hence, it suffices to show $f^*(F(X, D)) \subset F(Y', E'_{\text{red}})$. By replacing $(Y, E)$ with $(Y', E')$, we may now assume that $\alpha$ is induced by a morphism $f : Y \to X = \mathbb{A}^r \times S$. Then $\alpha$ factors in $\text{MCor}$ as

$$(Y, E) \xrightarrow{i} (\mathbb{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset) \to (\mathbb{A}^1, 0)^{\otimes r} \otimes (S, \emptyset),$$

where the first map is induced by the map

$$i = (pr_{\mathbb{A}^r} \circ f, id_Y) : Y \to \mathbb{A}^r \times Y,$$

and the second is induced by

$$id_{\mathbb{A}^r} \times (pr_S \circ f) : \mathbb{A}^r \times Y \to \mathbb{A}^r \times S.$$
Let that \( i \) is a section of the projection \( \mathcal{A}^r \times Y \to Y \). Thus, we are reduced to showing \( i^*(F((\mathcal{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset))) \subset F(Y, E_{\text{red}}) \). By Proposition 3.2 this follows from the following claim.

**Claim 4.3.1.** Take \( a \in F((\mathcal{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset)) \). There exists an open neighborhood \( U \subset Y \) of the generic point of \( E \) such that for every function field \( K \) over \( k \) and every \( \delta = (\delta_0, \ldots, \delta_{r-1}, \delta_e) \in \text{mc}(U_K) \) with \( \xi := \delta_{r-1} \in E_{\text{red}}^0 \) and \( e = \dim(Y) \), we have

\[
(i^*(a))_K(\gamma)Y_{K, \delta} = 0 \quad \forall \gamma \in \mathcal{K}_e^M(O_{Y_K, \xi}, m_\xi)
\]

for the pairing from (3.0.4):

\[
(-, -)_{Y_{K, \delta}} : F(K(Y)) \otimes K^M_d(K(Y)) \to F(K).
\]

**Proof.** After replacing \( Y \) by an open neighborhood of the generic point of \( E \), we may assume that \( Y = \text{Spec}(A) \) is affine and \( E_{\text{red}} = \text{Spec}(A/(\pi)) \) for \( \pi \in A \) and, moreover, that writing

\[
\mathcal{A}^r \times Y = \text{Spec} A[x_1, \ldots, x_r], \quad (\mathcal{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset) = (A_Y, \{x_1 \cdots x_r = 0\})
\]

we have

\[
i(Y) = \bigcap_{1 \leq i \leq r} \{x_i - u_i \pi^{m_i} = 0\} \text{ with } m_i \in \mathbb{Z}_{\geq 0}, u_i \in A^r.
\]

Let \( \delta = (\delta_0, \ldots, \delta_e) \) be as in the claim and put \( \delta' = (\delta_0, \ldots, \delta_{r-1}) \in \text{mc}(U_K) \). Put \( \tilde{X}_K = \mathcal{A}^r \times Y_K \) and \( F = \{\pi = 0\} \subset \tilde{X}_K \). Note \( d := \dim(\tilde{X}_K) = e + r \). Let \( z_j \) for \( e \leq j \leq d - 1 \) be the generic point of

\[
Z_j = \bigcap_{1 \leq i \leq d - j} \{x_i - u_i \pi^{m_i} = 0\} \subset \tilde{X}_K
\]

which lies over \( \delta_e \), and \( w_j \) for \( e - 1 \leq j \leq d - 2 \) be the generic point of

\[
W_j = F \cap Z_{j+1} = \{\pi = x_1 = \cdots = x_{d-j-1} = 0\}
\]

which is contained in the closure of \( z_{j+1} \). Note \( \dim(Z_j) = \dim(W_j) = j \) and the section \( i \) induces isomorphisms

\[
Y_K \simeq Z_e \quad \text{and} \quad (E_{\text{red}})_K \simeq W_{e-1}.
\]

Let \( \sigma = (i(\delta'), w_e, \ldots, w_{d-2}, \eta_1, \nu) \in \text{mc}(\tilde{X}_K) \), where \( \nu \) is the generic point of \( \tilde{X}_K \) lying over \( \delta_e, \eta_1 \) is the generic point of \( D_1 = \{x_1 = 0\} \subset \tilde{X}_K \) contained in the closure of \( \nu \), and \( i(\delta') \in \text{mc}(W_{e-1}) \) is the image of \( \delta' \) under (4.3.2). Take any \( \gamma \in \mathcal{K}_e^M(O_{Y_K, \xi}, m_\xi) \) and put

\[
\beta = \left\{ \iota(\gamma), \frac{u_1 \pi^{m_1}}{u_1 \pi^{m_1}}, \ldots, \frac{u_r \pi^{m_r}}{u_r \pi^{m_r}} \right\} \in \mathcal{K}_d^M(O_{\tilde{X}_K, \nu}),
\]

where \( \iota: \mathcal{K}_e^M(O_{Y_K, \delta_e}) \to \mathcal{K}_d^M(O_{\tilde{X}_K, \nu}) \) is induced by the projection \( \tilde{X}_K \to Y_K \). For \( a \in F((\mathcal{A}^1, 0)^{\otimes r} \otimes (Y, \emptyset)) \) and its restriction \( a_K \in F((\mathcal{A}^1, 0)^{\otimes r} \otimes (Y_K, \emptyset)) \), we have

\[
0 = (a_K, \beta)_{\tilde{X}_K/K, \sigma} = -\sum_{\tau \in \tilde{X}_K^{(1)} \cup \{\eta_1\}} (a_K, \beta)_{\tilde{X}_K/K, (i(\delta'), w_e, ..., w_{d-2}, \tau, \nu)}
\]

\[
= -(a_K, \beta)_{\tilde{X}_K/K, (i(\delta'), w_e, ..., w_{d-2}, z_{d-1}, \nu)}
\]

\[
= \pm (a_K)_{Z_{d-1}, (\beta_1)_{Z_{d-1}/K, (i(\delta'), w_e, ..., w_{d-2}, z_{d-1}, \nu)}}.
\]

\[\text{5} \quad \text{Although } Y \text{ is assumed to be irreducible, } Y_K \text{ may not be so and possibly a finite product of schemes essentially smooth over } k, \text{ noting that } k \text{ is perfect.}\]
Reciprocity sheaves and logarithmic motives

\[ \beta_1 = \left\{ \frac{u_2 \pi^{m_2} - x_2}{u_2 \pi^{m_2}}, \ldots, \frac{u_r \pi^{m_r} - x_r}{u_r \pi^{m_r}} \right\} \in K^M_{d-1}(O_{Z_{d-1},z_{d-1}}), \]

where \( \iota_1 : K^M_e(O_{X_K,\delta}) \to K^M_e(O_{Z_{d-1},z_{d-1}}) \) is induced by the dominant map \( Z_{d-1} \to Y_K \) induced by the projection \( X_K \to Y_K \). The first equality follows from §3 (HS3) applied to \( D_1 \subset X_K \), noting that \( \beta \) lies in \( \overline{K}_d(O_{X_K,\eta_1},m_{\eta_1}) \) since \( (u_1 \pi^{m_1} - x_1)/u_1 \pi^{m_1} \in 1 + x_1 O_{X_K,\eta_1} \). The second follows from (HS4). The third equality holds since \( z_{d-1} \) is the unique \( \tau \in X_K^{(1)} \setminus \{ \eta_1 \} \) such that \( \tau > w_{d-2} \) and \( (a_K, \beta)_{X_K/K, (i(\delta'), w_e, \ldots, w_{d-2}, \tau, \nu)} \) may not vanish, which follows from (HS2), noting that \( \iota(\gamma)_{|F} = 0 \). The final equality follows from (HS2). When \( r = 1 \), the last term in the above formula is equal to \( ((a_K)|_{Y_K, \gamma})_{Y_K/K, \delta} \) by (4.3.2), so that the proof is complete. When \( r > 1 \), we further get

\[ 0 = ((a_K)|_{Z_{d-1}, \delta}, \beta_1)_{Z_{d-1}/K, (i(\delta'), w_e, \ldots, w_{d-2}, z_{d-1})} \]

\[ - \sum_{\tau \in Z_{d-1}^{(1)} \setminus \{ w_{d-2} \} \atop \tau > w_{d-3}} ((a_K)|_{Z_{d-1}, \delta}, \beta_1)_{Z_{d-1}/K, (i(\delta'), w_e, \ldots, w_{d-3}, \tau, z_{d-1})} \]

\[ = \pm ((a_K)|_{Z_{d-2}, \delta}, \beta_2)_{Z_{d-2}/K, (i(\delta'), w_e, \ldots, w_{d-3}, z_{d-2})}, \]

\[ \beta_2 = \left\{ \frac{u_3 \pi^{m_3} - x_3}{u_3 \pi^{m_3}}, \ldots, \frac{u_r \pi^{m_r} - x_r}{u_r \pi^{m_r}} \right\} \in K^M_{d-1}(O_{Z_{d-2},z_{d-2}}), \]

where \( \iota_2 : K^M_e(O_{X_K,\delta}) \to K^M_e(O_{Z_{d-2},z_{d-2}}) \) is induced by the dominant map \( Z_{d-2} \to Y_K \) induced by the projection \( X_K \to Y_K \). The above equalities hold by the same arguments as above, except that for the third equality there are a priori two \( \tau \in Z_{d-1}^{(1)} \setminus \{ w_{d-2} \} \) with \( \tau > w_{d-3} \) for which \( ((a_K)|_{Z_{d-1}, \delta}, \beta_1)_{Z_{d-1}/K, (i(\delta'), w_e, \ldots, w_{d-3}, \tau, z_{d-1})} \) may not vanish. One is \( z_{d-2} \) and the other is the generic point \( \eta_2 \) of \( Z_{d-1} \cap D_2 \) with \( D_2 = \{ x_2 = 0 \} \subset X_K \) which is contained in the closure of \( z_{d-1} \). But \((a_K)|_{Z_{d-1}, \delta}, \beta_1)_{Z_{d-1}/K, (i(\delta'), w_e, \ldots, w_{d-3}, \eta_2, z_{d-1})} = 0 \). Indeed, \( (a_K)|_{Z_{d-1}} \in F(\text{Spec}(O_{Z_{d-1}, \eta_2}), \eta_2) \) since \( Z_{d-1} \) and \( D_2 \) intersect transversally in \( X_K \). Hence, the vanishing follows from (HS3) applied to \( Z_{d-1} \cap D_2 \subset Z_{d-1} \), noting that \( ((u_2 \pi^{m_2} - x_2)/u_2 \pi^{m_2})|_{Z_{d-1}} \in 1 + x_2 O_{Z_{d-1}, \eta_2} \) so that \( \beta_1 \in K^M_{d-1}(O_{Z_{d-1}, \eta_2, m_{\eta_2}}) \). Repeating the same arguments, we finally get

\[ 0 = ((a_K)|_{Z_{e}, \iota_r(\gamma)}|_{Z_{e}/K, (i(\delta'), z_e}) = ((a_K)|_{Y_K, \gamma})_{Y_K/K, \delta}, \]

where \( \iota_r : K^M_e(O_{X_K,\delta}) \to K^M_e(O_{Z_{e},z_e}) \) is induced by the isomorphism \( Z_{e} \to Y_K \) induced by the projection \( X_K \to Y_K \) and the second equality follows from (4.3.2). This completes the proof of the claim and Theorem 4.2. \( \square \)

**Definition 4.4.** For \( F \in \text{MNST}_{\log}^i \) and an integer \( i \geq 0 \), consider the association

\[ H^i_{\log}(-, F) : \text{MCor}_{\text{is}}^{\text{fin}} \to \text{Ab}; (X, D) \to H^i(X_{Nis}, F(X, D_{\text{red}})). \]

By the definition this gives a presheaf on \( \text{MCor}_{\text{is}}^{\text{fin}} \), which we call the \( i \)th logarithmic cohomology with coefficient \( F \).

371
5. Invariance of logarithmic cohomology under blowups

Retain the notation of §4.

**Definition 5.1.** Let $\Lambda_{\text{fin}}^{\text{Is}}$ be the class of morphisms $\rho : (Y, E) \to (X, D)$ in $\text{MCor}_{\text{Is}}^{\text{fin}}$ satisfying the following conditions.

(a) $\rho$ is induced by a proper morphism $\rho : Y \to X$ inducing an isomorphism $Y \setminus E \cong X \setminus D$ and $E = \rho^* D$.

(b) Zariski locally on $X$, $\rho : Y \to X$ is the blowup of $X$ in a smooth center $Z \subset D$ which is normal crossing to $D$.

Here, a smooth $Z$ contained in $D$ is normal crossing to $D$ if, letting $D_1, \ldots, D_n$ be the irreducible components of $D$, there exists a subset $I \subset \{1, \ldots, n\}$ such that $Z \subset \bigcap_{i \in I} D_i$ and $Z$ is not contained in $D_j$ for any $j \notin I$ and intersects $\sum_{j \notin I} D_j$ transversally. Note that the condition is equivalent to that called strict normal crossing in [BPØ22, Definition 7.2.1].

**Theorem 5.2.** For $F \in \text{Cl}^\text{sp}_{\text{Nis}}$ and $\rho : Y \to X$ in $\Lambda_{\text{fin}}^{\text{Is}}$, we have

$$\rho^*: H_{\log}^i(X, F) \cong H_{\log}^i(Y, F) \quad \forall i \geq 0.$$  \hfill (5.2.1)

**Proof.** Write $Y = (Y, E)$ and $X = (X, D)$. First we prove the theorem for $i = 0$. We may assume that $D$ is reduced and $E = \rho^* D$. By [KMY21, Proposition 1.9.2 b)], $\rho$ is invertible in $\text{MCor}$, so that $\rho^*: F(X) \cong F(Y)$. Since this factors through $F(Y, E_{\text{red}})$ by Theorem 4.2, we get (5.2.1) for $i = 0$.

To show (5.2.1) for $i > 0$, it suffices to prove $R^i\rho_* F_{(Y, E_{\text{red}})} = 0$. The problem is Nisnevich local, so we may assume that $\rho$ is induced by a blowup $\rho : Y \to X$ in a smooth center $Z \subset D$ normal crossing to $D$. By [KS21, Corollary 9], Nisnevich locally around a point of $Z$, $(X, D)$ is isomorphic to

$$(\mathbb{A}^c, L_1 + \cdots + L_r) \otimes W \quad \text{with } W = (W, W^\infty) \in \text{MCor}_{\text{Is}},$$

where $\mathbb{A}^c = \text{Spec } k[t_1, \ldots, t_c]$ with $c = \text{codim}_Z(Z, X)$ and $L_i = V(t_i)$ for $i = 1, \ldots, r$ with $1 \leq r \leq c$, and $Z$ corresponds to $0 \times W$. Hence, the theorem follows from the following proposition. \hfill \square

**Proposition 5.3.** Let $F \in \text{Cl}^\text{sp}_{\text{Nis}}$ and $W = (W, W^\infty) \in \text{MCor}_{\text{Is}}$. Let $A^n = \text{Spec } k[t_1, \ldots, t_n]$ and put $L_i = V(t_i)$ for $1 \leq i \leq n$. Let $\rho : Y \to A^n$ be the blowup at the origin $0 \in A^n$ and $L_i \subset Y$ be the strict transforms of $L_i$ for $1 \leq i \leq n$ and $E = \rho^{-1}(0) \subset Y$. For any $1 \leq r \leq n$, we have

$$ R^i \rho_* F_{(Y, L_1 + \cdots + L_r + E)} \otimes W = 0 \quad \text{for } i \geq 1, \hfill (5.3.1)$$

where $\rho_W := \rho \times \text{id}_W : Y \times W \to A^2 \times W$.

**Lemma 5.4.** Proposition 5.3 holds for $n = 2$.

**Proof.** The case $r = 1$ is proved in [BRS22, Lemma 2.13] and we show the case $r = 2$.\footnote{The following argument is adopted from [BRS22, Lemma 2.13], but the present case is easier.} Put $D = L_1 + L_2$. By the case $i = 0$ of Theorem 5.2, we get

$$ F_{(A^2, D)} \otimes W \cong \rho_{W^*} F_{(Y, L_1 + L_2 + E)} \otimes W^* \hfill (5.4.1)$$

Set

$$ F := F_{(Y, L_1 + L_2 + E)} \otimes W^* $$

Put

$$ D = L_1 + L_2. \hfill (5.4.1)$$

\footnote{The following argument is adopted from [BRS22, Lemma 2.13], but the present case is easier.}
Reciprocity sheaves and logarithmic motives

and $A^2_w = A^2 \times W$ with the projection $p : A^2_w \to W$. Since $R^i \rho_{W*}\mathcal{F}$ for $i \geq 1$ is supported in $0 \times W$, we have

$$R^i \rho_{W*}\mathcal{F} = 0 \iff p_*R^i \rho_{W*}\mathcal{F} = 0$$

$$\iff (p_*R^i \rho_{W*}\mathcal{F})_w = 0 \quad \forall w \in W$$

$$\iff H^0(A^2_{w*}, R^i \rho_{W*}\mathcal{F}) = 0 \quad \forall w \in W,$$

where $W_w$ is the henselization of $W$ at $w$. Hence, it suffices to show $H^0(A^2_w, R^i \rho_{W*}\mathcal{F}) = 0$, assuming $W$ is henselian local. Then we have

$$H^i(A^2_{w*}, R^i \rho_{W*}\mathcal{F}) = 0, \quad \forall i, j \geq 1.$$

By (5.4.1) and [BRS22, Lemma 2.10],

$$H^i(A^2_{w*}, \rho_{W*}\mathcal{F}) = H^i(A^2_{w*}, F_{(A^2, D) \otimes W}) = 0.$$

Thus, the Leray spectral sequence yields

$$H^0(A^2_{w*}, R^i \rho_{W*}\mathcal{F}) = H^i(Y \times W, \mathcal{F}), \quad i \geq 0,$$

and we have to show that this group vanishes for $i \geq 1$. We can write

$$A^2 = \text{Spec } k[x, y] \quad \text{and} \quad L_1 = V(x), \; L_2 = V(y) \subset A^2.$$

Then we have

$$Y = \text{Proj } k[x, y][S, T]/(xT - yS) \subset A^2 \times P^1.$$

Denote by

$$\pi_0 : Y \hookrightarrow A^2 \times P^1 \to P^1 = \text{Proj } k[S, T]$$

the morphism induced by projection, and let $\pi : Y \times W \to P^1_w$ be its base change. Then $\pi_0$ induces an isomorphism $E \simeq P^1$, and we have

$$\tilde{L}_1 = \pi_0^{-1}(0), \quad \tilde{L}_2 = \pi_0^{-1}(\infty). \quad (5.4.2)$$

Set $s = S/T = x/y$ and write

$$P^1 \setminus \{\infty\} = A^1_s := \text{Spec } k[s], \quad P^1 \setminus \{0\} = \text{Spec } k[1/s].$$

Set $U := A^1_s \times W, \; V := (P^1 \setminus \{0\}) \times W$ and

$$U := (A^1_s, 0) \otimes W, \quad V := (P^1 \setminus \{0\}, \infty) \otimes W.$$

We have

$$\pi^{-1}(U) = A^1_s \times U, \quad \pi^{-1}(V) = A^1_s \times V,$$

and the restriction of $\pi$ to these open subsets is given by projection. Furthermore, $E \times W \subset Y$ is defined by $y = 0$ on $\pi^{-1}(U)$ and by $x = 0$ on $\pi^{-1}(V)$. In view of (5.4.2), we have

$$\mathcal{F}|_{\pi^{-1}(U)} = F_{(A^1_s, 0) \otimes U}, \quad \mathcal{F}|_{\pi^{-1}(V)} = F_{(A^1_s, 0) \otimes V}. \quad (5.4.3)$$

Thus, [BRS22, Lemma 2.10] yields

$$R^j \pi_*\mathcal{F} = 0 \quad \text{for } j \geq 1,$$

and it remains to show

$$H^i(P^1_w, \pi_*\mathcal{F}) = 0 \quad \text{for } i \geq 1, \quad (5.4.4)$$

373

https://doi.org/10.1112/S0010437X22007862 Published online by Cambridge University Press
where \( P^1_W = P^1 \times W \). For this consider the map
\[
a_0 : Y \to A^1_x \times P^1
\]
which is the closed immersion \( Y \hookrightarrow A^2 \times P^1 \) followed by the projection \( A^2 \to A^1_x \). Let \( a : Y \times W \to A^1_x \times P^1 \times W \) be its base change. In view of (5.4.2), the map \( a \) induces a morphism in \( \text{MCor} \),
\[
\alpha : (Y, \bar{L} + \bar{E} + E) \otimes W \to (A^1_x, 0) \otimes (P^1, \infty) \otimes W,
\]
which is an isomorphism over \( (A^1_x, 0) \otimes (P^1 \setminus \{0\}, \infty) \otimes W \). Setting
\[
F_1 := \text{Hom} \left( \mathbb{Z}[t] \left( A^1_x, 0 \right), F \right) \in \text{CT}_{\text{Nis}}^{\text{sp}},
\]
it induces a map of Nisnevich sheaves on \( P^1_W \),
\[
\pi_*(\alpha^*) : F_1 \otimes (P^1, \infty) \otimes W \to \pi_* F,
\]
which becomes an isomorphism over \( (P^1 - \{0\}) \times W \). Hence, (5.4.4) follows from
\[
H^i(P^1_W, F_1 \otimes (P^1, \infty) \otimes W) = 0 \quad \text{for} \ i \geq 1,
\]
which follows from [Sai20, Theorem 0.6].

**Lemma 5.5.** Let \( N > 2 \) be an integer and assume that Proposition 5.3 holds for \( n < N \). Let \((X, D) \in \text{MCor}_n \) and \( Z \subset X \) be a smooth integral closed subscheme with \( 2 \leq \text{codim}(Z, X) =: c < N \). Assume
\[
D = D_1 + \cdots + D_r + D' \quad \text{with} \quad r \leq c,
\]
where \( D_1, \ldots, D_r \) are distinct and reduced irreducible components of \( D \) containing \( Z \), and \( D' \) is an effective divisor on \( X \) such that none of the component of \( D' \) contains \( Z \) and \( Z \) is transversal to \( |D'| \). Let \( \rho : Y \to X \) be the blowup of \( X \) in \( Z \), let \( \bar{D}_i \) and \( D'_i \) be the strict transforms of \( D_i \) and \( D' \) respectively, and let \( E_Z = \rho^{-1}(Z) \). Then, for all \( W = (W, W^\infty) \in \text{MCor}_n \),
\[
R^i \rho_W^* F_{(Y, \bar{D}_1 + \cdots + \bar{D}_r + E_Z + D')} \otimes W = 0 \quad \text{for} \ i \geq 1,
\]
where \( \rho_W : Y \times W \to X \times W \) denotes the base change of \( \rho \).

**Proof.** This proof is adapted from [BRS22, Lemma 2.14]. The question is Nisnevich local around the points in \( Z \times W \). Let \( z \in Z \times W \) be a point and set \( A := O^k_{X, W, z} \). For \( V \subset Y \times W \), we denote \( V(z) := V \times_{X \times W} \text{Spec} \ A \). By assumption we find a regular system of local parameters \( t_1, \ldots, t_m \) of \( A \), such that
\[
(D_i \times W)(z) = V(t_i) \quad \text{for} \ 1 \leq i \leq r, \quad (Z \times W)(z) = V(t_1, \ldots, t_c),
\]
\[
(D'_i \times W)(z) = V(t^{c+1}_{c+1} \cdots t^{c+m}_{c+m}) \quad \text{with} \ c + 1 \leq m_0 \leq m,
\]
\[
(X \times W^\infty)(z) = V(t^{m_0+1}_{m_0+1} \cdots t^{m_1}_{m_1}) \quad \text{with} \ m_0 \leq m_1 \leq m.
\]
Letting \( K \) be the residue field of \( A \), we can choose a ring homomorphism \( K \hookrightarrow A \) which is a section of \( A \to K \). Then we obtain an isomorphism
\[
K \{ t_1, \ldots, t_m \} \cong A.
\]
Let \( \rho_1 : \tilde{A}^c \to A^c \) be the blowup in 0. By the above,
\[
\rho_W : (Y, \bar{D}_1 + \cdots + \bar{D}_r + E_Z + D') \otimes W \to (X, D) \otimes W
\]

374
Reciprocity sheaves and logarithmic motives

is Nisnevich locally around \( z \) isomorphic over \( k \) to the morphism

\[
(\mathbb{A}^c, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}' \rightarrow (\mathbb{A}^c, L_1 + \cdots + L_r) \otimes \mathcal{W}',
\]

\[
(\mathcal{W}' = A_K^{m-c}, \left( \prod_{i=c+1}^{m_1} t^e_i \right))
\]

induced by a map \((\mathbb{A}^c, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \rightarrow (\mathbb{A}^c, L_1 + \cdots + L_r)\) as in Proposition 5.3. Hence, the statement follows from the proposition for \( n = c < N \).

\[ \square \]

**Proof of Proposition 5.3.** The proof is by induction on \( n \geq 2 \). The case \( n = 2 \) follows from Lemma 5.4. Assume that \( n > 2 \) and that the proposition is proven for \( A^m \) with \( m < n \). For \( r = 1 \), Proposition 5.3 is proved in [BRS22, Theorem 2.12]. Assume that \( r \geq 2 \). Let \( Z := L_1 \cap L_2 \subset A^m \) and \( \tilde{Z} \subset Y \) be the strict transform of \( Z \). Denote by \( \rho' : Y' \rightarrow Y \) the blowup of \( Y \) in \( \tilde{Z} \), let \( \tilde{L}_1', E' \subset Y' \) be the strict transforms of \( \tilde{L}, E \) respectively, and let \( E'' = (\rho')^{-1}(\tilde{Z}) \). Note that \( \tilde{Z} = \tilde{L}_1 \cap \tilde{L}_2 \) intersecting transversally with \( \tilde{L}_3 + \cdots + \tilde{L}_r + E \) and \( \text{codim}(\tilde{Z}, Y) = 2 \). Hence, by Lemma 5.5,

\[
R^i\rho^*_W F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + E' + E'') \otimes \mathcal{W} = 0 \quad \text{for } i \geq 1.
\]

Since Theorem 5.2 has been proved for \( i = 0 \), we have

\[
\rho^*_F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + E' + E'') \otimes \mathcal{W} = F(Y', \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W}.
\]

Hence, we obtain

\[
R^i\rho^*_W F(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W} = R^i(\rho' \rho^*) W^*_F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + E' + E'') \otimes \mathcal{W}.
\]

(5.5.1)

Denote by \( \sigma : Y \rightarrow A^n \) the blowup in \( Z \), let \( \tilde{L}_i \subset Y \) be the strict transform of \( L_i \), and let \( \Xi = \sigma^{-1}(Z) \). By Lemma 5.5 we get

\[
R^i\sigma_* W^*_F(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + \Xi) \otimes \mathcal{W} = 0 \quad \text{for } i \geq 1.
\]

(5.5.2)

Denote by \( \sigma' : Y' \rightarrow Y \) the blowup in \( \tilde{Z} = \sigma^{-1}(0) \subset \Xi \), let \( \tilde{L}_i', \Xi' \subset Y' \) be the strict transforms of \( \tilde{L}_i, \Xi \) respectively, and let \( \Xi'' = \sigma'^{-1}(\tilde{Z}) \). Note that \( \tilde{Z} \subset \tilde{L}_3 \cap \cdots \cap \tilde{L}_n \cap \Xi \) and \( \text{codim}(\tilde{Z}, Y') = n - 1 \) and \( \tilde{Z} \) intersects transversally with \( \tilde{L}_1 + \tilde{L}_2 \). Thus, by Lemma 5.5 and the case \( i = 0 \) of Theorem 5.2, we obtain

\[
R^i\sigma^*_W F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + \Xi' + \Xi'') \otimes \mathcal{W} = F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + E') \otimes \mathcal{W}.
\]

(5.5.3)

Finally, by [BRS22, Lemma 2.15], there is an isomorphism of \( A^n \times W \)-schemes

\[
(Y', \tilde{L}_i', \ldots, \tilde{L}_r', \Xi', \Xi'') \cong (Y', \tilde{L}_i, \ldots, \tilde{L}_r, E', E'').
\]

(5.5.4)

Altogether we obtain, for \( i \geq 1 \),

\[
R^i\rho^*_W F(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + E) \otimes \mathcal{W} = R^i(\rho' \rho^*) W^*_F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + E' + E'') \otimes \mathcal{W},
\]

by (5.5.1),

\[
= R^i(\sigma \sigma') W^*_F(Y', \tilde{L}_1' + \cdots + \tilde{L}_r' + \Xi' + \Xi'') \otimes \mathcal{W},
\]

by (5.5.4),

\[
= R^i\sigma^*_W F(Y, \tilde{L}_1 + \cdots + \tilde{L}_r + \Xi) \otimes \mathcal{W},
\]

by (5.5.3),

\[
= 0,
\]

by (5.5.2).

This completes the proof of the proposition. \[ \square \]
Remark 5.6. For simplicity, we write

\[ H^i_{\log}(-, F) = H^i_{\log}(-, \omega^m F) \quad \text{for } F \in \mathbb{RSC}_{\text{Nis}}. \]

By [RS21a, Corollary 6.8], if \( \text{ch}(k) = 0 \) and \( F = \Omega^1 \), we have

\[ H^i_{\log}(-, \Omega^1) = H^i(X, \Omega^1(\log |D|)) \quad \text{for } (X, D) \in \mathbb{MCor}_{\text{is}}. \]

Hence, \( H^i_{\log}(-, F) \) for \( F \in \mathbb{RSC}_{\text{Nis}} \) is a generalization of cohomology of sheaves of logarithmic differentials.

6. Relation with logarithmic sheaves with transfers

In this section we use the same notation as [BPØ22].

Let \( \mathbf{lSm} \) be the category of log smooth and separated fs log schemes of finite type over the base field \( k \) and \( \mathbf{Sm}_{\text{Log}} \subseteq \mathbf{lSm} \) be the full subcategory consisting of objects whose underlying schemes are smooth over \( k \). Let \( \mathbf{lCor} \) be the category with the same objects as \( \mathbf{lSm} \) and whose morphisms are log correspondences defined in [BPØ22, Definition 2.1.1]. Let \( \mathbf{lCor}_{\text{Sm}_{\text{Log}}} \subseteq \mathbf{lCor} \) be the full subcategory consisting of all objects in \( \mathbf{Sm}_{\text{Log}} \).

Let \( \mathbf{PSh}^{\text{tr}} \) be the category of additive presheaves of abelian groups on \( \mathbf{lCor} \) and \( \mathbf{Shv}^{\text{tr}}_{d\text{Nis}} \subset \mathbf{PSh}^{\text{tr}} \) be the full subcategory consisting of those \( F \) whose restrictions to \( \mathbf{Sm}_{\text{Log}} \) are dividing Nisnevich sheaves (see [BPØ22, Definition 3.1.4]). It is shown in [BPØ22, Theorem 1.2.1 and Proposition 4.7.5] that \( \mathbf{Shv}^{\text{tr}}_{d\text{Nis}} \) is a Grothendieck abelian category and there is an equivalence

\[ \mathbf{Shv}^{\text{tr}}_{d\text{Nis}} \simeq \mathbf{Shv}^{\text{tr}}_{d\text{Nis}}(\mathbf{Sm}_{\text{Log}}), \tag{6.0.1} \]

where the right-hand side denotes the full subcategory of the category \( \mathbf{PSh}^{\text{tr}}(\mathbf{Sm}_{\text{Log}}) \) of additive presheaves of abelian groups on \( \mathbf{lCor}_{\text{Sm}_{\text{Log}}} \) consisting of those \( F \) whose restrictions to \( \mathbf{Sm}_{\text{Log}} \) are dividing Nisnevich sheaves.

We now construct a functor

\[ \mathcal{L}og : \mathbf{MNST}_{\text{Log}} \to \mathbf{Shv}^{\text{tr}}_{d\text{Nis}}. \tag{6.0.2} \]

For \( \mathfrak{X} = (X, M) \in \mathbf{Sm}_{\text{Log}} \), we put \( \mathfrak{X}^{\text{MP}} = (X, \partial \mathfrak{X}) \), where \( \partial \mathfrak{X} \subset X \) is the closed subscheme consisting of the points where the log structure \( M \) is not trivial. By [BPØ22, Lemma A.5.10], \( \partial \mathfrak{X} \) with reduced structure is a normal crossing divisor on \( X \), so that we can view \( \mathfrak{X}^{\text{MP}} \) as an object of \( \mathbf{MCor}_{\text{is}} \). For \( F \in \mathbf{MPST}_{\text{Log}} \) and \( \mathfrak{X} \in \mathbf{Sm}_{\text{Log}} \), we put

\[ F^\text{log}(\mathfrak{X}) = F(\mathfrak{X}^{\text{MP}}). \tag{6.0.3} \]

Take \( \mathfrak{Y} \in \mathbf{Sm}_{\text{Log}} \) and \( \alpha \in \mathbf{lCor}(\mathfrak{Y}, \mathfrak{X}) \). By [BPØ22, Definition 2.1.1 and Remark 2.1.2(iii)], we have

\[ \alpha \in \mathbf{MCor}^{\text{fin}}((Y, n \cdot \partial \mathfrak{Y}), (X, \partial \mathfrak{X})) \quad \text{for some } n > 0, \]

where \( n \cdot \partial \mathfrak{Y} \to Y \) is the \( n \)th thickening of \( \partial \mathfrak{Y} \to Y \). By the assumption \( F \in \mathbf{MPST}_{\text{Log}} \), the induced map

\[ F^\text{log}(\mathfrak{X}) = F(\mathfrak{X}^{\text{MP}}) \xrightarrow{\alpha^*} F(Y, n \cdot \partial \mathfrak{Y}) \]

factors through \( F^\text{log}(\mathfrak{Y}) = F(Y, \partial \mathfrak{Y}) \subset F(Y, n \cdot \partial \mathfrak{Y}) \) and we get a map

\[ \alpha^* \text{log} : F^\text{log}(\mathfrak{X}) \to F^\text{log}(\mathfrak{Y}). \]
Reciprocity sheaves and logarithmic motives

Moreover, for a map $\gamma: F \to G$ in $\text{MPST}_{\log}$, the diagram

$$
\begin{array}{ccc}
F^{\log} (X) & \xrightarrow{\gamma} & G^{\log} (X) \\
\downarrow \alpha^{\log} & & \downarrow \alpha^{\log} \\
F^{\log} (\mathcal{Y}) & \xrightarrow{\gamma} & G^{\log} (\mathcal{Y})
\end{array}
$$

is obviously commutative. Hence, the assignment $X \mapsto F^{\log} (X)$ gives an object $F^{\log}$ of $\text{PSh}^{\text{ltr}} (\text{Sm} \text{Sm})$ and we get a functor

$$
\text{Log} : \text{MPST}_{\log} \to \text{PSh}^{\text{ltr}} (\text{Sm} \text{Sm}),
$$

(6.0.4)

By the definitions of sheaves ([KMSY21a, Definition 1], [BPØ22, Definition 3.1.4] and [KMSY21a, Proposition 1.9.2]), this induces a functor

$$
\text{MNST}_{\log} \to \text{Shv}^{\text{ltr}}_{\text{dNis}} (\text{Sm} \text{Sm})
$$

which induces the desired functor (6.0.2) using (6.0.1). By the construction, for $F \in \text{MNST}_{\log}$ and $\mathfrak{X} \in \text{Sm} \text{Sm}$ with $\mathfrak{X} = \mathfrak{X}^\text{MP} \in \text{MCor}_{\text{lst}}$, we have

$$
H^i_{\text{Nis}} (X, F_X) = H^i_{\text{sNis}} (\mathfrak{X}, F^{\log}) (F^{\log} = \text{Log} (F)),
$$

(6.0.5)

where the right-hand side is the cohomology for the strict Nisnevich topology (see [BPØ22, Definition 4.3.1]).

**Theorem 6.1.** For $F \in \text{CI}^{\text{r,sp}}_{\text{Nis}}$, $F^{\log} = \text{Log} (F) \in \text{Shv}^{\text{ltr}}_{\text{dNis}}$ is strictly $\Box$-invariant in the sense of [BPØ22, Definition 5.2.2]. For $\mathfrak{X} \in \text{Sm} \text{Sm}$ with $\mathfrak{X} = \mathfrak{X}^\text{MP} \in \text{MCor}_{\text{lst}}$, we have a natural isomorphism

$$
H^i_{\text{Nis}} (X, F_X) \simeq \text{Hom}_{\text{logDM}_{\text{eff}}} (M (\mathfrak{X}), F^{\log} [i]),
$$

(6.1.1)

where $\text{logDM}_{\text{eff}}$ is the triangulated category of logarithmic motives defined in [BPØ22, Definition 5.2.1].

**Proof.** Let $\mathfrak{X}^\text{sm}_{\text{div}}$ be the category of log modifications $\mathcal{Y} \to \mathfrak{X}$ such that $\mathcal{Y} \in \text{Sm} \text{Sm}$ (see [BPØ22, Definition A.11.12]) and $\mathfrak{X}^\text{sm}_{\text{div}} \subset \mathfrak{X}^\text{sm}_{\text{div}}$ be the full subcategory given by those maps $\mathcal{Y} \to \mathfrak{X}$ that are isomorphic to compositions of log modifications along smooth centers (see [BPØ22, Definitions 4.4.4 and A.14.10]). We have isomorphisms

$$
H^i_{\text{Nis}} (X, F_X) \overset{(6.0.5)}{=} H^i_{\text{sNis}} (\mathfrak{X}, F^{\log}) \overset{(*)}{=} \lim_{\mathcal{Y} \in \mathfrak{X}^\text{sm}_{\text{div}}} H^i_{\text{sNis}} (\mathcal{Y}, F^{\log})
$$

$$
\overset{(**)}{=} \lim_{\mathcal{Y} \in \mathfrak{X}^\text{sm}_{\text{div}}} H^i_{\text{sNis}} (\mathcal{Y}, F^{\log}) \overset{(***)}{=} H^i_{\text{dNis}} (\mathfrak{X}, F^{\log}),
$$

where $(*2)$ follows from [BPØ22, Corollary 4.4.5] and $(**3)$ from [BPØ22, Theorem 5.1.8], and $(*)$ is a consequence of Theorem 5.2 in view of (6.0.5) and the fact that a log modification of $\mathfrak{X} = (X, \mathcal{M}) \in \text{Sm} \text{Sm}$ along smooth center is induced Zariski locally by a blowup of $X$ in an intersection of irreducible components of $\partial X$ so that it corresponds to a morphism in $\Lambda^\text{fin}_{\text{ls}}$ from Definition 5.1.

Hence, the strict $\Box$-invariance of $F^{\log}$ follows from [Sai20, Theorem 0.6]. Finally, (6.1.1) follows from [BPØ22, Proposition 5.2.3].

377
We now consider the composite functor

\[ \log': \text{RSC}_{\text{Nis}} \xrightarrow{\mathcal{L}og^\text{CI}_{\text{sp}}} \text{CI}_{\text{tr}d\text{Nis}} \xrightarrow{\log} \text{CI}_{d\text{Nis}}, \]

where \( \text{CI}_{d\text{Nis}} \subset \text{Shv}_{d\text{Nis}} \) is the full subcategory consisting of strictly \( \square \)-invariant objects. By [BM12, Theorem 5.7], \( \text{CI}_{d\text{Nis}} \) is a Grothendieck abelian category.

**Lemma 6.2.** \( \log \) and \( \log' \) have the same essential image.

**Proof.** This follows directly from the construction and Corollary 2.6(3). \( \square \)

In what follows, we let \( \log' : \text{RSC}_{\text{Nis}} \rightarrow \text{CI}_{d\text{Nis}} \) denote \( \log' \) defined as above. By (6.0.3), we have

\[ F^\log(X, \text{triv}) = F(X) \quad \text{for } F \in \text{RSC}_{\text{Nis}}, \ X \in \text{Sm}, \]

where \((X, \text{triv})\) denotes the log scheme with the trivial log structure.

**Theorem 6.3.** \( \log \) is exact and fully faithful.

**Proof.** First we prove the full faithfulness. Faithfulness follows from (6.2.2). Let \( F, G \in \text{RSC}_{\text{Nis}} \) and \( \gamma : F^\log \rightarrow G^\log \) be a map in \( \text{Shv}_{d\text{Nis}} \). By (6.2.2) it induces maps \( \gamma_X : F(X) \rightarrow G(X) \) for all \( X \in \text{Sm} \). They are compatible with the action of \( \text{Cor} \) since by [BPØ22, Example 2.1.3(3)],

\[ \text{Cor}(Y, X) = \text{Cor}(Y, \text{triv})(X, \text{triv}) \quad \text{for } X, Y \in \text{Sm}. \]

Thus, \( \gamma_X \) for \( X \in \text{Sm} \) give a map \( \gamma_{\text{RSC}_{\text{Nis}}} : F \rightarrow G \) in \( \text{RSC}_{\text{Nis}} \). To see \( \log(\gamma_{\text{RSC}_{\text{Nis}}}) = \gamma \), it suffices by (6.0.1) to show that \( \log(\gamma_{\text{RSC}_{\text{Nis}}}) \) and \( \gamma \) induce the same map \( F^\log(X) \rightarrow G^\log(X) \) for \( X \in \text{Sm} \). If \( X \) has the trivial log structure, this follows immediately from the construction of \( \gamma_{\text{RSC}} \). The general case follows from this in view of the commutative diagram

\[
\begin{array}{ccc}
F^\log(X) & \xrightarrow{\gamma} & G^\log(X) \\
\downarrow j^* & & \downarrow j^* \\
F^\log(X \setminus \partial X, \text{triv}) & \xrightarrow{\gamma} & G^\log(X \setminus \partial X, \text{triv})
\end{array}
\]

where \( j^* \) are induced by the natural map \( (X \setminus \partial X, \text{triv}) \rightarrow X \) of log schemes and are injective by the construction and the semipurity of \( \mathcal{L}og \). This completes the proof of the full faithfulness.

Next we show the exactness of \( \log \). It suffices to show the following claim.

**Claim 6.3.1.** Given an exact sequence \( 0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0 \) in \( \text{RSC}_{\text{Nis}} \), the induced sequence

\[ 0 \rightarrow F^\log(X) \rightarrow G^\log(X) \rightarrow H^\log(X) \rightarrow 0 \]

is exact for every \( X \in \text{Sm} \) with \( X \) henselian local.

Indeed, by the definition of \( \log \), this is reduced to the exactness of

\[ 0 \rightarrow \mathcal{L}og^\text{CI}_F(X^\text{MP}) \rightarrow \mathcal{L}og^\text{CI}_G(X^\text{MP}) \rightarrow \mathcal{L}og^\text{CI}_H(X^\text{MP}) \rightarrow 0, \]

which follows from Corollary 2.6(2). This completes the proof of Theorem 6.3. \( \square \)

**Acknowledgements**

The author would like to thank Kay Rülling, F. Binda and A. Merici for many valuable discussions and comments. He is also grateful to A. Merici to whom he owes crucial ideas for § 2.
Reciprocity sheaves and logarithmic motives

The author also thanks the referee for a very careful reading and numerous suggestions improving this paper.

References


Shuji Saito sshuji@msb.biglobe.ne.jp
Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Tokyo 153-8914, Japan

https://doi.org/10.1112/50010437x22007862 Published online by Cambridge University Press

379