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# FREE $E_0$-SEMIGROUPS

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ABSTRACT. Given a strongly continuous semigroup of isometries $U$ acting on a Hilbert space $\mathcal{H}$, we construct an $E_0$-semigroup $\alpha^U$, the *free $E_0$-semigroup over $U$*, acting on the algebra of all bounded linear operators on full Fock space over $\mathcal{H}$. We show how the semigroup $\alpha^{U\oplus V}$ can be regarded as the free product of $\alpha^U$ and $\alpha^V$. In the case where $U$ is pure of multiplicity $n$, the semigroup $\alpha^U$, called the *free flow of rank $n$*, is shown to be completely spatial with Arveson index $+\infty$. We conclude that each of the free flows is cocycle conjugate to the CAR/CCR flow of rank $+\infty$.

0. **Introduction.** An $E_0$-semigroup is a continuous semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of normal, unital $*$-endomorphisms of a von Neumann algebra $\mathcal{M}$. More precisely, each $\alpha_t$ is a normal, unital $*$-endomorphism of $\mathcal{M}$, $\alpha_{s+t} = \alpha_s \circ \alpha_t$ whenever $s, t \geq 0$, $\alpha_0$ is the identity endomorphism, and for each $x \in \mathcal{M}$ and each $\sigma$-weakly continuous linear functional $\rho$ on $\mathcal{M}$, the map $t \longmapsto \rho\big(\alpha_t(x)\big)$ is continuous. Powers initiated the study of these semigroups in [9], and to date, even in the case where $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a separable, complex Hilbert space $\mathcal{H}$, there are very few concrete examples.

In this paper, we will introduce a family of $E_0$-semigroups called *free $E_0$-semigroups*, which are the free objects in the category of $E_0$-semigroups. We will show that a certain subfamily of the free $E_0$-semigroups, the *free flows*, consists entirely of $E_0$-semigroups which are completely spatial and have numerical index $+\infty$. Using Arveson's classification of completely spatial $E_0$-semigroups, we are then able to conclude that each of the free flows is cocycle conjugate to the CAR flow of rank $+\infty$.

The construction of $E_0$-semigroups which we present is modeled after similar constructions by Powers ([9]) and Arveson ([1]). In each of these constructions, one begins with a strongly continuous semigroup of isometries $U = \{U_t : t \geq 0\}$ acting on $\mathcal{H}$. By making use of an appropriate set of commutation relations, one can effectively "quantize" $U$ to produce an $E_0$-semigroup $\alpha^U$ acting on $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is a Fock space over $\mathcal{H}$. The relations used by Powers are the canonical anticommutation relations, or CARs, and the Hilbert space $\mathcal{K}$ which underlies the resulting CAR $E_0$-semigroup is the antisymmetric Fock space over $\mathcal{H}$. Arveson makes use of the canonical commutation relations (CCRs), and the underlying Hilbert space $\mathcal{K}$ in this case is the symmetric Fock space over $\mathcal{H}$. Our construction of free $E_0$-semigroups is the full Fock space analogue of these constructions, and makes use of the Cuntz relations.

In each of the above constructions the semigroup $\alpha^U$ is *spatial*; that is, there is a strongly continuous semigroup of isometries $V = \{V_t : t \geq 0\}$ acting on $\mathcal{K}$ which

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intertwines $\alpha$ with the identity in the sense that $\alpha_t^U(A)V_t = V_tA$ for each $A \in \mathcal{B}(\mathcal{K})$ and each $t \geq 0$. Indeed, one such $V$ is simply the second quantization of the original semigroup $U$.

Much attention has been paid to the case where $U$ is a pure semigroup of isometries. If $U$ is pure of multiplicity $n$ ($1 \leq n \leq \infty$), the $E_0$-semigroup which results from any of the constructions referred to above is called a *flow of rank n*. In [9], Powers showed that the CCR flow of rank $n$ is conjugate to the CAR flow of rank $n$, and both Powers and Arveson have defined numerical indices for spatial $E_0$-semigroups which recover the rank of these flows. In computing the index of the CCR flows, Arveson ([3]) explicitly described the set of all intertwining semigroups for the CCR flow of rank $n$, and made precise the notion that this particular $E_0$-semigroup has an abundance of intertwining semigroups by showing that it satisfies a certain technical condition which he called *complete spatialness*. He concluded by showing that his numerical index was a complete cocycle conjugacy invariant for completely spatial $E_0$-semigroups, and consequently that every completely spatial $E_0$-semigroup is cocycle conjugate to a CAR/CCR flow.

In Section 3.1 of this paper we will show that the numerical index of the free flow of rank $n$ is $+\infty$ whenever $n \geq 1$. This is a radical departure from the CAR/CCR case, where the corresponding flow of rank $n$ has index $n$. We then verify that each of the free flows is completely spatial, and hence cocycle conjugate to the CAR/CCR flow of rank $+\infty$. We also take a moment in Section 1.3 to justify the use of the word *free* by showing how free $E_0$-semigroups are related to free products.

We close the introduction with a few remarks on notation, most of which is standard. We use the symbols $\mathbf{C}$, $\mathbf{R}$, and $\mathbf{Z}$ to denote the complex numbers, real numbers, and integers, respectively. The symbol $\mathbf{N}$ denotes the natural numbers $\{1, 2, 3, \ldots, \}$, and we define $\mathbf{N}_0$ to be the set $\{0, 1, 2, \ldots, \}$. All Hilbert spaces are assumed to be over the complex numbers, and are also assumed to be separable. The inner product we use is linear in the first slot and conjugate linear in the second slot. The identity operator on $\mathcal{H}$ will be denoted $I_{\mathcal{H}}$, and abbreviated $I$ when the context is clear.

## 1. Free $E_o$-semigroups.

1.1. *Preliminaries.* Let $\mathcal{H}$ be a complex Hilbert space. For each $n \geq 1$, let $\mathcal{H}^{\otimes n}$ denote the $n$-fold full tensor product of $\mathcal{H}$, and let $\mathcal{H}^{\otimes 0} = \mathbf{C}$. The *full Fock space over $\mathcal{H}$* is the Hilbert space $\mathcal{F}(\mathcal{H})$ defined by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}.$$

The distinguished vector $1 \oplus 0 \oplus 0 \oplus 0 \oplus \cdots \in \mathcal{F}(\mathcal{H})$ is called the *vacuum vector*, and shall be denoted by $\Omega_{\mathcal{H}}$, or just $\Omega$ if the context is clear. We will denote the projection of $\mathcal{F}(\mathcal{H})$ onto the subspace $\mathcal{H}^{\otimes n}$ by $P_n$.

To each $f \in \mathcal{H}$ we associate a *left creation operator* $l(f)$ and a *right creation operator* $r(f)$ acting on $\mathcal{F}(\mathcal{H})$, defined by

$$l(f)\Omega = r(f)\Omega = f$$

$$l(f)(f_1 \otimes \cdots \otimes f_n) = f \otimes f_1 \otimes \cdots \otimes f_n$$
$$r(f)(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_n \otimes f, \quad n \geq 1, f_1, \ldots, f_n \in \mathcal{H}.$$

On easily checks that their adjoints, called *annihilation operators*, satisfy

$$l(f)^*\Omega = r(f)^*\Omega = 0$$
$$l(f)^*(f_1 \otimes \cdots \otimes f_n) = \langle f_1, f\rangle f_2 \otimes \cdots \otimes f_n$$
$$r(f)^*(f_1 \otimes \cdots \otimes f_n) = \langle f_n, f\rangle f_1 \otimes \cdots \otimes f_{n-1} \quad n \geq 1, f_1, \ldots, f_n \in \mathcal{H}.$$

The maps $l, r: \mathcal{H} \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ are linear, isometric, satisfy the Cuntz relations

$$l(g)^* l(f) = \langle f, g\rangle I$$
$$r(g)^* r(f) = \langle f, g\rangle I$$

and the commutation relations

$$l(g)r(f) - r(f)l(g) = 0$$
$$l(g)^* r(f) - r(f)l(g)^* = \langle f, g\rangle P_0, \quad f, g \in \mathcal{H}. \tag{1.1}$$

It is easy to extend the domain of $l$ and $r$ to all vectors in $\mathcal{F}(\mathcal{H})$ which have bounded support; that is, vectors $f \in \mathcal{F}(\mathcal{H})$ for which $P_n f = 0$ for all but finitely may $n$. Simply define

$$l(\Omega) = r(\Omega) = I$$
$$l(f_1 \otimes \cdots \otimes f_n) = l(f_1) \cdots l(f_n)$$
$$r(f_1 \otimes \cdots \otimes f_n) = r(f_n) \cdots r(f_1), \quad n \geq 1, f_1, \ldots, f_n \in \mathcal{H},$$

and extend linearly. We will denote the set of all vectors in $\mathcal{F}(\mathcal{H})$ which have bounded support by $\mathcal{F}_b(\mathcal{H})$.

When working with the symmetric Fock space over $\mathcal{H}$, denoted $e^{\mathcal{H}}$, one has the luxury of a canonical isomorphism $e^{\mathcal{H}_1} \otimes e^{\mathcal{H}_2} \cong e^{\mathcal{H}_1 \oplus \mathcal{H}_2}$. The following proposition gives a full Fock space analogue of this isomorphism. We now recognize this as a special case of [5, Definition 1.5.1], which we have reproduced in Definition 1.3.2 of this paper.

PROPOSITION 1.1.1. *Suppose $\mathcal{H}$ is expressed as an internal direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$, and consider $\mathcal{F}(\mathcal{H}_1)$ as a subspace of $\mathcal{F}(\mathcal{H})$ in the natural way. Define another subspace $E$ of $\mathcal{F}(\mathcal{H})$ by*

$$E = \mathbf{C} \oplus \bigoplus_{n=1}^{\infty} (\mathcal{H}_2 \otimes \mathcal{H}^{\otimes(n-1)}).$$

*Then there is a unique unitary operator*

$$W: \mathcal{F}(\mathcal{H}_1) \otimes E \longrightarrow \mathcal{F}(\mathcal{H})$$

*which satisfies*

$$W(h \otimes f) = r(f)h, \quad h \in \mathcal{F}(\mathcal{H}_1), f \in E \cap \mathcal{F}_b(\mathcal{H}). \tag{1.2}$$

PROOF. Suppose $h = h_1 \otimes \cdots \otimes h_m \in \mathcal{F}(\mathcal{H}_1)$ for some $m \geq 0$, $h_1, \ldots, h_m \in \mathcal{H}_1$, with the understanding that $h = \Omega$ if $m = 0$. Similarly, let $f = f_1 \otimes \cdots \otimes f_a \in E$ for some $a \geq 0$, where $f_1 \in \mathcal{H}_2$ if $a \geq 1$ and $f_i \in \mathcal{H}$ for all $i \geq 2$. Suppose $h' = h_1' \otimes \cdots \otimes h_n' \in \mathcal{F}(\mathcal{H}_1)$ and $f' = f_1' \otimes \cdots \otimes f_b' \in E$ are another such pair of vectors. We will show that

$$\langle r(f)h, r(f')h' \rangle = \langle h \otimes f, h' \otimes f' \rangle, \tag{1.3}$$

from which we are assured the existence of an isometry $W: \mathcal{F}(\mathcal{H}_1) \otimes E \longrightarrow \mathcal{F}(\mathcal{H})$ satisfying (1.2).

Equation (1.3) is obvious if $m = n$ and $a = b$. If either $m \neq n$ or $a \neq b$, we have

$$\langle h \otimes f, h' \otimes f' \rangle = \langle h, h' \rangle \langle f, f' \rangle = 0,$$

so we must establish that $\langle r(f)h, r(f')h' \rangle = 0$ in this case. This equation clearly holds if $m + a \neq n + b$, so we may assume that $m + a = n + b$, and without loss of generality $m < n$. Then

$$\begin{aligned} \langle r(f)h, r(f')h' \rangle &= \langle h_1 \otimes \cdots \otimes h_m \otimes f_1 \otimes \cdots \otimes f_a, h_1' \otimes \cdots \otimes h_n' \otimes f_1' \otimes \cdots \otimes f_b' \rangle \\ &= \langle h_1, h_1' \rangle \cdots \langle h_m, h_m' \rangle \langle f_1, h_{m+1}' \rangle \cdots \langle f_a, f_b' \rangle \\ &= 0, \end{aligned}$$

since $\langle f_1, h_{m+1}' \rangle = 0$.

It remains to show that $W$ is surjective. For this, we define a subspace $\mathcal{H}_x$ of $\mathcal{F}(\mathcal{H})$ for each $x \in \sqcup_{n=0}^{\infty} \{1, 2\}^n$ by

$$\mathcal{H}_x = \begin{cases} \mathbf{C} & \text{if } x \in \{1,2\}^0 \\ \mathcal{H}_{x_1} \otimes \cdots \otimes \mathcal{H}_{x_n} & \text{if } x = (x_1, \ldots, x_n) \in \{1,2\}^n \text{ for some } n \geq 1. \end{cases}$$

A moment's thought shows that $\mathcal{F}(\mathcal{H}) = \bigoplus_x \mathcal{H}_x$, so it suffices to show that the range of $W$ contains each of the subspaces $\mathcal{H}_x$. The case $x \in \{1, 2\}^0$ is trivial, since $W(\Omega \otimes \Omega) = \Omega$. Suppose $x = (x_1, \ldots, x_n)$ for some $n \geq 1$. If $x_i = 1$ for all $i$, then

$$\mathcal{H}_x = \mathcal{H}_1^{\otimes n} \subset \mathcal{F}(\mathcal{H}_1) = W\big(\mathcal{F}(\mathcal{H}_1) \otimes \Omega\big).$$

Otherwise, let $j = \min\{i : x_{i+1} = 2\}$. Then

$$\begin{aligned} \mathcal{H}_x &\subseteq \mathcal{H}_1^{\otimes j} \otimes \mathcal{H}_2 \otimes \mathcal{H}_{x_{j+2}} \otimes \cdots \otimes \mathcal{H}_{x_n} \\ &\subseteq \overline{\operatorname{span}}\, r(\mathcal{H}_2 \otimes \mathcal{H}^{n-j-1}) \mathcal{H}_1^{\otimes j} \\ &\subseteq \operatorname{ran} W. \end{aligned}$$

■

As a final preliminary, we observe that to each linear contraction $T: \mathcal{H} \longrightarrow \mathcal{K}$ between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we can associate a contraction $\tilde{T}: \mathcal{F}(\mathcal{H}) \longrightarrow \mathcal{F}(\mathcal{K})$, the *second quantization* of $T$, defined by

$$\begin{aligned} \tilde{T}\Omega_{\mathcal{H}} &= \Omega_{\mathcal{K}} \\ \tilde{T}(f_1 \otimes \cdots \otimes f_n) &= Tf_1 \otimes \cdots \otimes Tf_n, \quad n \geq 1, f_1, \ldots, f_n \in \mathcal{H}. \end{aligned}$$

The map $T \longmapsto \tilde{T}$ is unital, strongly continuous, and preserves both involution and composition of operators. Moreover, these quantized operators interact nicely with creation and annihilation operators, as demonstrated in the following lemma.

LEMMA 1.1.2. *Suppose $T\colon \mathcal{H} \longrightarrow \mathcal{K}$ is a linear contraction. Then for each $f \in \mathcal{F}_b(\mathcal{H})$ we have*

$$\tilde{T}l(f) = l(\tilde{T}f)\tilde{T}$$
$$\tilde{T}r(f) = r(\tilde{T}f)\tilde{T}.$$

*If $T$ is an isometry, we further have*

$$\tilde{T}l(f)^* = l(\tilde{T}f)^*\tilde{T}$$
$$\tilde{T}r(f)^* = r(\tilde{T}f)^*\tilde{T}.$$

PROOF. The first statement is routine and left to the reader. For the second statement, suppose $T$ is an isometry. Relying heavily on the fact that $(\tilde{T})^* = (T^*)\tilde{}$, we have

$$\tilde{T}l(f)^* = (l(f)\tilde{T}^*)^* = (l(\tilde{T}^*\tilde{T}f)\tilde{T}^*)^* = \big(\tilde{T}^*l(\tilde{T}f)\big)^* = l(\tilde{T}f)^*\tilde{T}. \qquad \blacksquare$$

The following corollary is a trivial restatement of Proposition 1.1.1, using the language of the previous lemma.

COROLLARY 1.1.3. *Suppose $L\colon \mathcal{H} \longrightarrow \mathcal{K}$ is an isometry. Let $E_L \subseteq \mathcal{F}(\mathcal{K})$ be the subspace $E$ of Proposition 1.1.1 with respect to the decomposition $\mathcal{K} = L\mathcal{H} \oplus (L\mathcal{H})^\perp$; that is, let*

$$E_L = \mathbf{C} \oplus \bigoplus_{n=1}^{\infty} (L\mathcal{H})^\perp \otimes \mathcal{K}^{\otimes(n-1)}.$$

*There is a unique unitary operator*

$$W_L\colon \mathcal{F}(\mathcal{H}) \otimes E_L \longrightarrow \mathcal{F}(\mathcal{K})$$

*such that*

$$W_L(h \otimes f) = r(f)\tilde{L}h, \quad h \in \mathcal{F}(\mathcal{H}),\ f \in E_L \cap \mathcal{F}_b(\mathcal{K}).$$

1.2. *Construction.*

THEOREM 1.2.1. *Suppose $U = \{U_t : t \geq 0\}$ is a strongly continuous semigroup of isometries on $\mathcal{H}$. There is a unique $E_0$-semigroup $\alpha^U = \{\alpha_t^U : t \geq 0\}$ on $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$, the free $E_0$-semigroup over $U$, which satisfies*

$$\alpha_t^U\big(l(f)\big) = l(U_t f), \quad f \in \mathcal{H},\ t \geq 0. \tag{1.4}$$

PROOF. Uniqueness of $\alpha^U$ follows from the fact that $\{l(f) : f \in \mathcal{H}\}$ generates $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ as a von Neumann algebra. (See [6].) To prove the existence of $\alpha^U$, we begin by observing that the theorem is trivial if each of the isometries $U_t$ is actually unitary. In this case the second quantization $\tilde{U} = \{\tilde{U}_t : t \geq 0\}$ of $U$ is a strongly continuous semigroup of unitary operators on $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$, and we can define a trivial $E_0$-semigroup $\alpha^U$ on $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ by

$$\alpha_t^U(A) = \tilde{U}_t A \tilde{U}_t^*, \quad A \in \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big),\ t \geq 0.$$

By Lemma 1.1.2 this is clearly the desired semigroup.

In the case where $U_t$ is a proper isometry for some (and hence all) $t > 0$, let $V$ be a unitary extension of $U$; that is, let $V = \{V_t : t \in \mathbf{R}\}$ be a strongly continuous one parameter unitary group acting on a Hilbert space $\mathcal{K}$ which extends $U$ in the sense that there is an isometry $L: \mathcal{H} \longrightarrow \mathcal{K}$ which intertwines $U$ and $V$:

$$V_t L = L U_t, \quad t \geq 0.$$

Let $\beta = \{\beta_t : t \in \mathbf{R}\}$ be the one parameter automorphism group acting on $\mathcal{B}\big(\mathcal{F}(\mathcal{K})\big)$ defined by

$$\beta_t(A) = \tilde{V}_t A \tilde{V}_t^*, \quad A \in \mathcal{B}\big(\mathcal{F}(\mathcal{K})\big),\ t \in \mathbf{R}.$$

By Lemma 1.1.2 we have

$$\beta_t\big(l(g)\big) = l(V_t g), \quad g \in \mathcal{K},\ t \in \mathbf{R},$$

so in particular

$$\beta_t\big(l(Lf)\big) = l(V_t L f) = l(L U_t f), \quad f \in \mathcal{H},\ t \geq 0.$$

Thus the von Neumann algebra $\mathcal{A}$ generated by $\{l(Lf) : f \in \mathcal{H}\}$ is invariant under the semigroup $\{\beta_t : t \geq 0\}$. We will show that the equation

$$\theta\big(l(f)\big) = l(Lf), \quad f \in \mathcal{H}$$

extends to a $*$-isomorphism of $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ onto $\mathcal{A}$. The $E_0$-semigroup $\alpha^U$ is then defined by $\alpha_t^U = \theta^{-1} \circ \beta_t \circ \theta$, $t \geq 0$.

Let $W_L: \mathcal{F}(\mathcal{H}) \otimes E_L \longrightarrow \mathcal{F}(\mathcal{K})$ be the unitary operator of Corollary 1.1.3. We claim that

$$\theta(T) = W_L(T \otimes I) W_L^*, \quad T \in \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$$

defines the desired $*$-isomorphism of $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ onto $\mathcal{A}$. For this, simply note that for $f \in E_L \cap \mathcal{F}_b(\mathcal{K})$, $g, h \in \mathcal{F}(\mathcal{H})$, we have

$$\begin{aligned} W_L\big(l(g) \otimes I\big)(h \otimes f) &= W_L\big(l(g)h \otimes f\big) \\ &= r(f)\tilde{L} l(g) h \\ &= r(f) l(Lg) \tilde{L} h \quad \text{(Lemma 1.1.2)} \\ &= l(Lg) r(f) \tilde{L} h \\ &= l(Lg) W_L(h \otimes f). \end{aligned}$$

Again this suffices, since $\{l(g) : g \in \mathcal{H}\}$ generates $\mathcal{B}(\mathcal{F}(\mathcal{H}))$ as a von Neumann algebra. (See [6].) ∎

COROLLARY 1.2.2. *Suppose $U = \{U_t : t \geq 0\}$ is a strongly continuous semigroup of isometries on $\mathcal{H}$. Then the free $E_0$-semigroup $\alpha^U = \{\alpha_t^U : t \geq 0\}$ satisfies*

$$\alpha_t^U(A) = W_{U_t}(A \otimes I)W_{U_t}^*, \quad A \in \mathcal{B}(\mathcal{F}(\mathcal{H})),\ t \geq 0, \tag{1.5}$$

*where $W_{U_t}\colon \mathcal{F}(\mathcal{H}) \otimes E_{U_t} \longrightarrow \mathcal{F}(\mathcal{H})$ is the unitary operator of Corollary 1.1.3.*

PROOF. Simply verify that equation (1.5) holds whenever $A = l(g)$ for some $g \in \mathcal{H}$. The calculation is identical to the one just given at the end of the proof of Theorem 1.2.1. ∎

1.3. *Free products.* Given two strongly continuous semigroups of isometries $U$ and $V$ acting on Hilbert spaces $\mathcal{H}_U$ and $\mathcal{H}_V$, respectively, one can define a semigroup $U \oplus V$ acting on $\mathcal{H}_U \oplus \mathcal{H}_V$ in the obvious way: $(U \oplus V)_t(f \oplus g) = U_t f \oplus V_t g$. Similarly, given two $E_0$-semigroups $\alpha$ and $\beta$ acting on $\mathcal{B}(\mathcal{H}_\alpha)$ and $\mathcal{B}(\mathcal{H}_\beta)$, respectively, one can show there is a unique $E_0$-semigroup $\alpha \otimes \beta$ acting on $\mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ which satisfies $(\alpha \otimes \beta)_t(A \otimes B) = \alpha_t(A) \otimes \beta_t(B)$. Arveson has shown that the construction of the CCR $E_0$-semigroups carries the direct sum operation for semigroups of isometries into the tensor product operation for $E_0$-semigroups ([1]). More precisely, he has shown that the map $\gamma\colon U \longmapsto \gamma^U$ defined by the construction of the CCR $E_0$-semigroups is a functor from the (appropriately defined) category of strongly continuous semigroups of isometries to the (appropriately defined) category of $E_0$-semigroups, and that under this functor, the $E_0$-semigroup $\gamma^{U \oplus V}$ is naturally isomorphic to $\gamma^U \otimes \gamma^V$.

We would like to carry out this program for the construction $U \longmapsto \alpha^U$ of the free $E_0$-semigroups, showing that $\alpha^{U \oplus V}$ is naturally isomorphic to the free product of $\alpha^U$ and $\alpha^V$. Unfortunately, the theory of free products of $E_0$-semigroups has not yet been developed. Nevertheless, in a very real sense the semigroup $\alpha^{U \oplus V}$ *is* the free product of $\alpha^U$ and $\alpha^V$, as we shall describe in this section.

To begin with, we will say a few words about the free product of Hilbert spaces and the reduced free product of von Neumann algebras. The definitions that follow are from [5], where one can find a more complete discussion of this material.

DEFINITION 1.3.1. Let $(\mathcal{H}_i, \xi_i)_{i \in I}$ be a family of Hilbert spaces $\mathcal{H}_i$ with distinguished vectors $\xi_i \in \mathcal{H}_i$. The *Hilbert space free product* $*_{i \in I}(\mathcal{H}_i, \xi_i)$ is $(\mathcal{H}, \xi)$, where $\mathcal{H}$ is the Hilbert space

$$\mathcal{H} = \mathbf{C}\xi \oplus \bigoplus_{n \geq 1} \Big( \bigoplus_{(i_1 \neq i_2 \neq \cdots \neq i_n)} \overset{\circ}{\mathcal{H}}_{i_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{i_n} \Big).$$

Here $\overset{\circ}{\mathcal{H}}_i$ denotes the orthocomplement of $\xi_i$ in $\mathcal{H}_i$.

REMARK. One can check that $*_{i \in I}\big(\mathcal{F}(\mathcal{H}_i), \Omega_i\big) = \big(\mathcal{F}(\oplus_{i \in I}\mathcal{H}_i), \Omega\big)$, where $\Omega_i$ is the vacuum vector in $\mathcal{F}(\mathcal{H}_i)$ for each $i \in I$.

DEFINITION 1.3.2. Suppose $(\mathcal{A}_i, \mathcal{H}_i, \xi_i)_{i\in I}$ is a family of von Neumann algebras $\mathcal{A}_i$ acting on Hilbert spaces $\mathcal{H}_i$ with distinguished vectors $\xi_i \in \mathcal{H}_i$. Let $(\mathcal{H}, \xi) = *_{i\in I}(\mathcal{H}_i, \xi_i)$, and for each $i \in I$ let $\mathcal{H}(i)$ denote the subspace of $\mathcal{H}$ defined by

$$\mathcal{H}(i) = \mathbf{C}\xi \oplus \bigoplus_{n\geq 1}\Big(\bigoplus_{(i\neq i_1\neq i_2\neq\cdots\neq i_n)} \mathring{\mathcal{H}}_{i_1} \otimes\cdots\otimes \mathring{\mathcal{H}}_{i_n}\Big).$$

Define unitary operators $V_i\colon \mathcal{H}_i \otimes \mathcal{H}(i) \longrightarrow \mathcal{H}$ by

$$\begin{gathered}\xi_i \otimes \xi \longmapsto \xi\\ \mathring{\mathcal{H}}_i \otimes\xi \longrightarrow \mathring{\mathcal{H}}_i\\ \xi_i \otimes (\mathring{\mathcal{H}}_{i_1} \otimes\cdots\otimes \mathring{\mathcal{H}}_{i_n}) \longrightarrow \mathring{\mathcal{H}}_{i_1} \otimes\cdots\otimes \mathring{\mathcal{H}}_{i_n}\\ \mathring{\mathcal{H}}_i \otimes(\mathring{\mathcal{H}}_{i_1} \otimes\cdots\otimes \mathring{\mathcal{H}}_{i_n}) \longrightarrow \mathring{\mathcal{H}}_i \otimes \mathring{\mathcal{H}}_{i_1} \otimes\cdots\otimes \mathring{\mathcal{H}}_{i_n}\,.\end{gathered}$$

For each $i \in I$, let $\lambda_i\colon A_i \longrightarrow \mathcal{B}(\mathcal{H})$ be the representation defined by

$$\lambda_i(a) = V_i(a \otimes I_{\mathcal{H}(i)})V_i^*, \quad a \in \mathcal{A}_i.$$

The *reduced free product von Neumann algebra* $\mathcal{A} = *_{i\in I}(\mathcal{A}_i, \mathcal{H}_i, \xi_i)$ is the von Neumann algebra

$$\mathcal{A} = \Big(\bigcup_{i\in I} \lambda_i(\mathcal{A}_i)\Big)''.$$

Now, suppose $\mathcal{U}$ is a collection of strongly continuous semigroups of isometries. For each $U \in \mathcal{U}$, let $\mathcal{H}_U$ be the Hilbert space on which $U$ acts. Let $\mathcal{H} = \oplus_{U\in\mathcal{U}}\mathcal{H}_U$, and let $\iota_U$ be the inclusion map of $\mathcal{H}_U$ into $\mathcal{H}$. Then the semigroup $W = \oplus_{U\in\mathcal{U}}U$ is the unique semigroup of isometries on $\mathcal{H}$ which satisfies

$$W_t\iota_U f = \iota_U U_t f, \quad U \in \mathcal{U}, f \in \mathcal{H}_U, t \geq 0.$$

We will show that $\alpha^W$ behaves like the "free product" of the family $(\alpha^U)_{U\in\mathcal{U}}$ in the sense that

1. $\alpha^W$ acts on the reduced free product von Neumann algebra
$$*_{U\in\mathcal{U}}\Big(\mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big), \mathcal{F}(\mathcal{H}_U), \Omega_U\Big),$$
2. $\alpha_t^W \circ \lambda_U = \lambda_U \circ \alpha^U$ for each $U \in \mathcal{U}$, where $\lambda_U$ is the representation of $\mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big)$ on $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ used in defining $*_{U\in\mathcal{U}}\Big(\mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big), \mathcal{F}(\mathcal{H}_U), \Omega_U\Big)$.

In computing $*_{U\in\mathcal{U}}\Big(\mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big), \mathcal{F}(\mathcal{H}_U), \Omega_U\Big)$, the subspace of $\mathcal{F}(\mathcal{H})$ which corresponds to $\mathcal{H}(i)$ in Definition 1.3.2 is precisely the subspace $E_{\iota_U}$ of Corollary 1.1.3, and the unitary operator which corresponds to $V_i$ is $W_{\iota_U}$. Thus the representation $\lambda_U\colon \mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big) \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ is simply

$$\lambda_U(A) = W_{\iota_U}(A \otimes I)W_{\iota_U}^*, \quad A \in \mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big),$$

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from which we obtain

$$\lambda_U\big(l(f)\big) = l(\iota_U f), \quad f \in \mathcal{H}_U. \tag{1.6}$$

Since $\{l(\iota_U f) : U \in \mathcal{U}, f \in \mathcal{H}_U\}$ generates $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ as a von Neumann algebra, we have

$$\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big) = *_{U\in\mathcal{U}}\Big(\mathcal{B}\big(\mathcal{F}(\mathcal{H}_U)\big), \mathcal{F}(\mathcal{H}_U), \Omega_U\Big).$$

The second statement follows easily from (1.6):

$$\begin{aligned}(\alpha_t^W \circ \lambda_U)\big(l(f)\big) &= \alpha_t^W\big(l(\iota_U f)\big)\\ &= l(W_t \iota_U f)\\ &= l(\iota_U U_t f)\\ &= \lambda_U\big(l(U_t f)\big)\\ &= (\lambda_U \circ \alpha_t^U)\big(l(f)\big), \quad f \in \mathcal{H}_U,\ t \geq 0.\end{aligned}$$

2. **Free product systems.** Continuous tensor product systems, or *product systems* for short, were introduced by Arveson in [3], and play an integral role in the theory of $E_0$-semigroups. For example, if $\phi: E \longrightarrow \mathcal{B}(\mathcal{H})$ is an essential representation of a product system $E$, there is a natural way of associating an $E_0$-semigroup with $\phi$. Conversely, for each $E_0$-semigroup $\alpha$ there is an associated concrete product system $\mathcal{E}_\alpha$. These two processes interact nicely: if $\alpha$ is the $E_0$-semigroup associated with the essential representation $\phi: E \longrightarrow \mathcal{B}(\mathcal{H})$, then the product systems $E$ and $\mathcal{E}_\alpha$ are canonically isomorphic. We refer the reader to [3] for details.

In this chapter we make use of a strongly continuous semigroup of isometries $U$ acting on $\mathcal{H}$ to construct a product system $E = E^U$, the *free product system over* $U$, along with an essential representation $\phi = \phi^U: E^U \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ in such a way that the $E_0$-semigroup associated with $\phi^U$ is $\alpha^U$, the free $E_0$-semigroup over $U$.

We begin by characterizing the intertwining space $\mathcal{E}_t$ of the endomorphism $\alpha_t^U$; that is, the linear subspace of $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ defined by

$$\mathcal{E}_t = \Big\{T \in \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big) : \alpha_t^U(A)T = TA, \quad A \in \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)\Big\}.$$

It is a standard result that such an intertwining space is a Hilbert space in the inner product $T^*S = \langle S, T\rangle I$. (See [11].) Observe that the corresponding Hilbert space norm on $\mathcal{E}_t$ coincides with the operator norm.

For the remainder of this paper, we will use the abbreviations $E_t$ and $W_t$ for the Hilbert space $E_{U_t}$ and the unitary operator $W_{U_t}$ of Corollary 1.1.3.

LEMMA 2.0.3. *For each $t > 0$, define $\phi_t: E_t \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ by*

$$\phi_t(f)h = W_t(h \otimes f), \quad f \in E_t,\ h \in \mathcal{F}(\mathcal{H}).$$

*Then $\phi_t$ maps $E_t$ onto the Hilbert space $\mathcal{E}_t$ of intertwining operators for $\alpha_t^U$. Moreover, $\phi_t$ defines a unitary operator from $E_t$ to $\mathcal{E}_t$; that is,*

$$\phi_t(g)^*\phi_t(f) = \langle f, g\rangle I, \quad f, g \in E_t. \tag{2.1}$$

REMARK. Observe that $\phi_t(f) = r(f)\tilde{U}_t$ when $f \in E_t \cap \mathcal{F}_b(\mathcal{H})$.

PROOF. To see that $\phi_t(E_t) \subset \mathcal{E}_t$, we make use of Corollary 1.1.3: for $A \in \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$, $f \in E_t$, $h \in \mathcal{F}(\mathcal{H})$,

$$\begin{aligned} \alpha_t^U(A)\phi_t(f)h &= \big(W_t(A \otimes I)W_t^*\big)\big(W_t(h \otimes f)\big) \\ &= W_t(A \otimes I)(h \otimes f) \\ &= W_t(Ah \otimes f) \\ &= \phi_t(f)Ah. \end{aligned}$$

Equation (2.1) is easily verified: if $f, g \in E_t$, $h_1, h_2 \in \mathcal{F}(\mathcal{H})$, then

$$\begin{aligned} \langle \phi_t(f)h_1, \phi_t(g)h_2 \rangle &= \langle W_t(h_1 \otimes f), W_t(h_2 \otimes g) \rangle \\ &= \langle h_1 \otimes f, h_2 \otimes g \rangle \\ &= \langle f, g \rangle \langle h_1, h_2 \rangle. \end{aligned}$$

Finally, we show that $\phi_t$ maps $E_t$ onto $\mathcal{E}_t$. Since $\phi_t(E_t)$ is a closed subspace of the Hilbert space $\mathcal{E}_t$, it suffices to show that its orthogonal complement in $\mathcal{E}_t$ is $\{0\}$. Suppose $R \in \mathcal{E}_t$ is orthogonal to $\phi_t(E_t)$. Then $R^*\phi_t(E_t) = \{0\}$, so $R^*$ is zero on

$$\overline{\operatorname{span}}\, \phi_t(E_t)\mathcal{F}(\mathcal{H}) = W_t\big(\mathcal{F}(\mathcal{H}) \otimes E_t\big) = \mathcal{F}(\mathcal{H}).$$

Thus $R = 0$. ∎

The following lemma provides a useful characterization of the Hilbert space $E_t$.

LEMMA 2.0.4. $E_t = \bigcap_{h \in \mathcal{H}} \ker l(U_t h)^*$.

PROOF. The inclusion $E_t \subseteq \bigcap_{h \in \mathcal{H}} \ker l(U_t h)^*$ is obvious, since

$$E_t = \mathbf{C} \oplus \bigoplus_{n=1}^{\infty} (U_t \mathcal{H})^{\perp} \otimes \mathcal{H}^{\otimes (n-1)}. \tag{2.2}$$

For the reverse inclusion, we will establish that

$$E_t^{\perp} \subseteq \Big(\bigcap_{h \in \mathcal{H}} \ker l(U_t h)^*\Big)^{\perp}.$$

For this, suppose $n \geq 1$ and $h_1, h_2, \ldots, h_n \in \mathcal{H}$. Note that vectors of the form $U_t h_1 \otimes h_2 \otimes \cdots \otimes h_n$ span $E_t^{\perp}$. If $f \in \bigcap_{h \in \mathcal{H}} \ker l(U_t h)^*$, then

$$\begin{aligned} \langle f, U_t h_1 \otimes h_2 \otimes \cdots \otimes h_n \rangle &= \langle f, l(U_t h_1)(h_2 \otimes \cdots \otimes h_n) \rangle \\ &= \langle l(U_t h_1)^* f, h_2 \otimes \cdots \otimes h_n \rangle \\ &= 0. \end{aligned}$$

∎

The total space of the free product system over $U$ is defined by

$$E = \{(t, f) \in (0, \infty) \times \mathcal{F}(\mathcal{H}) : f \in E_t\},$$

and the projection $p: E \longrightarrow (0,\infty)$ is given simply by $p(t,f) = t$, with $p^{-1}(t)$ inheriting its Hilbert space structure naturally from $E_t$. If we endow $E$ with the relative norm topology, then $E$ is a closed subset of $(0,\infty) \times \mathcal{F}(\mathcal{H})$, and is thus a standard Borel space. To see this, suppose $\{(t_n,f_n)\}$ is a sequence in $E$ which converges to some point $(t,f)$ in $(0,\infty) \times \mathcal{F}(\mathcal{H})$. We need to show that $f \in E_t$. For this, it suffices by Lemma 2.0.4 to prove that $l(U_t h)^* f = 0$ for any $h \in \mathcal{H}$. But this is simple, since

$$l(U_t h)^* f = \lim_{n\longrightarrow\infty} l(U_{t_n} h)^* f_n = 0,$$

the last equality holding because $f_n \in E_{t_n}$ for each $n$.

To complete the definition of $E$, we will define a Borel isomorphism $\psi: E \longrightarrow \mathcal{E}$, where $\mathcal{E}$ is the concrete product system associated with the free $E_0$-semigroup over $U$. This isomorphism will be defined so that it restricts to a unitary operator on each fiber. Not only will this assure the measurability of the Hilbert space operations in $E$, but by pulling back the multiplication in $\mathcal{E}$ we can define a multiplication in $E$ which makes $E$ a product system.

Since the total space of $\mathcal{E}$ is given by

$$\mathcal{E} = \Big\{(t,T) \in (0,\infty) \times \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big) : T \in \mathcal{E}_t\Big\},$$

we can simply use the maps $\phi_t: E_t \longrightarrow \mathcal{E}_t$, $t > 0$, to define $\psi$:

$$\psi(t,f) = \big(t, \phi_t(f)\big), \quad t > 0,\, f \in E_t.$$

We claim that $\psi$ is a Borel isomorphism. Since both $E$ and $\mathcal{E}$ are standard Borel spaces and $\psi$ is a bijection, it suffices to show that $\psi$ is measurable. Recall that $\mathcal{E}$ inherits its Borel structure from $(0,\infty) \times \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ by first endowing $\mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ with the Borel structure generated by the strong operator topology, and subsequently endowing $(0,\infty) \times \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ with the product Borel structure. Consequently, it suffices to show that $\psi$ is continuous when we consider $E$ and $\mathcal{E}$ as topological subspaces of $(0,\infty) \times \mathcal{F}(\mathcal{H})$ and $(0,\infty) \times \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$, respectively. Proving this amounts to showing that $(t,f) \in E \longmapsto \phi_t(f)$ is strongly continuous. As a first step in this direction, we establish the following claim.

CLAIM 2.0.5. *If $0 < s \leq t$, $f \in E_s$, and $h \in \mathcal{F}(\mathcal{H})$, then*

$$\|\phi_s(f)h - \phi_t(f)h\| \leq \|f\|\,\|\tilde{U}_s h - \tilde{U}_t h\|.$$

PROOF. The claim is trivial if $h = 0$, so assume $h \neq 0$. Suppose $\epsilon > 0$. By letting $f' = \sum_{n=0}^{N} P_n f$ for sufficiently large $N$, we can choose $f' \in E_s \cap \mathcal{F}_b(\mathcal{H})$ such that $\|f - f'\| < \frac{\epsilon}{2\|h\|}$ and $\|f'\| \leq \|f\|$. Then

$$\begin{aligned}
\|\phi_s(f)h - \phi_t(f)h\| &\leq \|\phi_s(f)h - \phi_s(f')h\| + \|\phi_s(f')h - \phi_t(f')h\| + \|\phi_t(f')h - \phi_t(f)h\| \\
&\leq 2\|f - f'\|\,\|h\| + \|r(f')\tilde{U}_s h - r(f')\tilde{U}_t h\| \\
&< \epsilon + \|r(f')\tilde{U}_s(h - \tilde{U}_{t-s} h)\| \\
&= \epsilon + \|\phi_s(f')(h - \tilde{U}_{t-s} h)\| \\
&= \epsilon + \|f'\|\,\|h - \tilde{U}_{t-s} h\| \\
&\leq \epsilon + \|f\|\,\|\tilde{U}_s h - \tilde{U}_t h\|.
\end{aligned}$$

■

Now suppose $s, t > 0, f \in E_s, g \in E_t$, and $h \in \mathcal{F}(\mathcal{H})$. If $s \le t$, then

$$\begin{aligned}\|\phi_s(f)h - \phi_t(g)h\| &\le \|\phi_s(f)h - \phi_t(f)h\| + \|\phi_t(f)h - \phi_t(g)h\| \\ &\le \|f\|\,\|\tilde{U}_s h - \tilde{U}_t h\| + \|f - g\|\,\|h\|,\end{aligned}$$

and if $t \le s$, we similarly have

$$\|\phi_s(f)h - \phi_t(g)h\| \le \|g\|\,\|\tilde{U}_s h - \tilde{U}_t h\| + \|f - g\|\,\|h\|.$$

Thus for all $s, t > 0$ we have

$$\|\phi_s(f)h - \phi_t(g)h\| \le \max\{\|f\|, \|g\|\}\|\tilde{U}_s h - \tilde{U}_t h\| + \|f - g\|\,\|h\|.$$

From this equation it is clear that if $(s,f) \longrightarrow (t,g)$ in $E$, then $\phi_s(f)h \longrightarrow \phi_t(g)h$ in $\mathcal{F}(\mathcal{H})$. Thus $(s,f) \longmapsto \phi_s(f)$ is strongly continuous.

Define multiplication in $E$ to be the pullback along $\psi$ of multiplication in $\mathcal{E}$. The corollary to the following lemma gives an explicit formulation of this multiplication.

LEMMA 2.0.6. *If $f \in E_s$, $g \in E_t$ for some $s, t > 0$, then*

1. $\phi_s(f)g \in E_{s+t}$
2. $\phi_{s+t}\big(\phi_s(f)g\big) = \phi_s(f)\phi_t(g)$.

PROOF. 1. By Lemma 2.0.4 we must show that $l(U_{s+t}h)^*\phi_s(f)g = 0$ for each $h \in \mathcal{H}$. We have

$$\begin{aligned}l(U_{s+t}h)^*\phi_s(f)g &= \alpha_s\big(l(U_t h)^*\big)\phi_s(f)g \\ &= \phi_s(f)l(U_t h)^*g \\ &= 0,\end{aligned}$$

the last equality holding since $g \in E_t$.

2. We may assume that $f$ and $g$ are vectors of bounded support. In this case

$$\begin{aligned}\phi_{s+t}\big(\phi_s(f)g\big) &= \phi_{s+t}\big(r(f)\tilde{U}_s g\big) \\ &= r\big(r(f)\tilde{U}_s g\big)\tilde{U}_{s+t} \\ &= r(f)r(\tilde{U}_s g)\tilde{U}_s\tilde{U}_t \\ &= r(f)\tilde{U}_s r(g)\tilde{U}_t \quad \text{(Lemma (1.1.2))} \\ &= \phi_s(f)\phi_t(g).\end{aligned}$$ ∎

COROLLARY 2.0.7. *The multiplication on $E$ is given by*

$$(s,f)(t,g) = \big(s + t, \phi_s(f)g\big) \tag{2.3}$$

PROOF. Multiplication in $\mathcal{E}$ is given by $(s,S)(t,T) = (s+t, ST)$. Thus

$$\begin{aligned}\psi\big((s,f)(t,g)\big) &= \psi(s,f)\psi(t,g) \quad \text{(by definition)}\\ &= \big(s, \phi_s(f)\big)\big(t, \phi_t(g)\big)\\ &= \big(s+t, \phi_s(f)\phi_t(g)\big)\\ &= \Big(s+t, \phi_{s+t}\big(\phi_s(f)g\big)\Big)\\ &= \psi\big(s+t, \phi_s(f)g\big).\end{aligned}$$

Since $\psi$ is a bijection, this implies equation (2.3). ∎

It is now a simple matter to define the representation $\phi: E \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ alluded to at the beginning of this section; simply define $\phi(t,f) = \phi_t(f)$. It is routine to check that $\phi$ is an essential representation of $E$ whose associated $E_0$-semigroup is $\alpha^U$.

3. **The free flows.** In this chapter we will study the free $E_0$-semigroup $\alpha^U$ in the case where $U$ is pure of multiplicity $n$ for some $n \in \{1, 2, \ldots, \infty\}$. We call this $E_0$-semigroup the *free flow of rank n*. The main result of this chapter is Corollary 3.2.19, which states that $\alpha^U$ is cocycle conjugate to the CCR/CAR flow of rank $+\infty$ whenever $n \geq 1$. This result follows immediately from Arveson's classification of completely spatial $E_0$-semigroups ([3]), once we establish that $\alpha^U$ is completely spatial (Theorem 3.2.2) and compute its numerical index $d_*(\alpha^U)$ (Theorem 3.1.2). Each of these theorems requires a complete understanding of the strongly continuous semigroups which intertwine $\alpha^U$ with the identity representation; we will classify these intertwining semigroups along the way.

3.1. *Numerical index.* We begin with a brief discussion on the computation of Arveson's numerical index. Suppose $\alpha$ is an $E_0$-semigroup acting on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The numerical index $d_*(\alpha)$ is defined to be the dimension of the product system $\mathcal{E}_\alpha$ associated with $\alpha$. More generally, if $\phi: E \longrightarrow \mathcal{B}(\mathcal{H})$ is an essential representation of an abstract product system $E$ whose associated $E_0$-semigroup is $\alpha$, then $d_*(\alpha) = \dim E$. This is true because $\phi$ can be used to implement an isomorphism of $E$ with $\mathcal{E}_\alpha$, and dimension is an isomorphism invariant of product systems.

One method of computing the dimension of a product system is given in the following fact, which encapsulates some of Arveson's results from [3]. Recall that a *unit* of a product system $p: E \longrightarrow (0,\infty)$ is a measurable, nonzero multiplicative cross section $\mathrm{u}: t \in (0,\infty) \longmapsto \mathrm{u}(t) \in p^{-1}(t)$.

FACT 3.1.1. Let $E$ be a product system and let $\mathcal{U}_E$ be the set of units of $E$. Suppose $\mathcal{K}$ is a Hilbert space and $(\lambda, \xi) \in \mathbf{C} \times \mathcal{K} \longmapsto \mathrm{u}_{(\lambda,\xi)} \in \mathcal{U}_E$ is a bijection which satisfies

$$\langle \mathrm{u}_{(\lambda,\xi)}(t), \mathrm{u}_{(\mu,\eta)}(t)\rangle = e^{t(\lambda+\bar{\mu}+\langle \xi,\eta\rangle)} \tag{3.1}$$

for all $(\lambda,\xi), (\mu,\eta) \in \mathbf{C} \times \mathcal{K}$ and all $t > 0$. Then $\dim E = \dim \mathcal{K}$.

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Next we will give a precise definition of the free flow of rank $n$. Let $\mathcal{C}$ be a Hilbert space of dimension $n$, where $n \in \{1, 2, \ldots, \infty\}$. We will denote by $\mathrm{L}^2\big([0,\infty)\ ;\ \mathcal{C}\big)$ the Hilbert space of all measurable functions $f\colon [0,\infty) \longrightarrow \mathcal{C}$ which satisfy

$$\int_0^\infty \|f(x)\|^2\, dx < \infty,$$

where of course we identify any two functions which are equal almost everywhere. The equivalence class of a function $f$ will be denoted $[f]$. The inner product on $\mathrm{L}^2\big([0,\infty)\ ;\ \mathcal{C}\big)$ is defined by

$$\langle [f], [g] \rangle = \int_0^\infty \langle f(x), g(x)\rangle\, dx.$$

On this Hilbert space there is a strongly continuous semigroup of isometries $U = \{U_t : t \geq 0\}$, the *unilateral shift of multiplicity n*, defined by

$$(U_t f)(x) = \begin{cases} 0 & \text{if } x < t \\ f(x-t) & \text{if } x \geq t, \end{cases} \qquad t \geq 0,\ f \in \mathrm{L}^2\big([0,\infty)\ ;\ \mathcal{C}\big),\ x \geq 0.$$

The *free flow of rank n* is the free $E_0$-semigroup $\alpha^U$ of Theorem 1.2.1.

THEOREM 3.1.2. *If $\alpha$ is a free flow of positive rank, then $d_*(\alpha) = +\infty$.*

PROOF. Let $\mathcal{C}$ be a separable Hilbert space of positive dimension, and let $U = \{U_t : t \geq 0\}$ be the unilateral shift semigroup on $\mathcal{H} = \mathrm{L}^2\big([0,\infty)\ ;\ \mathcal{C}\big)$. Let $\alpha = \{\alpha_t : t \geq 0\}$ be the free $E_0$-semigroup over $U$; that is, $\alpha$ is the free flow of rank $\dim \mathcal{C}$. Let $E$ be the free product system over $U$, and let $\phi\colon E \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ be the essential representation of $E$ on full Fock space given in Chapter 2. Since $\alpha$ is the $E_0$-semigroup associated with $\phi$, it follows from our earlier remarks that $d_*(\alpha) = \dim E$. Let $\mathcal{U}_E$ be the set of units of $E$. To prove Theorem 3.1.2, we will define an infinite-dimensional Hilbert space $\mathcal{K}$ and a bijection $(\lambda, \xi) \in \mathbf{C} \times \mathcal{K} \longmapsto \mathrm{u}_{(\lambda,\xi)} \in \mathcal{U}_E$ which satisfies equation (3.1).

It is useful to think of full Fock space over $\mathcal{H}$ as an $\mathrm{L}^2$-space of functions. For this we need to identify a variety of Hilbert spaces. First, we make the usual identification of $\mathrm{L}^2([0,\infty)\ ;\ \mathcal{C})^{\otimes n}$ with $\mathrm{L}^2\big([0,\infty)^n\ ;\ \mathcal{C}^{\otimes n}\big)$; that is, if $[f_1], \ldots, [f_n] \in \mathrm{L}^2\big([0,\infty)\ ;\ \mathcal{C}\big)$, then we identify $[f_1] \otimes \cdots \otimes [f_n]$ with $[f_1 \otimes \cdots \otimes f_n]$, where $f_1 \otimes \cdots \otimes f_n$ is the function $[0,\infty)^n \longrightarrow \mathcal{C}^{\otimes n}$ whose value at a point $(x_1, \ldots, x_n)$ is $f_1(x_1) \otimes \cdots \otimes f_n(x_n)$. This identification is even valid when $n = 0$ if we interpret $[0,\infty)^0$ as some one-point space $\{\omega\}$ and $\mathcal{C}^{\otimes 0}$ as $\mathbf{C}$.

Before making the next identification we need to set some notation. For $n = 0, 1, 2, \ldots$, let $X_n = [0,\infty)^n$. Define $\mathcal{B}_n$ to be the Borel $\sigma$-algebra on $X_n$, and let $\mu_n$ be Lebesgue measure on $X_n$. In the degenerate case $n = 0$ we simply mean $\mathcal{B}_0 = \big\{\emptyset, \{\omega\}\big\}$ and $\mu_0(\{\omega\}) = 1$. Let $(X, \mathcal{B}, \mu)$ be the disjoint union of the measure spaces $(X_n, \mathcal{B}_n, \mu_n)$; that is,

- $X = \bigsqcup_{n=0}^\infty X_n$,
- $\mathcal{B} = \{F \subseteq X : F \cap X_n \in \mathcal{B}_n, n \geq 0\}$, and
- $\mu(F) = \sum_{n=0}^\infty \mu_n(F \cap X_n)$, $F \in \mathcal{B}$.

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We identify $\mathcal{F}(\mathcal{H})$ with the subspace of $\mathrm{L}^2(X\,;\,\mathcal{F}(\mathcal{C}))$ consisting of all equivalence classes of functions $f$ which satisfy $f(X_n) \subseteq \mathcal{C}^{\otimes n}$ for each $n \geq 0$. We will say that such a function $f$ *represents* an element of $\mathcal{F}(\mathcal{H})$. Note that the vacuum vector $\Omega_{\mathcal{H}}$ is identified with the function which is zero on $X_n$ for each $n \geq 1$, and whose value on the one-point space $X_0 = \{\omega\}$ is the vacuum vector $\Omega_{\mathcal{C}}$.

With these identifications, the subspace $E_t \subset \mathcal{F}(\mathcal{H})$ is realized as

$$E_t = \mathcal{H}^{\otimes 0} \oplus \bigoplus_{n=1}^{\infty} \mathrm{L}^2\big([0,t) \times [0,\infty)^{n-1}\,;\,\mathcal{C}^{\otimes n}\big).$$

For notational convenience, we define $X_0(t) = \{\omega\}$, $X_n(t) = [0,t) \times [0,\infty)^{n-1}$ for each $n \geq 1$, and $X(t) = \sqcup_{n=o}^{\infty} X_n(t)$, so that

$$E_t = \bigoplus_{n=0}^{\infty} \mathrm{L}^2\big(X_n(t)\,;\,\mathcal{C}^{\otimes n}\big), \quad t > 0. \tag{3.2}$$

We will say that a function $f\colon X \longrightarrow \mathcal{F}(\mathcal{C})$ *represents* an element of $E_t$ if it represents an element of $\mathcal{F}(\mathcal{H})$ and is supported on $X(t)$.

We are now ready to analyze the units of $E$. This analysis will give us a complete understanding of the intertwining semigroups for $\alpha$, since the representation $\phi\colon E \longrightarrow \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ implements a bijection $\mathfrak{u} \longmapsto \phi \circ \mathfrak{u}$ from $\mathcal{U}_E$ onto the (strongly continuous) semigroups which intertwine $\alpha$ with the identity. In fact, this correspondence makes one unit of $E$ quite conspicuous, namely, the unit $\mathfrak{w}(t) = (t,\Omega)$ corresponding to the semigroup $\{\tilde{U}_t : t \geq 0\}$.

Suppose $\mathfrak{v}$ is a unit of $E$. Then there is a measurable map $v\colon (0,\infty) \longrightarrow \mathcal{F}(\mathcal{H})$ such that $\mathfrak{v}(t) = \big(t, v(t)\big)$ and $v(t) \in E_t$ for each $t > 0$. By [3, Theorem 4.1], there is a unique complex number $\lambda$ such that

$$\langle v(t),\Omega\rangle = \langle \mathfrak{v}(t),\mathfrak{w}(t)\rangle = e^{\lambda t}, \quad t > 0.$$

If we define $u(t) = e^{-\lambda t}v(t)$ and $\mathfrak{u}(t) = \big(t, u(t)\big)$, then $\mathfrak{u}$ is also a unit of $E$, and $\langle u(t),\Omega\rangle \equiv 1$. We will focus our attention on $\mathfrak{u}$.

CLAIM 3.1.3. *If $0 < s < t$, then the projection of $u(t)$ onto $E_s$ is $u(s)$.*

PROOF. Suppose $0 < s < t$. For each $n \geq 0$ and $r > 0$, let $u^n(r) = P_n\big(u(r)\big)$, the projection of $u(r)$ onto $\mathcal{H}^{\otimes n}$ in $\mathcal{F}(\mathcal{H})$, and let $Q_r$ be the projection of $\mathcal{F}(\mathcal{H})$ onto $E_r$. By Corollary 2.0.7 and the multiplicativity of $\mathfrak{u}$ we have

$$\begin{aligned} u(t) &= \phi_s\big(\mathfrak{u}(s)\big)u(t-s) \\ &= \Big(\sum_{i=0}^{\infty} r\big(u^i(s)\big)\tilde{U}_s\Big)\Big(\sum_{j=0}^{\infty} u^j(t-s)\Big) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} r\big(u^{n-k}(s)\big)\tilde{U}_s u^k(t-s), \end{aligned}$$

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and by projecting onto $\mathcal{H}^{\otimes n}$ we obtain

(3.3) $$u^n(t) = \sum_{k=0}^{n} r\big(u^{n-k}(s)\big)\tilde{U}_s u^k(t-s), \quad n \geq 0.$$

One can easily verify that $E_s$ is a reducing subspace for $r(f)$ whenever $f \in E_s \cap \mathcal{F}_b(\mathcal{H})$, and consequently

$$Q_s u^n(t) = \sum_{k=0}^{n} r\big(u^{n-k}(s)\big)Q_s\tilde{U}_s u^k(t-s), \quad n \geq 0.$$

But $Q_s\tilde{U}_s = P_0$, so

$$\begin{aligned} Q_s u^n(t) &= r\big(u^n(s)\big)u^0(t-s) \\ &= u^n(s), \quad n \geq 0, \end{aligned}$$

the last equality holding because $\langle u(t-s), \Omega\rangle = 1$. Summing on $n$ gives the desired result. ∎

A consequence of this claim is that the functions $u(t)$, $t > 0$, are coherent in the sense that there is a measurable function $\hat{u}: X \longrightarrow \mathcal{F}(C)$ such that for each $t > 0$, $u(t)$ is represented by $\hat{u}$ on $X(t)$; that is, $u(t) = [\hat{u} \cdot \chi_{X(t)}]$ for each $t > 0$, and $\hat{u}(X_n) \subseteq C^{\otimes n}$ for each $n \geq 0$. The next claim tells us that this function $\hat{u}$ is translation invariant.

CLAIM 3.1.4. *For each $s > 0$ we have*

$$\hat{u}(x) = \hat{u}(x+s) \quad a.e.\ d\mu(x);$$

*where*

$$x + s = \begin{cases} \Omega & \text{if } x = \Omega \\ (x_1 + s, \ldots, x_n + s) & \text{if } x = (x_1, \ldots, x_n) \text{ for some } n \geq 1. \end{cases}$$

PROOF. Fix $s > 0$, and suppose $t > s$. Since $u(t)$ is represented by the function $\hat{u} \cdot \chi_{X(t)}$, we have that $\tilde{U}_s^* u(t)$ is represented by the function whose value at a point $x$ is $\hat{u}(x+s)$ if $x \in X(t-s)$, and 0 otherwise. But $u(t) = \phi_s\big(u(s)\big)u(t-s)$, so

$$\begin{aligned} \tilde{U}_s^* u(t) &= \tilde{U}_s^* \phi_s\big(u(s)\big)u(t-s) \\ &= \phi_s(\Omega)^* \phi_s\big(u(s)\big)u(t-s) \\ &= \langle u(s), \Omega\rangle u(t-s) \\ &= u(t-s), \end{aligned}$$

which shows that $\tilde{U}_s^* u(t)$ is also represented by the function $\hat{u} \cdot \chi_{X_{t-s}}$. Thus $\hat{u}(x+s) = \hat{u}(x)$ for almost every $x$ in $X(t-s)$. Since $t$ was arbitrary, we have $\hat{u}(x+s) = \hat{u}(x)$ almost everywhere $d\mu(x)$. ∎

Let us pause for a moment to informally examine the function $\hat{u}$ when restricted to $X_n$ for very small $n$. Hopefully this will help to motivate what follows. For $n = 0$, we have arranged things so that $\hat{u}(\omega) = \Omega$. For $n = 1$, $\hat{u}\restriction_{X_1}$ is a translation invariant measurable function from $[0,\infty)$ to the Hilbert space $\mathcal{C}$, so it should not seem unreasonable to expect that there is a vector $f_1 \in \mathcal{C}$ such that $\hat{u}(x) = f_1$ for almost every $x \in [0,\infty)$. (A generalization of this assertion is proved in Corollary 3.1.14.)

In dimension two, one can begin to see what is going on. Expanding and simplifying equation (3.3) for $n = 2$ gives

$$u^2(t) = u^2(s) + r\big(u^1(s)\big)\tilde{U}_s u^1(t-s) + \tilde{U}_s u^2(t-s), \quad 0 < s < t. \tag{3.4}$$

Now $u^2(t)$ is supported in $[0,t) \times [0,\infty)$ (that is, $u^2(t)$ can be represented by a function whose support is contained in $[0,t) \times [0,\infty)$), and the three functions on the right-hand side of equation (3.4) have mutually disjoint supports contained in the sets $[0,s) \times [0,\infty)$, $[s,t) \times [0,s)$, and $[s,t) \times [s,\infty)$, respectively. Hence by restricting equation (3.4) to the set $[s,t) \times [0,s)$, we obtain

$$\begin{aligned} \hat{u}(x_1,x_2) &= \hat{u}(x_1 - s) \otimes \hat{u}(x_2) \\ &= f_1 \otimes f_1 \quad \text{a.e. } x_1,x_2 \in [s,t) \times [0,s). \end{aligned}$$

But this is true whenever $0 < s < t$, so $\hat{u}(x_1,x_2) = f_1 \otimes f_1$ for almost every $(x_1,x_2) \in X_2$ satisfying $x_1 > x_2$.

Above the diagonal, on the set $R_2 = \{(x_1,x_2) : x_1 \leq x_2\}$, $\hat{u}$ is not dependent on the one-dimensional case. Nevertheless, $\hat{u}$ is still translation invariant on $R_2$, and a moment's thought leads one to expect the existence of a function $[f_2] \in L^2(X_1\ ;\ \mathcal{C}^{\otimes 2})$ such that $\hat{u}(x_1,x_2) = f_2(x_2 - x_1)$ for almost every $(x_1,x_2) \in R_2$.

For $n = 3$, we obtain a similar result. By expanding and simplifying equation (3.3), then letting $s$ and $t$ vary, it becomes clear that $\hat{u}\restriction_{X_3}$ is completely determined by the one and two-dimensional cases, except on the set

$$R_3 = \big\{(x_1,x_2,x_3) \in X_3 : x_1 \leq \min\{x_2,x_3\}\big\}.$$

For example, by restricting the functions in equation (3.3) to the set $[s,t) \times [s,\infty) \times [0,s)$, all functions on the right vanish except for $r\big(u^1(s)\big)\tilde{U}_s u^2(t-s)$, so

$$\begin{aligned} \hat{u}(x_1,x_2,x_3) &= \hat{u}(x_1 - s, x_2 - s) \otimes \hat{u}(x_3) \\ &= \hat{u}(x_1,x_2) \otimes \hat{u}(x_3) \quad \text{a.e. } (x_1,x_2,x_3) \in [s,t) \times [s,\infty) \times [0,\infty). \end{aligned}$$

Similarly, by restricting to $[s,t) \times [0,s) \times [0,\infty)$, all functions on the right vanish except for $r\big(u^2(s)\big)\tilde{U}_s u^1(t-s)$, so

$$\begin{aligned} \hat{u}(x_1,x_2,x_3) &= \hat{u}(x_1 - s) \otimes \hat{u}(x_2,x_3) \\ &= \hat{u}(x_1) \otimes \hat{u}(x_2,x_3) \quad \text{a.e. } (x_1,x_2,x_3) \in [s,t) \times [0,s) \times [0,\infty). \end{aligned}$$

The union of all sets of these forms as $s$ and $t$ vary is the complement of $R_3$.

On $R_3$ itself, the translation invariance of $\hat{u}$ leads one to expect the existence of a function $[f_3] \in L^2(X_2 ; \mathcal{C}^{\otimes 3})$ such that

$$\hat{u}(x_1, x_2, x_3) = f_3(x_2 - x_1, x_3 - x_1) \quad \text{a.e. } (x_1, x_2, x_3) \in R_3.$$

With this motivation, we proceed as follows. Let

$$\mathcal{K} = \bigoplus_{n=1}^{\infty} L^2(X_{n-1} ; \mathcal{C}^{\otimes n}) \subset L^2(X ; \mathcal{F}(\mathcal{C})).$$

We will define a bijection

$$(\lambda, \xi) \in \mathbf{C} \times \mathcal{K} \longmapsto \mathrm{u}_{(\lambda,\xi)} \in \mathcal{U}_E$$

such that equation (3.1) holds. Since $\mathcal{K}$ is infinite-dimensional whenever $\dim \mathcal{C} \geq 1$, this will prove that $d_*(\alpha) = +\infty$. For $\xi \in \mathcal{K}$, let $f : X \longrightarrow \mathcal{F}(\mathcal{C})$ represent $\xi$ in the sense that $\xi = [f]$ and $f(X_{n-1}) \subseteq \mathcal{C}^{\otimes n}$ for each $n \geq 1$. We will define a measurable function $\hat{f} : X \longrightarrow \mathcal{F}(\mathcal{C})$ (motivated by the function $\hat{u}$ above) which satisfies $\hat{f}(X_n) \subseteq \mathcal{C}^{\otimes n}$ for each $n \geq 0$, and

$$\int_{X(t)} \|\hat{f}(x)\|^2 \, d\mu(x) < \infty, \quad t > 0,$$

so that $\hat{f} \cdot \chi_{X(t)}$ represents an element of $E_t$. The unit $\mathrm{u}_{(\lambda,\xi)}$ is then defined by $\mathrm{u}_{(\lambda,\xi)}(t) = \big(t, e^{\lambda t} u_f(t)\big)$, where $u_f(t) = [\hat{f} \cdot \chi_{X(t)}]$.

To define the map $f \longmapsto \hat{f}$ described above we first create a partition of the measure space $X$. This partition arises from analyzing the supports of the functions in equation (3.3) and letting $s$ and $t$ vary, as was done in our informal examination of $\hat{u}$ restricted to $X_3$. Indeed, the sets $R_2$ and $R_3$ which we defined in this previous discussion are elements of the partition.

DEFINITION 3.1.5. Let $\mathcal{N} = \bigsqcup_{b=0}^{\infty} \mathbf{N}^b$, with the understanding that $\mathbf{N}^0$ is the one-point space $\{\omega\}$.

DEFINITION 3.1.6. For each $\mathbf{p} \in \mathcal{N}$ we will now define a subset $R_{\mathbf{p}}$ of $X$. To begin with, define $R_\omega = X_0 = \{\omega\}$. Next, if $\mathbf{p} = p \in \mathbf{N}^1$, we define

$$R_p = \{(x_1, \ldots, x_p) \in [0, \infty)^p : x_1 \leq x_i, i \geq 2\}. \tag{3.5}$$

Finally, if $\mathbf{p} = (p_1, \ldots, p_b)$ for some $b \geq 1$, let $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, and let $n = q_b$. Define

$$R_{\mathbf{p}} = \{(x_1, \ldots, x_n) \in R_{p_1} \times \cdots \times R_{p_b} : x_1 > x_{1+q_1} > \cdots > x_{1+q_{b-1}}\}. \tag{3.6}$$

LEMMA 3.1.7. *$\{R_{\mathbf{p}} : \mathbf{p} \in \mathcal{N}\}$ is a measurable partition of $X$.*

PROOF. The sets $R_{\mathbf{p}}$, $\mathbf{p} \in \mathcal{N}$ are certainly disjoint measurable subsets of $X$. Suppose $x \in X$. We will define $\mathbf{p} \in \mathcal{N}$ such that $x \in R_{\mathbf{p}}$. If $x = \omega$, then $x \in R_\omega$, so we may assume that $x = (x_1, \ldots, x_n)$ for some $n \geq 1$. Let

$$p_1 = \begin{cases} n & \text{if } x_i \geq x_1 \text{ for each } i \\ \min\{i : x_{i+1} < x_1\} & \text{otherwise.} \end{cases}$$

Suppose $p_1, \ldots, p_k$ have been defined for some $k \geq 1$. If $p_1 + \cdots + p_k = n$, then let $\mathbf{p} = (p_1, \ldots, p_k)$. Otherwise, define

$$p_{k+1} = \begin{cases} n - (p_1 + \cdots + p_k) & \text{if } x_i \geq x_{1+p_1+\cdots+p_k} \text{ for all } i \\ \min\{i : x_{i+1} < x_{1+p_1+\cdots+p_k}\} - (p_1 + \cdots + p_k) & \text{otherwise.} \end{cases}$$

Then $x \in R_{\mathbf{p}}$. ∎

We now define the map $f \longmapsto \hat{f}$ by defining the value of $\hat{f}$ on $R_{\mathbf{p}}$ for each $\mathbf{p} \in \mathcal{N}$. Suppose $f$ represents an element of $\mathcal{K}$. To begin with, define $\hat{f}(\omega) = \Omega_C$ (even if $f = 0$). Next, if $x \in R_p$ for some $p \in \mathbf{N}$, define

$$\hat{f}(x) = \begin{cases} f(\omega) & \text{if } p = 1 \\ f(x_2 - x_1, x_3 - x_1, \ldots, x_p - x_1) & \text{if } p \geq 2. \end{cases} \tag{3.7}$$

Finally, if $\mathbf{p} = (p_1, \ldots, p_b)$ for some $b \geq 1$ and $x \in R_{\mathbf{p}}$, let $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, let $n = q_b$, and consider $R_{\mathbf{p}}$ as a subset of $R_{p_1} \times \cdots \times R_{p_b}$, as was done in the definition of $R_{\mathbf{p}}$. (See (3.6)). We use the definition of $\hat{f}$ on each of the sets $R_{p_i}$ to define

$$\hat{f}(x) = \hat{f}^{\otimes b}(x), \quad x \in R_{\mathbf{p}}, \mathbf{p} = (p_1, \ldots, p_b). \tag{3.8}$$

More precisely,

$$\begin{aligned} \hat{f}(x) &= \hat{f}(x_1, \ldots, x_{q_1}) \otimes \hat{f}(x_{1+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes \hat{f}(x_{1+q_{b-1}}, \ldots, x_n) \\ &= f(x_2 - x_1, \ldots, x_{q_1} - x_1) \otimes f(x_{2+q_1} - x_{1+q_1}, \ldots, x_{q_2} - x_{1+q_1}) \otimes \cdots \\ &\quad \otimes f(x_{2+q_{b-1}} - x_{1+q_{b-1}}, \ldots, x_n - x_{1+q_{b-1}}), \quad x \in R_{\mathbf{p}}, \mathbf{p} = (p_1, \ldots, p_b), \end{aligned} \tag{3.9}$$

with the understanding that for each $i$ such that $p_i = 1$, the corresponding term

$$f(x_{2+q_{i-1}} - x_{1+q_{i-1}}, \ldots, x_{q_i} - x_{1+q_{i-1}})$$

really means $f(\omega)$, and that the right-hand side is considered to be an element of $\mathcal{C}^{\otimes n}$ under the usual isomorphism of $\mathcal{C}^{\otimes p_1} \otimes \cdots \otimes \mathcal{C}^{\otimes p_b}$ with $\mathcal{C}^{\otimes n}$.

REMARK 3.1.8. Note that $\hat{f}$ is translation invariant in the sense of Claim 3.1.4.

LEMMA 3.1.9. *The map* $[f] \in \mathcal{K} \longmapsto [\hat{f}]$ *is well-defined.*

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PROOF. Suppose $f, g: X \longrightarrow \mathcal{F}(C)$ are measurable functions which represent elements of $\mathcal{K}$ and $f(x) = g(x)$ a.e. $d\mu(x)$. We must show that $\hat{f}(x) = \hat{g}(x)$ a.e. $d\mu(x)$. By Lemma 3.1.7, it is enough to show that $f(x) = g(x)$ a.e. $x \in R_{\mathbf{p}}$ for each $\mathbf{p} \in \mathcal{N}$. First observe that $\hat{f}(\omega)$ and $\hat{g}(\omega)$ are both defined to be the vacuum vector $\Omega$, so we have that $\hat{f} = \hat{g}$ on $X_0 = R_\omega$. Also, recall from (3.7) that $\hat{f}(x) = f(\omega)$ and $\hat{g}(x) = g(\omega)$ for all $x \in X_1$. Since $\mu(\{\omega\}) = 1$ and $f = g$ a.e. $d\mu$, it must be that $f(\omega) = g(\omega)$. Thus the functions $\hat{f}$ and $\hat{g}$ are identical on $X_1 = R_1$.

Next we show that $\hat{f} = \hat{g}$ a.e. on $R_p$ for each $p \geq 2$. Fix $p \geq 2$ and define

$$T_p(x_1, \ldots, x_p) = (x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_p), \qquad (x_1, \ldots, x_p) \in X_p. \tag{3.10}$$

The map $T_p$ is a measure-preserving affine isomorphism from $X_p$ onto $R_p$. Let $A = \{x \in R_p : \hat{f}(x) \neq \hat{g}(x)\}$. Then

$$\begin{aligned}
\mu(A) &= \int_{R_p} \chi_A \, d\mu \\
&= \int_{X_p} \chi_A \circ T_p \, d\mu \\
&= \int_{X_p} \chi_A(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_p) \, d\mu(x) \\
&= \int_0^\infty \int_{X_{p-1}} \chi_A(s, s + y_1, s + y_2, \ldots, s + y_{p-1}) \, d\mu(y) \, ds \quad \text{(Fubini's theorem)},
\end{aligned}$$

where in the last equality, $y$ represents the point $(y_1, \ldots, y_{p-1})$. But for each $s \geq 0$ the point $(s, s + y_1, s + y_2, \ldots, s + y_{p-1})$ is in $A$ if and only if $f(y) \neq g(y)$, and since $f = g$ almost everywhere on $X_{p-1}$, this implies that the integral

$$\int_{X_{p-1}} \chi_A(s, s + y_1, s + y_2, \ldots, s + y_{p-1}) \, d\mu(y)$$

is zero for each $s \geq 0$. Thus $\mu(A) = 0$.

Finally, we show that $\hat{f} = \hat{g}$ a.e. on each $R_{\mathbf{p}}$ where $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b$ for some $b \geq 1$. Fix such a $\mathbf{p}$. Since $\hat{f} = \hat{g}$ a.e. on $R_{p_i}$ for each $i = 1, 2, \ldots, b$, we have that $\hat{f}^{\otimes b} = \hat{g}^{\otimes b}$ a.e. on $R_{p_1} \times \cdots \times R_{p_b}$. But $R_{\mathbf{p}}$ is a subset of $R_{p_1} \times \cdots \times R_{p_b}$, and $\hat{f}$ (resp. $\hat{g}$) is defined on $R_{\mathbf{p}}$ to be the restriction of $\hat{f}^{\otimes b}$ (resp. $\hat{g}^{\otimes b}$) to $R_{\mathbf{p}}$, so $\hat{f} = \hat{g}$ a.e. on $R_{\mathbf{p}}$. ∎

Just as it was useful to partition $X$ into subspaces $R_{\mathbf{p}}$, $\mathbf{p} \in \mathcal{N}$, it will also be useful to partition each $X(t)$ into subspaces $R_{\mathbf{p}}(t)$, $\mathbf{p} \in \mathcal{N}$. For this, we simply define

$$R_{\mathbf{p}}(t) = R_{\mathbf{p}} \cap X(t), \qquad t > 0, \mathbf{p} \in \mathcal{N}.$$

The following lemma is the main step toward showing that $\hat{f} \cdot \chi_{X(t)}$ represents an element of $E_t$ whenever $f$ represents an element of $\mathcal{K}$ and $t > 0$.

LEMMA 3.1.10. *Suppose $f$ and $g$ represent elements of $\mathcal{K}$. Then for any $b \geq 0$, $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b$, and $t > 0$, we have*

$$\int_{R_{\mathbf{p}}(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) = \frac{t^b}{b!} \kappa_{p_1} \cdots \kappa_{p_b}, \tag{3.11}$$

*where $\kappa_i = \int_{X_{i-1}} \langle f(x), g(x) \rangle \, d\mu(x)$.*

PROOF. Suppose $f$ and $g$ represent elements of $\mathcal{K}$, $t > 0$, and $\mathbf{p} \in \mathbf{N}^b$ for some $b \geq 0$. If $b = 0$, simply note that both sides of (3.11) are equal to one. If $b \geq 1$, let $\mathbf{p} = (p_1, \ldots, p_b)$, let $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, and let $n = q_b$. For each $i = 1, 2, \ldots, b$, we have a measure preserving bijection $T_{p_i} : X_{p_i} \longrightarrow R_{p_i}$ as in (3.10). Consequently, the map $T_{\mathbf{p}} = T_{p_1} \times \cdots \times T_{p_b}$ is measure preserving on $X_n = X_{p_1} \times \cdots \times X_{p_b}$, and $T_{\mathbf{p}}$ maps the set

$$D_{\mathbf{p}}(t) = \{x \in X_n(t) : t > x_1 > x_{1+q_1} > \cdots > x_{1+q_{b-1}} \geq 0\}$$

bijectively onto $R_{\mathbf{p}}(t)$. Thus

$$\int_{R_{\mathbf{p}}(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) = \int_{D_{\mathbf{p}}(t)} \Big\langle \hat{f}\big(T_{\mathbf{p}}(x)\big), \hat{g}\big(T_{\mathbf{p}}(x)\big) \Big\rangle \, d\mu(x). \tag{3.12}$$

By the definition of $\hat{f}$ on $R_{\mathbf{p}}$ we have

$$\hat{f}\big(T_{\mathbf{p}}(x)\big) = \hat{f}\big(T_{p_1}(x_1, \ldots, x_{q_1})\big) \otimes \hat{f}\big(T_{p_2}(x_{1+q_1}, \ldots, x_{q_2})\big) \otimes \cdots$$
$$\otimes \hat{f}\big(T_{p_b}(x_{1+q_{b-1}}, \ldots, x_n)\big), \quad x \in D_{\mathbf{p}}(t).$$

Also, for any $p \geq 1$, $(x_1, \ldots, x_p) \in X_p$, we have

$$\begin{aligned} \hat{f}(T_p(x_1, \ldots, x_p)) &= \hat{f}(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_p) \\ &= f(x_2, x_3, \ldots, x_p), \end{aligned}$$

with the understanding that $f(x_2, x_3, \ldots, x_p)$ really means $f(\omega)$ if $p = 1$. (See (3.7).) Combining these last two equations yields

$$\hat{f}\big(T_{\mathbf{p}}(x)\big) = f(x_2, \ldots, x_{q_1}) \otimes f(x_{2+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes f(x_{2+q_{b-1}}, \ldots, x_n), \quad x \in D_{\mathbf{p}}(t).$$

Of course a similar statement holds for $g$. Using this equation and the fact that the vectors $f(x_{2+q_{i-1}}, \ldots, x_{q_i})$ and $g(x_{2+q_{i-1}}, \ldots, x_{q_i})$ are in $\mathcal{C}^{\otimes p_i}$ for each $i = 1, 2, \ldots, b$, we have

$$\begin{aligned} &\Big\langle \hat{f}\big(T_{\mathbf{p}}(x)\big), \hat{g}\big(T_{\mathbf{p}}(x)\big) \Big\rangle \\ &\quad = \langle f(x_2, \ldots, x_{q_1}) \otimes \cdots \\ &\qquad\quad \otimes f(x_{2+q_{b-1}}, \ldots, x_n), g(x_2, \ldots, x_{q_1}) \otimes \cdots \otimes g(x_{2+q_{b-1}}, \ldots, x_n) \rangle \\ &\quad = \prod_{i=1}^{b} \langle f(x_{2+q_{i-1}}, \ldots, x_{q_i}), g(x_{2+q_{i-1}}, \ldots, x_{q_i}) \rangle, \quad x \in D_{\mathbf{p}}(t). \end{aligned} \tag{3.13}$$

Let $y_i = (x_{2+q_{i-1}}, \ldots, x_{q_i}) \in X_{p_i - 1}$ for each $i = 1, 2, \ldots, b$. Substituting (3.13) into (3.12) with this change of variables gives

$$\begin{aligned} &\int_{R_{\mathbf{p}}(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) \\ &\quad = \int_0^t \int_0^{x_1} \cdots \int_0^{x_{1+q_{b-2}}} \left[ \prod_{i=1}^{b} \int_{X_{p_i - 1}} \langle f(y_i), g(y_i) \rangle \, d\mu(y_i) \right] dx_{1+q_{b-1}} \cdots dx_{1+q_1} \, dx_1 \\ &\quad = \frac{t^b}{b!} \kappa_{p_1} \cdots \kappa_{p_b}. \end{aligned}$$

■

We now proceed to prove that $\hat{f}\cdot\chi_{X(t)}$ represents an element of $E_t$ whenever $f$ represents an element of $\mathcal{K}$ and $t > 0$. The first important observation to be made is that $\hat{f}$ maps $X_n$ into $\mathcal{C}^{\otimes n}$ for each $n \geq 0$. Certainly this is the case for $n = 0$ since $\hat{f}(\omega) = \Omega$. For $n \geq 1$, first note that (3.7) implies that

$$\hat{f}(R_n) \subseteq f(X_{n-1}) \subseteq \mathcal{C}^{\otimes n}. \tag{3.14}$$

Next, suppose $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b$ for some $b \geq 1$. Let $n = p_1 + \cdots + p_b$, so that $R_{\mathbf{p}} \subseteq X_n$. Then (3.14) and the definition of $\hat{f}$ on $R_{\mathbf{p}}$ given in (3.8) imply that

$$\begin{aligned} \hat{f}(R_{\mathbf{p}}) &\subseteq \hat{f}(R_{p_1}) \otimes \cdots \otimes \hat{f}(R_{p_b}) \\ &\subseteq \mathcal{C}^{\otimes p_1} \otimes \cdots \otimes \mathcal{C}^{\otimes p_b} \\ &= \mathcal{C}^{\otimes n}. \end{aligned}$$

Since the spaces $R_{\mathbf{p}}$, $\mathbf{p} \in \mathcal{N}$ partition $X$, it follows that $\hat{f}(X_n) \subseteq \mathcal{C}^{\otimes n}$ for all $n \geq 0$.

To prove that $\hat{f} \cdot \chi_{X(t)}$ represents an element of $E(t)$, it remains only to show that the function $\hat{f} \cdot \chi_{X(t)}$ is square-integrable over $X$; that is, that $\hat{f}$ is square-integrable over $X(t)$. For this, we prove a slightly more general statement.

LEMMA 3.1.11. *Suppose $\xi$, $\eta \in \mathcal{K}$ are represented by functions $f$ and $g$, respectively. Then*

$$\int_{X(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) = e^{t\langle \xi, \eta \rangle}, \quad t > 0.$$

PROOF. For each $i = 1, 2, \ldots$, let $\kappa_i = \int_{X_{i-1}} \langle f(x), g(x) \rangle \, d\mu(x)$, so that $\langle \xi, \eta \rangle = \sum_{i=1}^{\infty} \kappa_i$. Then

$$\begin{aligned} \int_{X(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) &= \sum_{\mathbf{p} \in \mathcal{N}} \int_{R_{\mathbf{p}}(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) \\ &= \sum_{b=0}^{\infty} \sum_{\mathbf{p} \in \mathbf{N}^b} \int_{R_{\mathbf{p}}(t)} \langle \hat{f}(x), \hat{g}(x) \rangle \, d\mu(x) \\ &= \sum_{b=0}^{\infty} \sum_{(p_1, \ldots, p_b) \in \mathbf{N}^b} \frac{t^b}{b!} \kappa_{p_1} \cdots \kappa_{p_b} \quad \text{(Lemma 3.1.10)} \\ &= \sum_{b=0}^{\infty} \frac{t^b}{b!} \Big( \sum_{i=1}^{\infty} \kappa_i \Big)^b \\ &= \sum_{b=0}^{\infty} \frac{t^b}{b!} \langle \xi, \eta \rangle^b \\ &= e^{t\langle \xi, \eta \rangle}. \end{aligned}$$

■

DEFINITION 3.1.12. Suppose $(\lambda, \xi) \in \mathbf{C} \times \mathcal{K}$ and $f: X \longrightarrow \mathcal{F}(\mathcal{C})$ represents $\xi$. Define $u_f: (0, \infty) \longrightarrow \mathcal{F}(\mathcal{H})$ by

$$u_f(t) = [\hat{f} \cdot \chi_{X(t)}], \quad t > 0,$$

and $\mathrm{u}_{(\lambda,\xi)}: (0, \infty) \longrightarrow E$ by

$$\mathrm{u}_{(\lambda,\xi)}(t) = \big(t, e^{\lambda t} u_f(t)\big), \quad t > 0.$$

Note that $\mathrm{u}_{(\lambda,\xi)}$ is well-defined by Lemma 3.1.9.

We claim that $\mathrm{u}_{(\lambda,\xi)}$ is a unit of $E$. From the definition of the spaces $X(t)$, $t > 0$, it is clear that $t \longmapsto u_f(t)$ is measurable, hence so is $\mathrm{u}_{(\lambda,\xi)}$. Also, since $\hat{f}(\omega) = \Omega$ (even if $f = 0$), we have $u_f(t) \neq 0$ for all $t > 0$, so $\mathrm{u}_{(\lambda,\xi)}$ is not the trivial cross section. It remains only to show that $\mathrm{u}_{(\lambda,\xi)}(s+t) = \mathrm{u}_{(\lambda,\xi)}(s)\mathrm{u}_{(\lambda,\xi)}(t)$ for all $s, t > 0$, where the multiplication is in the product system $E$. By Corollary 2.0.7 and a simple change of variables this is equivalent to the statement

$$u_f(t) = \phi_s\big(u_f(s)\big)u_f(t-s), \quad 0 < s < t. \tag{3.15}$$

For each $k \geq 0$, $t > 0$, let $u_f^k(t) = P_k\big(u_f(t)\big) = [\hat{f} \cdot \chi_{X_k(t)}]$. Then

$$\begin{aligned} \phi_s\big(u_f(s)\big)u_f(t-s) &= \Big(\sum_{i=0}^{\infty} r\big(u_f^i(s)\big)\tilde{U}_s\Big)\Big(\sum_{j=0}^{\infty} u_f^j(t-s)\Big) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} r\big(u_f^{n-k}(s)\big)\tilde{U}_s u_f^k(t-s), \end{aligned}$$

so (3.15) is equivalent to the statement

$$u_f^n(t) = \sum_{k=0}^{n} r\big(u_f^{n-k}(s)\big)\tilde{U}_s u_f^k(t-s), \quad n \geq 0,\ 0 < s < t. \tag{3.16}$$

Fix $n \geq 0$, $0 < s < t$. If $n = 0$ then both sides of (3.16) are equal to the vacuum vector $\Omega$, so we can assume that $n \geq 1$. To establish equality in this case we will choose appropriate representatives for the $n+2$ terms in (3.16) and show that equality holds on these representatives. Note that it would be sufficient to show that equality holds almost everywhere $d\mu$, but when we make the obvious choices for representatives we get equality everywhere.

By definition, $u_f^n(t)$ is represented by the function $\hat{f} \cdot \chi_{X_n(t)}$. Similarly, for each $k = 0, 1, \ldots, n$, the vector $u_f^{n-k}(s)$ is represented by $\hat{f} \cdot \chi_{X_{n-k}(s)}$ and $u_f^k(t-s)$ is represented by $\hat{f} \cdot \chi_{X_k(t-s)}$. Thus $r\big(u_f^{n-k}(s)\big)\tilde{U}_s u_f^k(t-s)$ is represented by $g_k$, where for $1 \leq k \leq n-1$ we define

$$g_k(x) = \begin{cases} 0 & \text{if } x \in X_i \text{ for some } i \neq n \\ \hat{f}(x_1 - s, \ldots, x_k - s) \otimes \hat{f}(x_{k+1}, \ldots, x_n) & \\ & \text{if } x = (x_1, \ldots, x_n) \in [s,t) \times [s,\infty)^{k-1} \times [0,s) \times [0,\infty)^{n-k-1} \\ 0 & \text{if } x \in X_n \setminus \big([s,t) \times [s,\infty)^{k-1} \times [0,s) \times [0,\infty)^{n-k-1}\big), \end{cases}$$

for $k = 0$ we define

$$g_0(x) = \begin{cases} 0 & \text{if } x \in X_i \text{ for some } i \neq n \\ \hat{f}(x) & \text{if } x \in [0,s) \times [0,\infty)^{n-1} \\ 0 & \text{if } x \in X_n \setminus \big([0,s) \times [0,\infty)^{n-1}\big), \end{cases}$$

and for $k = n$ we define

$$g_n(x) = \begin{cases} 0 & \text{if } x \in X_i \text{ for some } i \neq n \\ \hat{f}(x_1 - s, \ldots, x_n - s) & \text{if } x \in [s,t) \times [s,\infty)^{n-1} \\ 0 & \text{if } x \in X_n \setminus \big([s,t) \times [s,\infty)^{n-1}\big). \end{cases}$$

We claim that

$$\hat{f}\cdot\chi_{X_n(t)}=\sum_{k=0}^{n}g_k. \tag{3.17}$$

As usual, we will establish this equation by showing that the two functions are identical on $R_{\mathbf{p}}$ for each $\mathbf{p}\in\mathcal{N}$. Fix $\mathbf{p}\in\mathcal{N}$. The case $\mathbf{p}=\omega$ has already been considered, so we may assume that $\mathbf{p}=(p_1,\ldots,p_b)$ for some $b\geq 1$. Both sides of (3.17) are identically zero on $R_{\mathbf{p}}$ unless $p_1+\cdots+p_b=n$, so assume this is the case. Fix $x=(x_1,\ldots,x_n)\in R_{\mathbf{p}}$, and let $q_i=p_1+\cdots+p_i$ for each $i=0,1,\ldots,b$. Again, both sides of (3.17) are zero at $x$ if $x_1\geq t$, so we may assume that $x_1<t$. If $x_1<s$, then $g_k(x)=0$ for $k=1,2,\ldots,n$, and $g_0(x)=\hat{f}(x)$, so equality holds in this case. Similarly, if $x\in[s,t)\times[s,\infty)^{n-1}$, then $g_k(x)=0$ for $k=0,1,\ldots,n-1$, and

$$g_n(x)=\hat{f}(x_1-s,\ldots,x_n-s).$$

Since $x\in R_{\mathbf{p}}$ we have that $(x_1-s,\ldots,x_n-s)\in R_{\mathbf{p}}$ as well, so by (3.9) we have

$$\begin{aligned}
g_n(x)&=\hat{f}(x_1-s,\ldots,x_{q_1}-s)\otimes\hat{f}(x_{1+q_1}-s,\ldots,x_{q_2}-s)\otimes\cdots\\
&\qquad\otimes\hat{f}(x_{1+q_{b-1}}-s,\ldots,x_n-s)\\
&=f(x_2-x_1,\ldots,x_{q_1}-x_1)\otimes f(x_{2+q_1}-x_{1+q_1},\ldots,x_{q_2}-x_{1+q_1})\otimes\cdots\\
&\qquad\otimes f(x_{2+q_{b-1}}-x_{1+q_{b-1}},\ldots,x_n-x_{1+q_{b-1}})\\
&=\hat{f}(x).
\end{aligned}$$

Thus equality holds in (3.17) at the point $x$ in this case as well. Finally, if $x_1\geq s$ but $x$ is not in $[s,t)\times[s,\infty)^{n-1}$, let $j=\min\{i: x_{i+1}<s\}$. Then $1\leq j\leq n-1$, $g_k(x)=0$ for all $k\neq j$, and

$$g_j(x)=\hat{f}(x_1-s,\ldots,x_j-s)\otimes\hat{f}(x_{j+1},\ldots,x_n). \tag{3.18}$$

Now, since $x\in R_{\mathbf{p}}$ and $x_i\geq s>x_{j+1}$ for all $i\leq j$, it must be that $j=q_\ell$ for some $\ell\in\{1,2,\ldots,b-1\}$. (See (3.5) and (3.6).) Thus

$$(x_1-s,\ldots,x_j-s)\in R_{(p_1,\ldots,p_\ell)}$$

and

$$(x_{j+1},\ldots,x_n)\in R_{(p_{\ell+1},\ldots,p_b)}.$$

Consequently, by (3.9) we have

$$\begin{aligned}
&\hat{f}(x_1-s,\ldots,x_j-s)\\
&\quad=\hat{f}(x_1-s,\ldots,x_{q_1}-s)\otimes\hat{f}(x_{1+q_1}-s,\ldots,x_{q_2}-s)\otimes\cdots\otimes\hat{f}(x_{1+q_{\ell-1}}-s,\ldots,x_j-s)\\
&\quad=\hat{f}(x_1,\ldots,x_{q_1})\otimes\hat{f}(x_{1+q_1},\ldots,x_{q_2})\otimes\cdots\otimes\hat{f}(x_{1+q_{\ell-1}},\ldots,x_j)
\end{aligned}$$

and

$$\hat{f}(x_{j+1},\ldots,x_n)=\hat{f}(x_{1+q_\ell},\ldots,x_{q_{\ell+1}})\otimes\hat{f}(x_{1+q_{\ell+1}},\ldots,x_{q_{\ell+2}})\otimes\cdots\otimes\hat{f}(x_{1+q_{b-1}},\ldots,x_n).$$

Substituting these equations into (3.18) yields

$$g_j(x) = \hat{f}(x_1, \ldots, x_{q_1}) \otimes \cdots \otimes \hat{f}(x_{1+q_{b-1}}, \ldots, x_n).$$

By one last application of (3.9) this implies that $g_j(x) = \hat{f}(x)$, giving equality in (3.17) at the point $x$ in this final case as well. Thus $\mathfrak{u}_{(\lambda,\xi)}$ is a unit of $E$.

Equation (3.1) is now easily verified. Suppose $(\lambda, \xi)$ and $(\tau, \eta)$ are in $\mathbf{C} \times \mathcal{K}$. Let $f$ and $g$ be representatives of $\xi$ and $\eta$, respectively. By Lemma 3.1.11 we have $\langle u_f(t), u_g(t)\rangle = e^{t\langle \xi,\eta\rangle}$ for all $t > 0$, so

$$\begin{aligned}\langle \mathfrak{u}_{(\lambda,\xi)}(t), \mathfrak{u}_{(\tau,\eta)}(t)\rangle &= \langle e^{\lambda t} u_f(t), e^{\tau t} u_g(t)\rangle \\ &= e^{t(\lambda + \bar{\tau} + \langle \xi,\eta\rangle)}, \quad t > 0.\end{aligned}$$

Having established (3.1), it is now easy to see that the map $(\lambda, \xi) \in \mathbf{C} \times \mathcal{K} \longmapsto \mathfrak{u}_{(\lambda,\xi)} \in \mathcal{U}_E$ is injective. For this, suppose $\mathfrak{u}_{(\lambda,\xi)} = \mathfrak{u}_{(\tau,\eta)}$. Then for any $\zeta \in \mathcal{K}$ we have

$$\langle \mathfrak{u}_{(\lambda,\xi)}(t), \mathfrak{u}_{(0,\zeta)}(t)\rangle = \langle \mathfrak{u}_{(\tau,\eta)}(t), \mathfrak{u}_{(0,\zeta)}(t)\rangle, \quad t > 0,$$

which by (3.1) implies that

$$\lambda + \langle \xi, \zeta\rangle = \tau + \langle \eta, \zeta\rangle, \quad \zeta \in \mathcal{K}. \tag{3.19}$$

Choosing $\zeta = 0$ in (3.19) implies that $\lambda = \tau$. But then (3.19) reduces to the statement that $\langle \xi, \zeta\rangle = \langle \eta, \zeta\rangle$ for all $\zeta \in \mathcal{K}$, which implies that $\xi = \eta$ as well.

It remains only to show that every unit of $E$ can be realized as $\mathfrak{u}_{(\lambda,\xi)}$ for some $(\lambda, \xi) \in \mathbf{C} \times \mathcal{K}$. Earlier we observed that for any given unit $\mathfrak{v} \in \mathcal{U}_E$, there is a complex number $\lambda$ such that the unit $\mathfrak{u}$ defined by $\mathfrak{u}(t) = e^{-\lambda t}\mathfrak{v}(t)$ satisfies $\langle u(t), \Omega\rangle \equiv 1$, where $\mathfrak{u}(t) = \big(t, u(t)\big)$. We then established the existence of a translation-invariant measurable function $\hat{u}: X \longrightarrow \mathcal{F}(\mathcal{C})$ such that $\hat{u} \cdot \chi_{X(t)}$ represents the vector $u(t) \in E_t$ for each $t > 0$. Our goal is to find a function $f$, representing some element $\xi \in \mathcal{K}$, such that $\hat{f} = \hat{u}$ almost everywhere. For then $u_f(t) = u(t)$, so that $\mathfrak{u} = \mathfrak{u}_{(0,\xi)}$, and finally $\mathfrak{v} = \mathfrak{u}_{(\lambda,\xi)}$. The corollary to the following technical lemma will be used to define $f$.

LEMMA 3.1.13. *Suppose $n \geq 1$ and $g: X_n \longrightarrow [0, \infty)$ is a measurable function which satisfies the following two conditions:*

1. *For each $s > 0$, $g(t, x) = g(t + s, x)$ a.e. $(t, x) \in [0, \infty) \times X_{n-1} = X_n$.*
2. *$g$ is locally integrable; that is, $\int_K g\, d\mu < \infty$ for each compact $K \subset X_n$.*

*Then there is a measurable function $f: X_{n-1} \longrightarrow [0, \infty)$ such that*

$$g(t, x) = f(x) \quad \text{a.e. } (t, x) \in [0, \infty) \times X_{n-1}. \tag{3.20}$$

PROOF. For each $(t,x) \in [0,\infty) \times X_{n-1}$, where $x = (x_1,\ldots,x_{n-1})$, define

$$R(x) = [0,x_1] \times \cdots \times [0,x_{n-1}] \subset X_{n-1},$$

and define

$$R(t,x) = [0,t] \times R(x) \subset X_n.$$

Fix $x \in X_{n-1}$. Then for each $t > 0$ and $m \geq 1$ we have

$$\begin{aligned}
\frac{1}{t}\int_{R(t,x)} g\,d\mu &= \sum_{k=1}^{m} \frac{1}{t}\int_{\frac{(k-1)t}{m}}^{\frac{kt}{m}} \int_{R(x)} g(t,x)\,d\mu(x)\,dt \\
&= \sum_{k=1}^{m} \frac{1}{t}\int_{0}^{\frac{t}{m}} \int_{R(x)} g(t,x)\,d\mu(x)\,dt \quad \text{(by (1))} \\
&= \frac{1}{(t/m)}\int_{R(t/m,x)} g\,d\mu.
\end{aligned}$$

Consequently the function $t \longmapsto \frac{1}{t}\int_{R(t,x)} g\,d\mu$ is constant on the set of positive rational numbers. Since this function is continuous on $(0,\infty)$, it must be constant. Letting $x$ vary, we thus see that there is a function $h: X_{n-1} \longrightarrow [0,\infty)$ such that

$$th(x) = \int_{R(t,x)} g\,d\mu, \quad t \geq 0.$$

Define $f: X_{n-1} \longrightarrow \mathbf{R}$ by

$$f(x_1,\ldots,x_{n-1}) = \frac{\partial^{n-1} h}{\partial x_1 \cdots \partial x_{n-1}}(x_1,\ldots,x_{n-1}).$$

We claim that $f$ satisfies (3.20). To see this, define $\tilde{g}: X_n \longrightarrow \mathbf{R}$ by

$$\tilde{g}(t,x_1,\ldots,x_{n-1}) = f(x_1,\ldots,x_{n-1}).$$

We will show that for each rectangle

$$S = [x_1^0, x_2^0] \times [x_1^1, x_2^1] \times \cdots \times [x_1^{n-1}, x_2^{n-1}] \subset X_n$$

we have

$$\int_S \tilde{g}\,d\mu = \int_S g\,d\mu.$$

This will prove that $\tilde{g} = g$ a.e., giving (3.20).

We begin by showing that $\int_{R(t,x)} \tilde{g}\,d\mu = \int_{R(t,x)} g\,d\mu$ for each $(t,x) \in [0,\infty) \times X_{n-1}$. We have

$$\begin{aligned}
\int_{R(t,x)} \tilde{g}\,d\mu &= t\int_{R(x)} f\,d\mu \\
&= t\int_0^{x_{n-1}} \cdots \int_0^{x_1} \frac{\partial^{n-1} h}{\partial y_1 \cdots \partial y_{n-1}}(y_1,\ldots,y_{n-1})\,dy_1 \cdots dy_{n-1} \\
&= t\int_0^{x_{n-1}} \cdots \int_0^{x_2} \left( \frac{\partial^{n-2} h}{\partial y_2 \cdots \partial y_{n-1}}(y_1,\ldots,y_{n-1})\Big|_{y_1=0}^{y_1=x_1} \right) dy_2 \cdots dy_{n-1}
\end{aligned}$$

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$$= t\int_0^{x_{n-1}} \cdots \int_0^{x_2} \frac{\partial^{n-2}h}{\partial y_2\cdots\partial y_{n-1}}(x_1,y_2,\ldots,y_{n-1})\,dy_2\cdots dy_{n-1}$$

$$\vdots$$

$$= th(x)$$

$$= \int_{R(t,x)} g\,d\mu. \tag{3.21}$$

Now suppose $S \subset X_n$ is a rectangle of the form

$$S = [x_1^0, x_2^0] \times [x_1^1, x_2^1] \times \cdots \times [x_1^{n-1}, x_2^{n-1}].$$

For each function $\sigma\colon \{0,1,\ldots,n-1\} \longrightarrow \{1,2\}$, define

$$S_\sigma = [0, x^0_{\sigma(0)}] \times \cdots \times [0, x^{n-1}_{\sigma(n-1)}].$$

By a standard combinatorical inclusion-exclusion argument, for any integrable function $F$ on $[0,x_2^0] \times \cdots \times [0, x_2^{n-1}]$ we have

$$\int_S F = \sum_{j=0}^{n} (-1)^j \sum_{\sigma\in I_j} \int_{S_\sigma} F,$$

where

$$I_j = \left\{\sigma : \operatorname{card}(\sigma^{-1}\{1\}) = j\right\}.$$

Since each $S_\sigma$ is a rectangle of the form $R(t,x)$ for some $(t,x) \in [0,\infty) \times X_{n-1}$, this together with (3.21) implies that

$$\int_S \tilde{g}\,d\mu = \int_S g\,d\mu. \qquad \blacksquare$$

COROLLARY 3.1.14. *Suppose $n \geq 1$ and $g\colon X_n \longrightarrow C^{\otimes n}$ is a measurable function which satisfies the following two conditions:*

1. *For each $s > 0$, $g(t,x) = g(t+s,x)$ a.e. $(t,x) \in [0,\infty) \times X_{n-1} = X_n$.*
2. *$\int_{X_n(t)} \|g(y)\|^2\,d\mu(y) < \infty$, $t > 0$.*

*Then there is a measurable function $f\colon X_{n-1} \longrightarrow C^{\otimes n}$ such that*

*(1′) $g(t,x) = f(x)$ a.e. $(t,x) \in [0,\infty) \times X_{n-1}$*

*(2′) $\int_{X_{n-1}} \|f(x)\|^2\,d\mu(x) < \infty$.*

PROOF. Let $\theta\colon \mathbf{C} \longrightarrow [0,2\pi)$ be a measurable function such that $z = |z|e^{i\theta(z)}$ for each $z \in \mathbf{C}$. Let $\Xi$ be an orthonormal basis for $C^{\otimes n}$. For each $\xi \in \Xi$ and each $j \in \{1,2\}$, we define a measurable function $g_{j,\xi}\colon X_n \longrightarrow [0,\infty)$ by

$$g_{j,\xi}(y) = \begin{cases} |\langle g(y), \xi\rangle|^2 & \text{if } j = 1 \\ \theta\big(\langle g(y),\xi\rangle\big) & \text{if } j = 2, \end{cases}$$

so that

$$g(y) = \sum_{\xi\in\Xi} g_{1,\xi}(y)^{1/2} e^{ig_{2,\xi}(y)}\xi \quad \text{for each } y \in X_n.$$

We claim that each of the functions $g_{j,\xi}$ satisfies the conditions of Lemma 3.1.13. Condition (1) is satisfied by each $g_{j,\xi}$ since it is satisfied by $g$. For each $\xi \in \Xi$ the function $g_{2,j}$ is bounded and hence locally integrable. Also, for any $t > 0$ we have

$$\int_{X_n(t)} g_{1,\xi}\, d\mu \leq \int_{X_n(t)} \|g(y)\|^2\, d\mu(y) < \infty,$$

which implies that $g_{1,\xi}$ is locally integrable since every compact subset $K$ of $X_n$ is contained in $X_n(t)$ for some $t$. By Lemma 3.1.13 there are measurable functions $f_{j,\xi}: X_{n-1} \longrightarrow [0,\infty)$ such that

$$g_{j,\xi}(t,x) = f_{j,\xi}(x) \text{ a.e. } (t,x) \in [0,\infty) \times X_{n-1}, \quad j \in \{1,2\},\ \xi \in \Xi.$$

For each $\xi \in \Xi$, define $f_\xi: X_{n-1} \longrightarrow \mathcal{C}^{\otimes n}$ by

$$f_\xi(x) = f_{1,\xi}(x)^{1/2} e^{if_{2,\xi}(x)} \xi.$$

Then for each $\xi \in \Xi$ we have

$$\begin{aligned}
\int_{X_{n-1}} \|f_\xi(x)\|^2\, d\mu(x) &= \int_{X_{n-1}} f_{1,\xi}(x)\, d\mu(x) \\
&= \int_{X_n(1)} g_{1,\xi}(t,x)\, d\mu(t,x) \\
&= \int_{X_n(1)} |\langle g(y), \xi\rangle|^2\, d\mu(y) \qquad (3.22)
\end{aligned}$$

But

$$\int_{X_n(1)} |\langle g(y), \xi\rangle|^2\, d\mu(y) \leq \int_{X_n(1)} \|g(y)\|^2\, d\mu(y) < \infty,$$

so $[f_\xi] \in L^2(X_{n-1}\ ;\ \mathcal{C}^{\otimes n})$. By (3.22) we have

$$\begin{aligned}
\sum_{\xi\in\Xi} \|[f_\xi]\|^2 &= \sum_{\xi\in\Xi} \int_{X_n(1)} |\langle g(y), \xi\rangle|^2\, d\mu(y) \\
&= \int_{X_n(1)} \Big( \sum_{\xi\in\Xi} |\langle g(y), \xi\rangle|^2 \Big)\, d\mu(y) \\
&= \int_{X_n(1)} \|g(y)\|^2\, d\mu(y) \\
&< \infty.
\end{aligned}$$

Since $[f_\xi]$ and $[f_\eta]$ are orthogonal whenever $\xi$ and $\eta$ are different elements of $\Xi$, this implies that the series $\sum_{\xi\in\Xi}[f_\xi]$ converges in $L^2(X_{n-1}\ ;\ \mathcal{C}^{\otimes n})$. Let $f: X_{n-1} \longrightarrow \mathcal{C}^{\otimes n}$ be a measurable function such that

$$[f] = \sum_{\xi\in\Xi} [f_\xi].$$

We claim that $f$ satisfies ($1'$). For this, note that for almost every $(t,x) \in [0,\infty) \times X_{n-1}$ we have (using the separability of $\mathcal{C}$)

$$\begin{aligned}
g(t,x) &= \sum_{\xi\in\Xi} g_{1,\xi}(t,x)^{1/2} e^{ig_{2,\xi}(t,x)} \\
&= \sum_{\xi\in\Xi} f_{1,\xi}(x)^{1/2} e^{if_{2,\xi}(x)} \\
&= \sum_{\xi\in\Xi} f_\xi(x).
\end{aligned}$$

But $f(x) = \sum_{\xi\in\Xi} f_\xi(x)$ for almost every $x \in X_{n-1}$, so $g(t,x) = f(x)$ a.e. $d\mu(t,x)$. ■

We are now ready to define an element $\xi \in \mathcal{K}$ such that $\mathrm{u} = \mathrm{u}_{(0,\xi)}$. For each $n \geq 1$, let $T_n : X_n \longrightarrow R_n$ be the affine isomorphism

$$T_n(x_1, \ldots, x_n) = (x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_n).$$

Define $g_n : X_n \longrightarrow \mathcal{C}^{\otimes n}$ by $g_n = \hat{u} \circ T_n$. Note that

$$\begin{aligned} \int_{X_n(t)} \|g_n(x)\|^2 \, d\mu(x) &= \int_{R_n(t)} \|\hat{u}(x)\|^2 \, d\mu(x) \\ &\leq \int_{X(t)} \|\hat{u}(x)\|^2 \, d\mu(x) < \infty, \qquad t > 0, \end{aligned}$$

and that for each $s > 0$ we have

$$\begin{aligned} g_n(x) &= \hat{u}(x_1, x_1 + x_2, x_1 + x_3, \ldots, x_1 + x_n) \\ &= \hat{u}(x_1 + s, x_1 + x_2 + s, x_1 + x_3 + s, \ldots, x_1 + x_n + s) \quad \text{(Claim 3.1.4)} \\ &= g_n(x_1 + s, x_2, x_3, \ldots, x_n) \quad \text{a.e. } d\mu_n(x). \end{aligned}$$

Consequently, by Lemma 3.1.13 there are functions $[f_n] \in \mathrm{L}^2(X_{n-1} \,;\, \mathcal{C}^{\otimes n})$ such that

$$g_n(x_1, \ldots, x_n) = f_n(x_2, \ldots, x_n) \quad \text{a.e. } x \in X_n.$$

Observe that

$$\begin{aligned} \|[f_n]\|^2 &= \int_{X_{n-1}} \langle f_n(x), f_n(x) \rangle \, d\mu(x) \\ &= \int_{X_n(1)} \langle g_n(x), g_n(x) \rangle \, d\mu(x) \\ &= \int_{R_n(1)} \langle \hat{u}(x), \hat{u}(x) \rangle \, d\mu(x) \\ &\leq \|u^n(1)\|^2, \quad n \geq 1, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \|[f_n]\|^2 \leq \sum_{n=1}^{\infty} \|u^n(1)\|^2 \leq \|u(1)\|^2 < \infty.$$

As a result, if we define $f : X \longrightarrow \mathcal{F}(\mathcal{C})$ by $f \restriction_{X_{n-1}} = f_n$ for each $n \geq 1$, the function $f$ represents an element $\xi$ of $\mathcal{K}$.

We claim that $\hat{f} = \hat{u}$ a.e. $d\mu$. We will prove this by showing that $\hat{f}(x) = \hat{u}(x)$ almost everywhere on $R_{\mathbf{p}}(t)$ for each $\mathbf{p} \in \mathcal{N}$ and each $t > 0$. Fix $t > 0$. Since $\hat{f}(\omega) = \hat{u}(\omega) = \Omega_{\mathcal{C}}$, $\hat{f}$ and $\hat{u}$ agree on the one-point space $\{\omega\} = R_\omega(t)$. Next suppose $\mathbf{p} = (p)$ for some $p \geq 1$. Then for almost every $x = (x_1, \ldots, x_p) \in R_p(t)$ we have

$$\begin{aligned} \hat{u}(x) &= g_p(x_1, x_2 - x_1, x_3 - x_1, \ldots, x_p - x_1) \\ &= f_p(x_2 - x_1, x_3 - x_1, \ldots, x_p - x_1) \\ &= \hat{f}(x). \end{aligned} \tag{3.23}$$

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To prove that $\hat{f}(x) = \hat{u}(x)$ a.e. $x \in R_{\mathbf{p}}(t)$ for the general $\mathbf{p} \in \mathcal{N}$, we must make use of the multiplicativity of $\mathfrak{u}$. In particular, if $t_1, \ldots, t_a$ are positive real numbers which sum to $t$, then

$$\mathfrak{u}(t) = \mathfrak{u}(t_1) \cdots \mathfrak{u}(t_a).$$

By an easy induction using the results of Corollary 2.0.7 and Lemma 2.0.6 we have

$$u(t) = \phi_{t_1}\big(u(t_1)\big) \cdots \phi_{t_{a-1}}\big(u(t_{a-1})\big)u(t_a),$$

and hence one can expect to learn more about the function $\hat{u}$ by investigating the vector on the right-hand side of this last equation. Because of its usefulness in the next section, we will actually consider the more general product $\mathfrak{v}_1(t_1) \cdots \mathfrak{v}_a(t_a)$, where $\mathfrak{v}_1, \ldots, \mathfrak{v}_a$ are units of $E$. In this case we have

$$\mathfrak{v}_1(t_1) \cdots \mathfrak{v}_a(t_a) = \Big(t, \phi_{t_1}\big(v_1(t_1)\big) \cdots \phi_{t_{a-1}}\big(v_{a-1}(t_{a-1})\big)v_a(t_a)\Big), \tag{3.24}$$

where $\mathfrak{v}_i(s) = \big(s, v_i(s)\big)$ for each $s > 0$ and each $i = 1, 2, \ldots, a$. Our goal is to specify a function which represents the element $\phi_{t_1}\big(v_1(t_1)\big) \cdots \phi_{t_{a-1}}\big(v_{a-1}(t_{a-1})\big)v_a(t_a)$ of $E_t$.

To begin with, we would like to present another way of partitioning the space $X(t)$.

DEFINITION 3.1.15. Suppose $t$ is a positive real number. A *partition of $t$* is a vector $(t_1, \ldots, t_a)$, where $a \geq 1$, $t_i > 0$ for each $i = 1, 2, \ldots, a$, and $t_1 + \cdots + t_a = t$.

DEFINITION 3.1.16. Suppose $t > 0$ and $\mathbf{t} = (t_1, \ldots, t_a)$ is a partition of $t$. Let $s_i = t_1 + \cdots + t_i$ for each $i = 0, 1, \ldots, a$. For each $a$-tuple $\mathbf{m} = (m_1, \ldots, m_a)$ of nonnegative integers, define a subset $X_{\mathbf{m}}(\mathbf{t})$ of $X(t)$ by

$$\begin{aligned} X_{\mathbf{m}}(\mathbf{t}) = \Big([s_{a-1}, s_a) \times [s_{a-1}, \infty)^{m_1 - 1}\Big) \times \Big([s_{a-2}, s_{a-1}) \times [s_{a-2}, \infty)^{m_2 - 1}\Big) \times \cdots \\ \times\Big([s_1, s_2) \times [s_1, \infty)^{m_{a-1} - 1}\Big) \times \Big([0, s_1) \times [0, \infty)^{m_a - 1}\Big), \end{aligned}$$

with the understanding that for each $i$ such that $m_i = 0$, the corresponding factor $[s_{a-i}, s_{a-i+1}) \times [s_{a-i}, \infty)^{m_i - 1}$ should be omitted from the above product. If $\mathbf{m}$ is the $a$-tuple $(0, \ldots, 0)$, it is understood that $X_{\mathbf{m}}(\mathbf{t}) = X_0(t) = \{\omega\}$.

REMARK 3.1.17. Note that if $\mathbf{t}$ is the trivial partition $(t)$ of $t$ and $\mathbf{m} = (m)$ for some $m \geq 0$, then $X_{\mathbf{m}}(\mathbf{t}) = X_m(t)$. Thus this seemingly new notation is actually just an extension of our previous notation.

LEMMA 3.1.18. *Suppose $t$ is a positive real number and $\mathbf{t} = (t_1, \ldots, t_a)$ is a partition of $t$. Then*

$$\{X_{\mathbf{m}}(\mathbf{t}) : \mathbf{m} \in (\mathbf{N}_0)^a\}$$

*is a measurable partition of $X(t)$.*

PROOF. The sets $X_{\mathbf{m}}(\mathbf{t})$, $\mathbf{m} \in (\mathbf{N}_0)^a$ are clearly disjoint measurable subsets of $X(t)$. Suppose $x \in X(t)$. We will define $\mathbf{m} \in (\mathbf{N}_0)^a$ such that $x \in X_{\mathbf{m}}(\mathbf{t})$. If $x = \omega$, then $x \in X_{(0,\dots,0)}(\mathbf{t})$, so we may assume that $x = (x_1, \dots, x_n)$ for some $n \geq 1$. Let $s_i = t_1 + \cdots + t_i$ for each $i = 0, 1, \dots, a$. Define

$$m_1 = \begin{cases} n & \text{if } x \in [s_{a-1}, \infty)^n \\ \min\{i : x_{i+1} < s_{a-1}\} & \text{otherwise,} \end{cases}$$

and for each $k = 2, \dots, a$, define

$$m_k = \begin{cases} 0 & \text{if } n = m_1 + \cdots + m_{k-1} \\ n - (m_1 + \cdots + m_{k-1}) & \text{if } x_i \geq s_{a-k} \text{ for each } i \\ \min\{i : x_{i+1} < s_{a-k}\} - (m_1 + \cdots + m_{k-1}) & \text{otherwise.} \end{cases}$$

Let $\mathbf{m} = (m_1, \dots, m_a)$. Then $x \in X_{\mathbf{m}}(\mathbf{t})$. ∎

DEFINITION 3.1.19. Suppose $t > 0$ and $\mathbf{t} = (t_1, \dots, t_a)$ is a partition of $t$. Suppose also that for each $i = 1, 2, \dots, a$ we have a function $h_i : X \longrightarrow \mathcal{F}(\mathcal{C})$ which maps $X_n$ into $\mathcal{C}^{\otimes n}$ for each $n \geq 0$. Let $\mathbf{h} = (h_1, \dots, h_a)$. We define a function $F(\mathbf{h}, \mathbf{t}) : X \longrightarrow \mathcal{F}(\mathcal{C})$ as follows. To begin with, define $F(\mathbf{h}, \mathbf{t})$ to be identically zero on $X \setminus X(t)$. We will define $F(\mathbf{h}, \mathbf{t})$ on $X(t)$ by defining it on $X_{\mathbf{m}}(\mathbf{t})$ for each $a$-tuple $\mathbf{m}$ of nonnegative integers. Fix $\mathbf{m} = (m_1, \dots, m_a)$, let $n_i = m_1 + \cdots + m_i$ for each $i = 0, 1, \dots, a$, and let $n = n_a$. For each $x = (x_1, \dots, x_n) \in X_{\mathbf{m}}(\mathbf{t})$, define

$$F(\mathbf{h}, \mathbf{t})(x) = h_a(x_1, \dots, x_{n_1}) \otimes h_{a-1}(x_{1+n_1}, \dots, x_{n_2}) \otimes \cdots \otimes h_1(x_{1+n_{a-1}}, \dots, x_n).$$

LEMMA 3.1.20. *Suppose $\mathfrak{v}_1, \dots, \mathfrak{v}_a$ are units of $E$ such that for each $i = 1, 2, \dots, a$, we have*

- $\mathfrak{v}_i(t) = \big(t, v_i(t)\big)$, $t > 0$
- $\langle v_i(t), \Omega \rangle = 1$, $t > 0$
- *$\hat{v}_i : X \longrightarrow \mathcal{F}(\mathcal{C})$ is a measurable function such that for each $t > 0$, $v_i(t)$ is represented by $\hat{v}_i$ on $X(t)$.*

*Let $\mathbf{v} = (v_1, \dots, v_a)$, and denote by $\hat{\mathbf{v}}$ the $a$-tuple $(\hat{v}_1, \dots, \hat{v}_a)$. Suppose $t$ is a positive real number and $\mathbf{t} = (t_1, \dots, t_a)$ is a partition of $t$. Then the vector*

$$\phi_{t_1}\big(v_1(t_1)\big) \cdots \phi_{t_{a-1}}\big(v_{a-1}(t_{a-1})\big) v_a(t_a)$$

*in $E_t$ is represented by the function $F(\hat{\mathbf{v}}, \mathbf{t})$.*

PROOF. We have

$$\begin{aligned} &\phi_{t_1}\big(v_1(t_1)\big) \cdots \phi_{t_{a-1}}\big(v_{a-1}(t_{a-1})\big) v_a(t_a) \\ &\qquad = \Big(\sum_{m_a=0}^{\infty} r\big(v_1^{m_a}(t_1)\big)\tilde{U}_{t_1}\Big) \cdots \Big(\sum_{m_2=0}^{\infty} r\big(v_{a-1}^{m_2}(t_{a-1})\big)\tilde{U}_{t_{a-1}}\Big)\Big(\sum_{m_1=0}^{\infty} v_a^{m_1}(t_a)\Big) \\ &\qquad = \sum_{m_1,\dots,m_a \geq 0} r\big(v_1^{m_a}(t_1)\big)\tilde{U}_{t_1} \cdots r\big(v_{a-1}^{m_2}(t_{a-1})\big)\tilde{U}_{t_{a-1}} v_a^{m_1}(t_a). \end{aligned}$$

Fix $m_1, \ldots, m_a \geq 0$, let $\mathbf{m} = (m_1, \ldots, m_a)$, let $n_i = m_1 + \cdots + m_i$ for each $i = 1, 2, \ldots, a$, and let $n = n_a$. Using methods similar to those used prior to equation (3.17), we see that

$$r\big(v_1^{m_a}(t_1)\big)\tilde{U}_{t_1} \cdots r\big(v_{a-1}^{m_2}(t_{a-1})\big)\tilde{U}_{t_{a-1}} v_a^{m_1}(t_a)$$

is represented by the function which vanishes on $X \setminus X_{\mathbf{m}}(\mathbf{t})$, and whose value at a point $x \in X_{\mathbf{m}}(\mathbf{t})$ is given by

$$\hat{v}_a(x_1 - s_{a-1}, \ldots, x_{n_1} - s_{a-1}) \otimes \hat{v}_{a-1}(x_{1+n_1} - s_{a-2}, \ldots, x_{n_2} - s_{a-2}) \otimes \cdots \otimes \hat{v}_1(x_{1+n_{a-1}}, \ldots, x_n),$$

which by Claim 3.1.4 is equal to

$$\hat{v}_a(x_1, \ldots, x_{n_1}) \otimes \hat{v}_{a-1}(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \hat{v}_1(x_{1+n_{a-1}}, \ldots, x_n).$$

Summing on $\mathbf{m} \in (\mathbf{N}_0)^a$ completes the lemma. ∎

COROLLARY 3.1.21. *Suppose $\mathfrak{v}$ is a unit of $E$, $\mathfrak{v}(t) = \big(t, v(t)\big)$ for each $t > 0$, $\langle v(t), \Omega\rangle \equiv 1$, and $\hat{v}: X \longrightarrow \mathcal{F}(C)$ is a measurable function such that for each $t > 0$, $v(t)$ is represented by $\hat{v}$ on $X(t)$. Suppose also that $\mathbf{p} = (p_1, \ldots, p_b) \in \mathcal{N}^b$ for some $b \geq 1$, $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, and $n = q_b$. Then for almost every $x = (x_1, \ldots, x_n) \in R_{\mathbf{p}}$ we have*

$$\hat{v}(x) = \hat{v}(x_1, \ldots, x_{q_1}) \otimes \hat{v}(x_{1+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes \hat{v}(x_{1+q_{b-1}}, \ldots, x_n). \tag{3.25}$$

*That is, if we regard $R_{\mathbf{p}}$ as a subset of $R_{p_1} \times \cdots \times R_{p_b}$, then $\hat{v} = \hat{v}^{\otimes b}$ on $R_{\mathbf{p}}$, in the sense of equation (3.8).*

PROOF. For each $b$-tuple $\mathbf{t} = (t_1, \ldots, t_b)$ of positive rational numbers, we will show that (3.25) holds for almost every $x \in X_{\mathbf{p}}(\mathbf{t}) \cap R_{\mathbf{p}}$. Since there are only countably many such $\mathbf{t}$ and $R_{\mathbf{p}} = \bigcup_{\mathbf{t}}\big(X_{\mathbf{p}}(\mathbf{t}) \cap R_{\mathbf{p}}\big)$, this will establish the corollary.

Fix $\mathbf{t} = (t_1, \ldots, t_b)$, and let $\mathbf{v}$ be the $b$-tuple $(v, \ldots, v)$. Observe that by the multiplicativity of $\mathfrak{v}$ we have $v(t) = \phi_{t_1}\big(v(t_1)\big) \cdots \phi_{t_{b-1}}\big(v(t_{b-1})\big)v(t_b)$. By Lemma 3.1.20, this implies that $v(t)$ is represented by the function $F(\hat{\mathbf{v}}, \mathbf{t})$. But $v(t)$ is represented by $\hat{v}$ on $X(t)$, so it must be that the functions $\hat{v}$ and $F(\hat{\mathbf{v}}, \mathbf{t})$ are equal almost everywhere on $X(t)$. In particular, for almost every $x \in X_{\mathbf{p}}(\mathbf{t}) \cap R_{\mathbf{p}}$, we have

$$\begin{aligned} \hat{v}(x) &= F(\hat{\mathbf{v}}, \mathbf{t})(x) \\ &= \hat{v}(x_1, \ldots, x_{q_1}) \otimes \hat{v}(x_{1+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes \hat{v}(x_{1+q_{b-1}}, \ldots, x_n). \end{aligned}$$ ∎

It is now a simple matter to complete the proof that $\hat{f} = \hat{u}$ a.e. on $X(t)$. In (3.23) we established that $\hat{f} = \hat{u}$ a.e. on $R_p(t)$ for each $p \geq 1$, so it remains only to show that $\hat{f} = \hat{u}$ a.e. on $R_{\mathbf{p}}(t)$ for each $\mathbf{p} \in \mathcal{N}$ of the form $(p_1, \ldots, p_b)$ for some $b \geq 2$. Fix such a $\mathbf{p}$, let $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, and let $n = q_b$. Then for almost every $x \in R_{\mathbf{p}}(t)$ we have

$$\begin{aligned} \hat{u}(x) &= \hat{u}(x_1, \ldots, x_{q_1}) \otimes \hat{u}(x_{1+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes \hat{u}(x_{1+q_{b-1}}, \ldots, x_n) \quad \text{(Corollary 3.1.21)} \\ &= \hat{f}(x_1, \ldots, x_{q_1}) \otimes \hat{f}(x_{1+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes \hat{f}(x_{1+q_{b-1}}, \ldots, x_n) \quad \text{(equation (3.23))} \\ &= \hat{f}(x) \quad (\text{definition of } \hat{f} \text{ on } R_{\mathbf{p}}(t)). \end{aligned}$$ ∎

3.2. *Cocycle conjugacy.* In this section we will prove that the free flows are completely spatial. By the index result of Section 3.1 and Arveson's work in [3], this will imply that each of the free flows of positive rank is cocycle conjugate to the CAR/CCR flow of infinite rank. See [3] for a discussion of cocycle conjugacy.

For our purposes, we may define a completely spatial $E_0$-semigroup to be an $E_0$-semigroup $\alpha$ whose associated product system $\mathcal{E}_\alpha$ is *divisible*, defined as follows.

DEFINITION 3.2.1. Suppose $E$ is a product system with projection map $p: E \longrightarrow (0,\infty)$. For each $t > 0$, let $E(t) = p^{-1}(t)$, and let $F(t)$ be the subspace of $E(t)$ defined by

$$F(t) = \overline{\operatorname{span}}\{\mathrm{u}_1(t_1)\cdots \mathrm{u}_a(t_a) : a \geq 1, \mathrm{u}_1, \ldots, \mathrm{u}_a \in \mathcal{U}_E, (t_1, \ldots, t_a) \text{ is a partition of } t.\}$$

We say that $E$ is *divisible* if $F(t) = E(t)$ for each $t > 0$.

THEOREM 3.2.2. *Suppose $\alpha$ is a free flow of positive rank. Then $\alpha$ is completely spatial.*

PROOF. Resuming the notation of the previous section, we must show that the free product system $E$ is divisible. Our first task is to determine a condition on the spaces $E_t$, $t > 0$, which is equivalent to the divisibility of $E$. Suppose $a$ is positive integer, $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_a$ are units of $E$, $t$ is a positive real number, and $\mathbf{t} = (t_1, \ldots, t_a)$ is a partition of $t$. By the results of Section 3.1, we know that for each $i = 1, 2, \ldots, a$, there is a complex number $z_i$ and a vector $\xi_i \in \mathcal{K}$ such that $\mathrm{u}_i = \mathrm{u}_{(z_i,\xi_i)}$. Let $\mathbf{z} = (z_1, \ldots, z_a)$, and for each $i$, let $f_i: X \longrightarrow \mathcal{F}(C)$ be a function which represents $\xi_i$. Then $\mathrm{u}_i(s) = \big(s, e^{sz_i} u_{f_i}(s)\big)$ for each $s > 0$, and by (3.24) we have

$$\mathrm{u}_1(t_1)\cdots \mathrm{u}_a(t_a) = \Big(t, e^{\mathbf{z}\cdot\mathbf{t}} \phi_{t_1}\big(u_{f_1}(t_1)\big)\cdots \phi_{t_{a-1}}\big(u_{f_{a-1}}(t_{a-1})\big) u_{f_a}(t_a)\Big).$$

This leads us to make the following definition.

DEFINITION 3.2.3. For each $t > 0$, let $F_t$ be the subspace of $E_t$ defined by

$$F_t = \overline{\operatorname{span}}\Big\{\phi_{t_1}\big(u_{f_1}(t_1)\big)\cdots \phi_{t_{a-1}}\big(u_{f_{a-1}}(t_{a-1})\big)u_{f_a}(t_a) : a \geq 1, f_1, \ldots, f_a \text{ represent elements of } \mathcal{K}, \text{ and } (t_1, \ldots, t_a) \text{ is a partition of } t.\Big\}$$

REMARK 3.2.4. By Lemma 3.1.20, the vector $\phi_{t_1}\big(u_{f_1}(t_1)\big)\cdots \phi_{t_{a-1}}\big(u_{f_{a-1}}(t_{a-1})\big)u_{f_a}(t_a)$ is represented by the function $F(\hat{\mathbf{f}}, \mathbf{t})$, where $\mathbf{f}$ is the $a$-tuple $(f_1, \ldots, f_a)$, $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_a)$, and $\mathbf{t} = (t_1, \ldots, t_a)$. Thus

$$F_t = \overline{\operatorname{span}}\{[F(\hat{\mathbf{f}}, \mathbf{t})] : \mathbf{t} = (t_1, \ldots, t_a) \text{ is a partition of } t, \text{ and } \mathbf{f} = (f_1, \ldots, f_a) \text{ is an } a\text{-tuple of representatives of elements of } \mathcal{K}\}. \tag{3.26}$$

To prove that $E$ is divisible, we must show that $F_t = E_t$ for each $t > 0$. The proof of this statement will proceed as follows. Fix $t > 0$ for the remainder of this section, and suppose that $\zeta \in E_t \cap F_t^{\perp}$. First we will define a sequence of mutually orthogonal

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projections $Q_0, Q_1, Q_2, \dots$ with sum $I$, and prove that $Q_b\zeta \in E_t \cap F_t^\perp$ for each $b \geq 0$ (Proposition 3.2.10). We will then prove that $Q_b(F_t)$ is dense in $Q_b(E_t)$ for each $b \geq 0$ (Proposition 3.2.17). The fact that $\zeta = 0$ follows easily from these two statements, since for any $b \geq 0$ and $\xi \in F_t$, we have

$$0 = \langle Q_b\zeta, \xi \rangle = \langle Q_b\zeta, Q_b\xi \rangle,$$

so that $Q_b\zeta = 0$ for each $b \geq 0$.

DEFINITION 3.2.5. For each nonnegative integer $b$, let $W_b = \bigcup_{\mathbf{p} \in \mathbf{N}^b} R_\mathbf{p}$, and let $Q_b$ be the projection $[g] \in \mathcal{F}(\mathcal{H}) \longmapsto [g \cdot \chi_{W_b}]$.

REMARK 3.2.6. This notation should not be confused with the use of $Q$ in Claim 3.1.3. Also, observe that $\bigcup_{b \geq 0} W_b = X$, so that $Q_0 + Q_1 + Q_2 + \cdots = I$.

As a first step toward the proof of Proposition 3.2.10, let us demonstrate that $E_t$ is invariant under each of the projections $Q_b$, $b \geq 0$. For each $\mathbf{p} \in \mathcal{N}$, define $d(\mathbf{p})$ to be the dimension of the space $R_\mathbf{p}$; that is, $d(\omega) = 0$, and if $\mathbf{p} = (p_1, \dots, p_b)$ for some $b \geq 1$, then $d(\mathbf{p}) = p_1 + \cdots + p_b$. Since

$$E_t = \bigoplus_{n=0}^{\infty} \mathrm{L}^2\big(X_n(t)\,;\, C^{\otimes n}\big)$$

and each $X_n(t)$ is partitioned by the spaces $R_\mathbf{p}(t)$, where $\mathbf{p}$ ranges over all elements of $\mathcal{N}$ satisfying $d(\mathbf{p}) = n$, we can express

$$E_t = \bigoplus_{\mathbf{p} \in \mathcal{N}} \mathrm{L}^2\big(R_\mathbf{p}(t)\,;\, C^{\otimes d(\mathbf{p})}\big).$$

Thus

$$Q_b(E_t) = \bigoplus_{\mathbf{p} \in \mathbf{N}^b} \mathrm{L}^2\big(R_\mathbf{p}(t)\,;\, C^{\otimes d(\mathbf{p})}\big), \quad b \geq 0, \tag{3.27}$$

from which it is clear that $E_t$ is invariant under $Q_b$.

LEMMA 3.2.7. *If $f$ represents an element of $\mathcal{K}$, then for each $z \in \mathbf{C}$, $b \geq 0$ and $x \in W_b$, we have*

$$\widehat{zf}(x) = z^b \hat{f}(x), \tag{3.28}$$

*with the understanding that* $0^0 = 1$.

PROOF. Fix $z \in \mathbf{C}$ and $b \geq 0$, and suppose $\mathbf{p} \in \mathbf{N}^b$. We must show that (3.28) holds for each $x \in R_\mathbf{p}$. If $b = 0$ (so that $\mathbf{p} = \omega$ and $R_\mathbf{p} = \{\omega\}$), simply note that

$$\widehat{zf}(\omega) = \Omega_C = z^0 \hat{f}(\omega).$$

For the case $b = 1$, suppose $\mathbf{p} = (p)$. If $p = 1$, observe that for $x \in R_1$ we have

$$\widehat{zf}(x) = (zf)(\omega) = z\hat{f}(x).$$

If $p \geq 2$, then for each $x = (x_1, \ldots, x_p) \in R_p$ we have

$$\begin{aligned} \widehat{zf}(x) &= (zf)(x_2 - x_1, \ldots, x_p - x_1) \\ &= z\hat{f}(x). \end{aligned}$$

Finally, suppose $\mathbf{p} = (p_1, \ldots, p_b) \in \mathcal{N}$ for some $b \geq 1$. Let $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, and let $n = q_b$. Then for each $x = (x_1, \ldots, x_n) \in R_{\mathbf{p}}$ we have

$$\begin{aligned} \widehat{zf}(x) &= \widehat{zf}(x_1, \ldots, x_{q_1}) \otimes \widehat{zf}(x_{1+q_1}, \ldots, x_{q_2}) \otimes \cdots \otimes \widehat{zf}(x_{1+q_{b-1}}, \ldots, x_n) \\ &= \big(z\hat{f}(x_1, \ldots, x_{q_1})\big) \otimes \big(z\hat{f}(x_{1+q_1}, \ldots, x_{q_2})\big) \otimes \cdots \otimes \big(z\hat{f}(x_{1+q_{b-1}}, \ldots, x_n)\big) \\ &= z^b \hat{f}(x). \end{aligned}$$

∎

LEMMA 3.2.8. *Suppose* $\mathbf{p} = (p_1, \ldots, p_b) \in \mathcal{N}$, $\mathbf{t} = (t_1, \ldots, t_a)$ *is a partition of* $t$, *and* $\mathbf{m} = (m_1, \ldots, m_a)$ *is an* $a$-*tuple of nonnegative integers. Let* $q_i = p_1 + \cdots + p_i$ *for each* $i = 0, 1, \ldots, b$, *and let* $n_i = m_1 + \cdots + m_i$ *for each* $i = 0, 1, \ldots, a$. *If the intersection of the sets* $R_{\mathbf{p}}$ *and* $X_{\mathbf{m}}(\mathbf{t})$ *is nonempty, then there is a unique function*

$$\tau\colon \{0, 1, \ldots, a\} \longrightarrow \{0, 1, \ldots, b\}$$

*such that* $n_i = q_{\tau(i)}$ *for each* $i = 0, 1, \ldots, a$. *Moreover,* $\tau$ *is nondecreasing, and* $\tau(a) = b$.

PROOF. To begin with, observe that since the integers $q_0, q_1, \ldots, q_b$ are distinct, the function $\tau$ is unique. Now, suppose $x \in R_{\mathbf{p}} \cap X_{\mathbf{m}}(\mathbf{t})$. Since $R_{\mathbf{p}} \subseteq X_{q_b}$ and $X_{\mathbf{m}}(\mathbf{t}) \subseteq X_{n_a}$, it must be that $n_a = q_b$, so we can define $\tau(a) = b$. Also define $\tau(i) = b$ for each $i \in \{0, 1, \ldots, a\}$ such that $n_i = n_a$. Finally, if $i \in \{0, 1, \ldots, a\}$ is such that $n_i < n_a$, by the definition of $X_{\mathbf{m}}(\mathbf{t})$ we have that $x_{1+n_i} < x_j$ for each $j = 1, 2, \ldots, n_i$. Consequently $n_i = q_{\tau(i)}$ for some $\tau(i) \in \{0, 1, \ldots, b\}$. The function $\tau$ is nondecreasing since both of the sequences $n_0, n_1, \ldots, n_a$ and $q_0, q_1, \ldots, q_b$ are nondecreasing. ∎

LEMMA 3.2.9. *Suppose* $\mathbf{t} = (t_1, \ldots, t_a)$ *is a partition of* $t$ *and* $\mathbf{f} = (f_1, \ldots, f_a)$ *is an* $a$-*tuple of representatives of elements of* $\mathcal{K}$. *Then for each* $z \in \mathbf{C}$, $b \geq 0$ *and* $x \in W_b$, *we have*

$$F(\widehat{z\mathbf{f}}, \mathbf{t})(x) = z^b F(\hat{\mathbf{f}}, \mathbf{t})(x), \tag{3.29}$$

*with the understanding that* $0^0 = 1$.

PROOF. Fix $z \in \mathbf{C}$ and $b \geq 0$. If $b = 0$, simply note that $W_b = \{\omega\}$, and that both sides of (3.29) are equal to $\Omega_C$ when $x = \omega$. Suppose $b \geq 1$ and $x \in W_b$. Then $x \in R_{\mathbf{p}}$ for some $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b$. Let $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$, and let $n = q_b$. If $x$ is not in $X(t)$, then both sides of (3.29) are equal to zero, so we may assume that $x \in X(t)$. By Lemma 3.1.18, there is an $a$-tuple $\mathbf{m} = (m_1, \ldots, m_a)$ of nonnegative integers such that $x \in X_{\mathbf{m}}(\mathbf{t})$. Let $n_i = m_1 + \cdots + m_i$ for each $i = 0, 1, \ldots, a$; note that $n_a = n$. By the definition of $F(\widehat{z\mathbf{f}}, \mathbf{t})$ on $X_{\mathbf{m}}(\mathbf{t})$, we have

$$F(\widehat{z\mathbf{f}}, \mathbf{t})(x) = \widehat{zf_a}(x_1, \ldots, x_{n_1}) \otimes \widehat{zf_{a-1}}(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \widehat{zf_1}(x_{1+n_{a-1}}, \ldots, x_n). \tag{3.30}$$

By Lemma 3.2.8, there is a unique nondecreasing function

$$\tau\colon \{0,1,\ldots,a\} \longrightarrow \{0,1,\ldots,b\}$$

such that $\tau(a) = b$ and $n_i = q_{\tau(i)}$ for each $i = 0,1,\ldots,a$. Consequently, for each $i = 1,2,\ldots,a$, we have

$$(x_{1+n_{i-1}},\ldots,x_{n_i}) = (x_{1+q_{\tau(i-1)}},\ldots,x_{q_{\tau(i)}}) \in R_{(p_{1+\tau(i-1)},\ldots,p_{\tau(i)})} \subseteq W_{\tau(i)-\tau(i-1)}.$$

By Lemma 3.2.7 this implies that

$$\widehat{zf_{a-i+1}}(x_{1+n_{i-1}},\ldots,x_{n_i}) = z^{\tau(i)-\tau(i-1)}\hat{f}_{a-i+1}(x_{1+n_{i-1}},\ldots,x_{n_i}), \quad i = 1,2,\ldots,a.$$

Substituting this into (3.30) gives

$$\begin{aligned} F(\widehat{z\mathbf{f}},\mathbf{t})(x) &= \left(z^{\tau(1)-\tau(0)}\hat{f}_a(x_1,\ldots,x_{n_1})\right) \\ &\quad \otimes \left(z^{\tau(2)-\tau(1)}\hat{f}_{a-1}(x_{1+n_1},\ldots,x_{n_2})\right) \otimes \cdots \otimes \left(z^{\tau(a)-\tau(a-1)}\hat{f}_1(x_{1+n_{a-1}},\ldots,x_n)\right) \\ &= z^{\tau(a)-\tau(0)}F(\hat{\mathbf{f}},\mathbf{t})(x) \\ &= z^b F(\hat{\mathbf{f}},\mathbf{t})(x). \end{aligned}$$ ■

PROPOSITION 3.2.10. *If $\zeta \in E_t \cap F_t^{\perp}$, then $Q_b\zeta \in E_t \cap F_t^{\perp}$ for each $b \geq 0$.*

PROOF. We have already demonstrated that $Q_b\zeta \in E_t$ for each $b \geq 0$. Let $g$ be a function which represents $\zeta$, and for each $b \geq 0$ let $g^b = g \cdot \chi_{W_b}$, so that $g^b$ represents $Q_b\zeta$. Suppose $\mathbf{t} = (t_1,\ldots,t_a)$ is a partition of $t$ and $\mathbf{f} = (f_1,\ldots,f_a)$ is an $a$-tuple of representatives of elements of $\mathcal{K}$. Then for each $z \in \mathbf{C}$ we have

$$\begin{aligned} 0 &= \langle F(\widehat{z\mathbf{f}},\mathbf{t}), g\rangle \\ &= \sum_{b=0}^{\infty}\langle F(\widehat{z\mathbf{f}},\mathbf{t}), g^b\rangle \\ &= \sum_{b=0}^{\infty} z^b\langle F(\hat{\mathbf{f}},\mathbf{t}), g^b\rangle \quad \text{(by Lemma 3.2.9.)} \end{aligned}$$

Since a power series is identically zero if and only if each of its coefficients is zero, this implies that $\langle F(\hat{\mathbf{f}},\mathbf{t}), g^b\rangle = 0$ for each $b \geq 0$. But $\mathbf{f}$ and $\mathbf{t}$ were arbitrary, so by (3.26) we have that $Q_b\zeta \in F_t^{\perp}$ for each $b \geq 0$. ■

It remains only to show that $Q_b(F_t)$ is dense in $Q_b(E_t)$ for each $b \geq 0$. For this it is quite useful to give a tensor product decomposition of each of the subspaces $L^2\left(R_{\mathbf{p}}(t)\,;\,\mathcal{C}^{\otimes d(\mathbf{p})}\right)$ of $Q_b(E_t)$. (See (3.27).)

DEFINITION 3.2.11. Suppose $b$ is a positive integer and $t$ is a positive real number. Define $\Delta_b(t)$ to be the $b$-simplex

$$\Delta_b(t) = \{x = (x_1,\ldots,x_b) \in X_b(t) : t > x_1 > x_2 > \cdots > x_b \geq 0\}.$$

We will use the simplex $\Delta_b(t)$ to obtain a better understanding of the set $R_{\mathbf{p}}(t)$. Suppose $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b$ for some $b \geq 1$, and let $n = d(\mathbf{p})$. In the proof of Lemma 3.1.10 we defined the set $D_{\mathbf{p}}(t)$ by

$$D_{\mathbf{p}}(t) = \{x \in X_n(t) : t > x_1 > x_{1+q_1} > \cdots > x_{1+q_{b-1}} \geq 0\},$$

and we observed that if we define $T_{p_i}: X_{p_i} \longrightarrow R_{p_i}$ for each $i = 1, 2, \ldots, b$ by

$$T_{p_i}(y_1, \ldots, y_{p_i}) = (y_1, y_1 + y_2, y_1 + y_3, \ldots, y_1 + y_{p_i}),$$

then the map

$$T_{\mathbf{p}} = T_{p_1} \times \cdots \times T_{p_b}$$

restricts to a measure-preserving bijection from $D_{\mathbf{p}}(t)$ onto $R_{\mathbf{p}}(t)$.

Let $\pi = \pi_{\mathbf{p}}$ be the permutation on $X_n$ which maps $x = (x_1, \ldots, x_n)$ to the point

$$(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}, x_2, x_3, x_4, \ldots),$$

where $x_2, x_3, x_4, \ldots$ denotes the remaining $n - b$ components of $x$ in their original order. The permutation $\pi$ restricts to a measure-preserving bijection of $D_{\mathbf{p}}(t)$ onto

$$\Delta_b(t) \times X_{p_1 - 1} \times \cdots \times X_{p_b - 1},$$

and thus the composition $T_{\mathbf{p}} \circ \pi^{-1}$ induces a unitary operator

$$\mathrm{L}^2\big(\Delta_b(t) \times X_{p_1-1} \times \cdots \times X_{p_b-1} \; ; \; \mathcal{C}^{\otimes n}\big) \longrightarrow \mathrm{L}^2\big(R_{\mathbf{p}}(t) \; ; \; \mathcal{C}^{\otimes n}\big)$$

in the natural way. We will denote by $V_{\mathbf{p}}(t)$ the resulting unitary operator

$$\mathrm{L}^2\big(\Delta_b(t)\big) \otimes \mathrm{L}^2(X_{p_1-1} \; ; \; \mathcal{C}^{\otimes p_1}) \otimes \cdots \otimes \mathrm{L}^2(X_{p_b-1} \; ; \; \mathcal{C}^{\otimes p_b}) \longrightarrow \mathrm{L}^2\big(R_{\mathbf{p}}(t) \; ; \; \mathcal{C}^{\otimes n}\big)$$

obtained by identifying the Hilbert spaces

$$\mathrm{L}^2\big(\Delta_b(t) \times X_{p_1-1} \times \cdots \times X_{p_b-1} \; ; \; \mathcal{C}^{\otimes n}\big)$$

and

$$\mathrm{L}^2\big(\Delta_b(t)\big) \otimes \mathrm{L}^2(X_{p_1-1} \; ; \; \mathcal{C}^{\otimes p_1}) \otimes \cdots \otimes \mathrm{L}^2(X_{p_b-1} \; ; \; \mathcal{C}^{\otimes p_b})$$

in the usual way.

REMARK 3.2.12. Observe that each of the Hilbert spaces $\mathrm{L}^2(X_{p_i-1} \; ; \; \mathcal{C}^{\otimes p_i})$ is a subspace of $\mathcal{K}$.

We can formulate the operator $V_{\mathbf{p}}(t)$ explicitly as follows. Suppose $\eta = [h] \in \mathrm{L}^2\big(\Delta_b(t)\big)$ and $f_i$ represents an element $\xi_i \in \mathrm{L}^2(X_{p_i-1} \; ; \; \mathcal{C}^{\otimes p_i})$ for each $i = 1, 2, \ldots, b$. Then the vector

$$V_{\mathbf{p}}(t)(\eta \otimes \xi_1 \otimes \cdots \otimes \xi_b)$$

in $L^2(R_{\mathbf{p}}(t)\ ;\ C^{\otimes n})$ is represented by the function

$$g = (h \otimes f_1 \otimes \cdots \otimes f_b) \circ \pi \circ T_{\mathbf{p}}^{-1}$$

on $R_{\mathbf{p}}(t)$; that is, for $x \in R_{\mathbf{p}}(t)$ we have

$$\begin{aligned} g(x) &= h(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}) f_1(x_2 - x_1, \ldots, x_{q_1} - x_1) \otimes \cdots \\ &\qquad \otimes f_b(x_{2+q_{b-1}} - x_{1+q_{b-1}}, \ldots, x_n - x_{1+q_{b-1}}) \\ &= h(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}) \hat{f}_1(x_1, \ldots, x_{q_1}) \otimes \cdots \\ &\qquad \otimes \hat{f}_b(x_{1+q_{b-1}}, \ldots, x_n), \end{aligned} \tag{3.31}$$

where as usual $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$.

We now proceed to define a spanning set of vectors for $L^2\big(\Delta_b(t)\big)$, and then use this set and the operator $V_{\mathbf{p}}(t)$ to create a useful spanning set of vectors for $L^2\big(R_{\mathbf{p}}(t)\ ;\ C^{\otimes n}\big)$.

DEFINITION 3.2.13. Suppose $b$ is a positive integer and $t$ is a positive real number. Suppose also that $\mathbf{t} = (t_1, \ldots, t_{2b+1})$ is a partition of $t$, and $s_i = t_1 + \cdots + t_i$ for each $i = 0, 1, \ldots, 2b+1$. Define a subset $S_b(\mathbf{t})$ of $\Delta_b(t)$ by

$$S_b(\mathbf{t}) = [s_{2b-1}, s_{2b}) \times \cdots \times [s_3, s_4) \times [s_1, s_2).$$

LEMMA 3.2.14. *For each integer $b \geq 1$, the set*

$$\{[\chi_{S_b(\mathbf{t})}] : \mathbf{t} = (t_1, \ldots, t_{2b+1}) \textit{ is a partition of } t\}$$

*has dense linear span in* $L^2\big(\Delta_b(t)\big)$.

PROOF. Let $\mathcal{R} = \{S_b(\mathbf{t}) : (t_1, \ldots, t_{2b+1}) \text{ is a partition of } t\}$, and let

$$\mathcal{R}' = \{[r_{2b-1}, r_{2b}) \times \cdots \times [r_3, r_4) \times [r_1, r_2) : 0 \leq r_1 \leq r_2 \leq \cdots \leq r_{2b} \leq t\}$$

be the set of all rectangles which are contained in $\Delta_b(t)$ and whose sides are parallel to the coordinate axes. By an obvious approximation argument the sets $\{[\chi_A] : A \in \mathcal{R}\}$ and $\{[\chi_A] : A \in \mathcal{R}'\}$ have the same closed linear span in $L^2\big(\Delta_b(t)\big)$, so it suffices to show that the latter is a spanning set. Since the step functions are dense in $L^2(I)$ for any interval $I \subset \mathbf{R}$, it is clear by taking tensor products that for any $B \in \mathcal{R}'$ the Hilbert space $L^2(B)$ is spanned by $\{[\chi_A] : A \in \mathcal{R}', A \subset B\}$. But it is easy to express $\Delta_b(t)$ as a countable disjoint union $\bigcup_{i=1}^{\infty} B_i$ of elements of $\mathcal{R}'$, and doing so allows us to express

$$L^2\big(\Delta_b(t)\big) = \bigoplus_{i=1}^{\infty} L^2(B_i),$$

from which it is clear that $L^2\big(\Delta_b(t)\big)$ is spanned by $\{[\chi_A] : A \in \mathcal{R}'\}$. ∎

DEFINITION 3.2.15. For each integer $b \geq 1$ and each positive real number $t$, define $Z_b(t)$ to be the set

$$\{V_{\mathbf{p}}(t)([\chi_{S_b(\mathbf{t})}] \otimes \eta_1 \otimes \cdots \otimes \eta_b) : \mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b, \mathbf{t} = (t_1, \ldots, t_{2b+1}) \text{ is a partition of } t, \text{ and } \eta_i \in L^2(X_{p_i-1}\,;\, C^{\otimes p_i}) \text{ for each } i = 1, \ldots, b\}.$$

COROLLARY 3.2.16. *For each integer $b \geq 1$ and each positive real number $t$, $Z_b(t)$ is a spanning set for $Q_b(E_t)$.*

PROOF. This is immediate from (3.27), Lemma 3.2.14, and the fact that each $V_{\mathbf{p}}(t)$ is unitary. ∎

PROPOSITION 3.2.17. *$Q_b(F_t)$ is dense in $Q_b(E_t)$ for each $b \geq 0$.*

PROOF. The case $b = 0$ is trivial since $Q_0$ is the projection onto the vacuum vector $\Omega$, and $\Omega = u_0(t) \in F_t$. Suppose $b \geq 1$. By Corollary 3.2.16, it is enough to show that $Z_b(t) \subseteq Q_b(F_t)$. Suppose $\mathbf{p} = (p_1, \ldots, p_b) \in \mathbf{N}^b$, $\mathbf{t} = (t_1, \ldots, t_{2b+1})$ is a partition of $t$, and $f_i$ represents an element of $L^2(X_{p_i-1}\,;\, C^{\otimes p_i})$ for each $i = 1, 2, \ldots, b$. We will show that

$$V_{\mathbf{p}}([\chi_{S_b(\mathbf{t})}] \otimes [f_1] \otimes \cdots \otimes [f_b]) \in Q_b(F_t).$$

Let $I$ denote the set of all functions $\sigma\colon \{1, \ldots, b\} \longrightarrow \{0, 1\}$, and for each $j = 0, 1, \ldots, b$, let

$$I_j = \left\{\sigma \in I : \operatorname{card}(\sigma^{-1}\{0\}) = j\right\}.$$

For each $\sigma \in I$, let $\mathbf{f}_\sigma$ denote the $(2b+1)$-tuple

$$\mathbf{f}_\sigma = \big(0, \sigma(b)f_b, 0, \sigma(b-1)f_{b-1}, 0, \ldots, 0, \sigma(2)f_2, 0, \sigma(1)f_1, 0\big).$$

Finally, let $g$ be the function

$$g = \sum_{j=0}^{b} (-1)^j \sum_{\sigma \in I_j} F(\hat{\mathbf{f}}_\sigma, \mathbf{t}).$$

Observe that $g$ represents an element of $F_t$. We claim that

$$Q_b([g]) = V_{\mathbf{p}}([\chi_{S_b(\mathbf{t})}] \otimes [f_1] \otimes \cdots \otimes [f_b]).$$

To establish this equation we will verify the following two statements:

† If $x \in R_{\mathbf{p}}(t)$ and $(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}) \in S_b(\mathbf{t})$, then

$$g(x) = \hat{f}_1(x_1, \ldots, x_{q_1}) \otimes \cdots \otimes \hat{f}_b(x_{1+q_{b-1}}, \ldots, x_{q_b}),$$

where $q_i = p_1 + \cdots + p_i$ for each $i = 0, 1, \ldots, b$.

‡ If $x \in R_{\mathbf{p}'}(t)$ for some $\mathbf{p}' \in \mathbf{N}^b$ and $g(x) \neq 0$, then $\mathbf{p}' = \mathbf{p}$ and

$$(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}) \in S_b(\mathbf{t}).$$

(See (3.31).)

CLAIM 3.2.18. *Suppose* $\mathbf{m} = (m_1, \ldots, m_{2b+1})$ *is a* $(2b+1)$*-tuple of nonnegative integers,* $n_i = m_1 + \cdots + m_i$ *for each* $i = 0, 1, \ldots, 2b+1$*, and* $x \in X_{\mathbf{m}}(\mathbf{t})$*. If* $g(x) \neq 0$*, then* $m_i$ *is zero for all odd* $i$ *and nonzero for all even* $i$*, and*

$$g(x) = \hat{f}_1(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \hat{f}_b(x_{1+n_{2b-1}}, \ldots, x_{n_{2b}}). \tag{3.32}$$

PROOF. Suppose that $g(x) \neq 0$. For each $\sigma \in I$, we have

$$\begin{aligned} F(\hat{\mathbf{f}}_\sigma, \mathbf{t})(x) = \hat{0}(x_1, \ldots, x_{n_1}) &\otimes \widehat{\sigma(1)f_1}(x_{1+n_1}, \ldots, x_{n_2}) \\ &\otimes \hat{0}(x_{1+n_2}, \ldots, x_{n_3}) \otimes \cdots \otimes \hat{0}(x_{1+n_{2b-2}}, \ldots, x_{n_{2b-1}}) \\ &\otimes \widehat{\sigma(b)f_b}(x_{1+n_{2b-1}}, \ldots, x_{n_{2b}}) \otimes \hat{0}(x_{1+n_{2b}}, \ldots, x_{n_{2b+1}}). \end{aligned}$$

If $m_i$ were nonzero for some odd $i$, then the corresponding term $\hat{0}(x_{1+n_{i-1}}, \ldots, x_{n_i})$ would be zero. This would cause $F(\hat{\mathbf{f}}_\sigma, \mathbf{t})(x)$ to be zero for each $\sigma \in I$, contradicting the assumption that $g(x) \neq 0$. Thus $m_i = 0$ for all odd $i$. Similarly, for each $\sigma \in I$, $i = 1, 2, \ldots, b$, we have

$$\widehat{\sigma(i)f_i}(x_{1+n_{2i-1}}, \ldots, x_{n_{2i}}) = \begin{cases} 0 & \text{if } \sigma(i) = 0 \text{ and } m_{2i} \geq 1. \\ \hat{f}_i(x_{1+n_{2i-1}}, \ldots, x_{n_{2i}}) & \text{otherwise.} \end{cases}$$

Consequently

$$F(\hat{\mathbf{f}}_\sigma, \mathbf{t})(x) = \begin{cases} \hat{f}_1(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \hat{f}_b(x_{1+n_{2b-1}}, \ldots, x_{n_{2b}}) & \text{if } \sigma(i) = 1 \text{ whenever } m_{2i} \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

so that $g(x)$ is a multiple of

$$\hat{f}_1(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \hat{f}_b(x_{1+n_{2b-1}}, \ldots, x_{n_{2b}}).$$

More precisely,

$$g(x) = \Big(\sum_{j=0}^{b} (-1)^j c_j\Big) \hat{f}_1(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \hat{f}_b(x_{1+n_{2b-1}}, \ldots, x_{n_{2b}}) \tag{3.33}$$

where $c_j$ is the cardinality of the set

$$\{\sigma \in I_j : \sigma(i) = 1 \text{ whenever } m_{2i} \geq 1\}.$$

Let $k$ be the number of non-zero components of $\mathbf{m}$. Then for each $j = 0, 1, \ldots, b$, we have

$$c_j = \binom{b-k}{j}$$

But

$$\sum_{j=0}^{b} (-1)^j \binom{b-k}{j} = \begin{cases} 1 & \text{if } k = b \\ 0 & \text{if } k \leq b-1, \end{cases}$$

so the assumption that $g(x)$ is nonzero implies that $k = b$; that is, $m_i$ is nonzero for each even $i$. Equation (3.33) now reduces to the desired equation (3.32). ∎

If $x \in R_{\mathbf{p}}(t)$ and $(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}) \in S_b(\mathbf{t})$, then by defining

$$\mathbf{m} = (0, p_1, 0, p_2, 0, \ldots, 0, p_{b-1}, 0, p_b, 0)$$

we have $x \in X_{\mathbf{m}}(\mathbf{t})$, and by the claim

$$g(x) = \hat{f}_1(x_1, \ldots, x_{q_1}) \otimes \cdots \otimes \hat{f}_b(x_{1+q_{b-1}}, \ldots, x_{q_b}).$$

This establishes †.

To verify ‡, suppose $x \in R_{\mathbf{p}'}(t)$ for some $\mathbf{p}' = (p_1', \ldots, p_b') \in \mathbf{N}^b$ and $g(x) \neq 0$. By Lemma 3.1.18, there is a $(2b+1)$-tuple $\mathbf{m} = (m_1, \ldots, m_{2b+1})$ of nonnegative integers such that $x \in X_{\mathbf{m}}(\mathbf{t})$. Let $n_i = m_1 + \cdots + m_i$ for each $i = 0, 1, \ldots, 2b+1$. By the claim, $m_i = 0$ if $i$ is odd, $m_i \geq 1$ if $i$ is even, and

$$g(x) = \hat{f}_1(x_{1+n_1}, \ldots, x_{n_2}) \otimes \cdots \otimes \hat{f}_b(x_{1+n_{2b-1}}, \ldots, x_{n_{2b}}). \tag{3.34}$$

Let $q_i' = p_1' + \cdots + p_i'$ for each $i = 0, 1, \ldots, b$. By Lemma 3.2.8, there is a unique nondecreasing function $\tau\colon \{0, 1, \ldots, 2b+1\} \longrightarrow \{0, 1, \ldots, b\}$ such that $\tau(0) = 0$, $\tau(2b+1) = b$, and $n_i = q_{\tau(i)}'$ for each $i = 0, 1, \ldots, 2b+1$. Since $m_i \geq 1$ for $i$ even, we have

$$0 < n_2 < n_4 < \cdots < n_{2b},$$

so that

$$0 < q_{\tau(2)}' < q_{\tau(4)}' < \cdots < q_{\tau(2b)}'.$$

This clearly implies that $\tau(2i) = i$; that is, $q_i' = n_{2i}$ for $i = 0, 1, \ldots, b$. Substituting into equation (3.34) we have

$$\begin{aligned} g(x) &= \hat{f}_1(x_1, \ldots, x_{q_1'}) \otimes \cdots \otimes \hat{f}_b(x_{1+q_{b-1}'}, \ldots, x_{q_b'}) \\ &= f_1(x_2 - x_1, \ldots, x_{q_1'} - x_1) \otimes \cdots \otimes f_b(x_{2+q_{b-1}'} - x_{1+q_{b-1}'}, \ldots, x_{q_b'} - x_{1+q_{b-1}'}) \end{aligned}$$

But $f_i$ is supported in $X_{p_i-1}$, so the assumption $g(x) \neq 0$ implies that $p_i' = p_i$ for all $i$; that is, $\mathbf{p}' = \mathbf{p}$. Lastly, observe that since $q_i = n_{2i}$ for each $i = 0, 1, \ldots, b$, and $m_i = 0$ for $i$ odd, we have

$$\mathbf{m} = (0, p_1, 0, p_2, 0, \ldots, 0, p_b, 0).$$

The fact that $x \in X_{\mathbf{m}}(\mathbf{t})$ now immediately implies the desired result that

$$(x_1, x_{1+q_1}, \ldots, x_{1+q_{b-1}}) \in S_b(\mathbf{t}).$$ ∎

COROLLARY 3.2.19. *Each of the free flows of positive rank is cocycle conjugate to the* CCR/CAR *flow of rank* $+\infty$.

PROOF. This follows immediately from Arveson's classification of completely spatial $E_0$-semigroups ([3]), which states that if $\alpha$ is a completely spatial $E_0$-semigroup such that $d_*(\alpha) = n$, then $\alpha$ is cocycle conjugate to the CCR/CAR flow of rank $n$. ∎

## REFERENCES


1. W. Arveson, *Quantizing the Fredholm index*, Proceedings of the Great Plains Operator Theory Seminar, 1988.
2. ———, *An addition formula for the index of semigroups of endomorphisms of $\mathcal{B}(\mathcal{H})$*, Pacific J. Math. (1) **137**(1989), 19–36.
3. ———, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc. (409) **80**(1989).
4. J. Cuntz, *Simple $C^*$-algebras generated by isometries*, Comm. Math. Phys. **57**(1977), 173–185.
5. K. J. Dykema, D. V. Voiculescu and A. Nica, *Free Random Variables*, CRM Monograph Series, Amer. Math. Soc., Providence, Rhode Island, 1992.
6. D. Evans, *On $O_n$*, Publ. Res. Inst. Math. Sci. **16**(1980), 915–927.
7. N. M. Hugenholtz and R. V. Kadison, *Automorphisms and quasi-free states of the* CAR *algebra*, Comm. Math. Phys. **43**(1975), 181–197.
8. R. Powers, *A non-spatial continuous semigroup of $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$*, Publ. Res. Inst. Math. Sci. **23**(1987), 1053–1069.
9. ———, *An index theory for semigroups of $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$ and type $II_1$ factors*, Canad. J. Math. (1) **40**(1988), 86–114.
10. R. Powers and D. Robinson, *An index for continuous semigroups of $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$*, J. Funct. Anal. **84**(1989), 85–96.
11. J. E. Roberts, *Cross Products of von Neumann algebras by group duals*, Sympos. Math., Instituto Nazionale di alta Matematica, Academic Press, London, 1976.


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