Can. J. Math., Vol. XXIII, No. 1, 1971, pp. 12–21

VERTEX SUBGROUPS OF IRREDUCIBLE REPRESENTATIONS OF SOLVABLE GROUPS

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If $\rho(G)$ is a finite, real, orthogonal group of matrices acting on the real vector space V, then there is defined [**5**], by the action of $\rho(G)$, a convex subset of the unit sphere in V called a *fundamental region*. When the unit sphere is covered by the images under $\rho(G)$ of a fundamental region, we obtain a semi-regular figure.

The group-theoretical problem in this kind of geometry is to find when the fundamental region is unique. In this paper we examine the subgroups, $\rho(H)$, of $\rho(G)$ with a view of finding what subspace, W of V consists of vectors held fixed by all the matrices of $\rho(H)$. Any such subspace lies between two copies of a fundamental region and so contributes to a boundary of both. If enough of these boundaries might be found, the fundamental region would be completely described.

Now if $\rho(H)$ is a subgroup of $\rho(G)$ holding fixed precisely the vectors in a subspace of dimension one, then $\rho(H)$ provides a vertex of a fundamental region, and so we call such a subgroup a *vertex subgroup*. Clearly $\rho(H)$ has this property if and only if the irreducible component $I(H)$ (the identity representation of H) appears exactly once among the irreducible components of $\rho(H)$. Thus, even if $\rho(G)$ is complex, we may ask whether $\rho(G)$ has a vertex subgroup, although the question will have lost its geometrical significance.

In this paper we shall assume that $\rho(G)$ is absolutely irreducible. In Theorem I, G is assumed to be a p-group, and it is found that a vertex subgroup can be found unless $\rho(G)$ is of the "second kind", in which case something similar can be found. In Theorem II, G is assumed to be solvable, while $\rho(G)$ is assumed to be real, and irreducible, of odd degree. A vertex subgroup is shown to exist.

Acknowledgements. I wish to express my sincere thanks to Professor G. de B. Robinson for his guidance and interest in this work, and to Professor N. Itô who provided most of the proof of Theorem I.

THEOREM I. *Let G be a p-group, $\rho(G)$ an absolutely irreducible representation of G. The following holds:*

(i) *If $\rho(G)$ is not of the second kind, then G has a subgroup H such that the restricted representation $\rho(G) \downarrow H$ contains $I(H)$ exactly once;*

(ii) *If $\rho(G)$ is of the second kind, then G has a subgroup H such that $\rho(G) \downarrow H$ contains $I(H)$ with multiplicity* 2.

Received September 17, 1969.

Remarks. The statement of the theorem will be clearer when we have defined the term "kind". $\rho(G)$ is said to be "of the first kind" if $\rho(G)$, as a group of matrices with complex entries, is similar to a group of matrices with real entries; "of the third kind" if the character of $\rho(G)$ is complex; and "of the second kind" if $\rho(G)$ has real character, yet is not similar to a group of real matrices. This distinction is embodied in the relation

$$\sum_G \chi^\rho(g^2) = c(\rho)|G|,$$

where χ^ρ is the character of $\rho(G)$, $|G|$ the order of G, and $c(\rho) = 1, -1$, or 0 according as $\rho(G)$ is of the first, second, or third kind, respectively [**2**, p. 20]. Two observations are in order. The first is that if $\rho(G)$ is of the first or second kind, then G is a 2-group since [**1**] groups of odd order have non-trivial irreducible representations of only the third kind. Second, if $\rho(G)$ is of the second kind, then $\rho(G)$ appears with even multiplicity in any real representation of G [**2**, p. 20]. It follows, from Frobenius' reciprocity theorem, that a vertex subgroup can never be found in this case.

Proof of Theorem I. We proceed by induction, assuming that it holds for all proper subgroups of G. We may assume, without loss of generality, that $\rho(G)$ is a faithful representation of G.

Let A be a maximal, normal, abelian subgroup of G. Certainly A contains $Z(G)$, the centre of G, and so A is non-trivial. But also, because of our choice of A, $C_G(A) = A$ [**3**, p. 185, Lemma 3.12]. That is, the only elements of G commuting with A are already in A. It follows that the factor group G/A is isomorphic to a subgroup of the group of automorphisms, $\mathrm{Aut}(A)$, of A. We consider a number of cases.

Case (i). We suppose that A is cyclic and that G has odd order. Since A is cyclic, of order p^n, p an odd prime, $\mathrm{Aut}(A)$ is cyclic, and our remarks above imply that G is metacyclic, since G/A is necessarily cyclic. It follows that $G = \langle x, y \mid x^{p^a} = 1, y^{p^b} = x^r, yxy^{-1} = x^s\rangle$ with $s^{p^b} \equiv 1 \pmod{p^a}$, $(s, p) = 1$.

Let $N = \langle x\rangle$. We have $N \lhd G$, and Clifford's theorem easily yields that all the faithful, irreducible representations of G arise as induced representations $\sigma(N) \uparrow G$, with $\sigma(N)$ a faithful, irreducible (and hence linear) representation of N. Thus, in a natural way, the matrices of $\rho(G)$ may be taken in monomial form, with the matrices of N diagonal, and those of elements not in N entirely off-diagonal. If we decompose G as

$$G = N + yN + y^2N + \ldots + y^{p^b-1}N,$$

we see that the matrix for y takes the form

$$\begin{bmatrix} 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \omega & 0 & 0 & . & . & 0 \end{bmatrix}$$

where ω is the representative for y^{p^b} in $\sigma(N)$. Now the subgroup $\langle y\rangle$ is easily seen to be a subgroup of the required kind, provided that $\omega = 1$. That is, $y^{p^b} = 1$. Thus, it is sufficient to show that y can be chosen so that this is the case. But Solomon [**6**] proved that this can be done. Case (i) is now proved.

Case (ii). We suppose that A is cyclic and that $p = 2$. Here, if $|A| > 4$, $\mathrm{Aut}(A)$ is isomorphic to a direct product $C_2 \times C_{2^k}$, cyclic groups of orders 2 and 2^k, respectively. (The case $|A| = 4$ is trivial, associated with the quaternion or dihedral groups of order 8, and will not be discussed here.) This direct product is such that if h is a generator of A, α a generator of C_2, then $h^\alpha = h^{-1}$. Hence in this case G/A is either cyclic or a direct product of cyclic groups. Let us suppose first that G/A is cyclic, so that G is, again, metacyclic. Now A contains exactly one element of order 2, being cyclic. If G, also, has exactly one element of order 2, then [**1**, p. 131] G is either cyclic (a possibility easily disposed of) or a generalized quaternion group, Q_n. Now Q_n may be given as

$$Q_n = \langle x, y \mid x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, yxy^{-1} = x^{-1}\rangle.$$

Again, $\rho(G)$ is an induced representation $\sigma(A)\uparrow G$ ($A = \langle x\rangle$) and matrices for x and y may be chosen in the form

$$\begin{bmatrix}\omega & 0\\ 0 & \omega^{-1}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$

respectively, with ω a primitive 2^{n-1}th root of unity. It is a purely mechanical exercise to verify that

$$\sum_G \chi^\rho(g^2) = -|G|$$

so that $\rho(G)$ is of the second kind. A subgroup with the desired properties is provided by the identity element of G.

Suppose next that G has more than one element of order 2. Then, being metacyclic, it follows [**6**] that G is the semidirect product of A and a cyclic group, B. Again, $\rho(G)$ is an induced representation $\sigma(A)\uparrow G$, and, as in Case (i), B is a subgroup with the property that $\rho(G)\downarrow B$ holds invariant a subspace of dimension 1. This again proves the theorem unless $\rho(G)$ is of the second kind. But, by Frobenius' reciprocity theorem, $\rho(G)$ appears exactly once in the induced representation $I(B)\uparrow G$, which is real, precluding the possibility that $\rho(G)$ is of the second kind.

Finally, suppose that G/A is a direct product, $C_2 \times C_{2^s}$, of cyclic groups. We may take y_1 and y_2 to be generators (in G) of C_2 and C_{2^s}, respectively, so that if x is a generator for A, $y_1xy_1^{-1} = x^{-1}$. A is of index 2^{s+1} in G, and G may be decomposed as left A-cosets:

$$G = y_2A + y_2^2A + \ldots + y_2^{2^s}A + y_1y_2A + \ldots + y_1y_2^{2^s}A.$$

Consider the restriction $\rho(G)\downarrow A$. If $\omega_1(A)$ and $\omega_2(A)$ are two irreducible components of this representation, then, since A is abelian, both are of degree 1 and, since $A \lhd G$, $\omega_1(A)$ and $\omega_2(A)$ are in the same family of representations of A. Hence $\omega_1(A)$ is also a faithful representation of A, because $\rho(G)$ is a faithful representation of G, so that if x is represented by λ in $\omega_1(A)$, λ is a primitive $|A|$th root of unity. We claim that $\rho(G)$ is equivalent to the induced representation $\omega_1(A) \uparrow G$. Certainly, $\rho(G)$ is contained in this representation, by Frobenius' reciprocity theorem. Also, $\omega_1(A) \uparrow G$ has degree 2^{s+1}. Thus it will suffice to show that $\rho(G)$ has degree at least as great as 2^{s+1}. But if g is any element of G, and $gxg^{-1} = x^t$, then the representation $\beta(A)$ in which x is represented by λ^t must also, by Clifford's theorem, appear in $\rho(G) \downarrow A$. Since the factor group G/A is isomorphic to a subgroup of $\mathrm{Aut}(A)$, this implies the existence of at least 2^{s+1} distinct, irreducible components in $\rho(G) \downarrow A$, proving the equivalence of $\rho(G)$ and $\omega_1(A) \uparrow G$.

Denote by K_2 the normal subgroup of G generated by x and y_2. K_2 is not cyclic, and $y_2xy_2^{-1} \neq x^{-1}$, so that K_2 is not a generalized quaternion. Thus K_2 has more than one element of order 2 and y_2 can be chosen so that K_2 is the semidirect product of $\langle x \rangle$ and $\langle y_2 \rangle$. If K_1 is the subgroup generated by x and y_1, then either K_1 is a generalized quaternion, or else (see [**6**]) y_1 can be chosen so that K_1 is the semidirect product of $\langle x \rangle$ and $\langle y_1 \rangle$. We shall assume that y_1 and y_2 have been so chosen.

Suppose first that K_1 is not a generalized quaternion, so that $y_1^2 = y_2^{2^s} = 1$. Now for any integer r, the element y_1x^r might be taken as a generator in place of y_1 since

$$(y_1x^r)^2 = (y_1x^ry_1^{-1})x^r = x^{-r}x^r = 1$$

and

$$(y_1x^r)x(y_1x^r)^{-1} = y_1(x^rxx^{-r})y_1^{-1} = x^{-1}.$$

We claim now that, for some r, y_2 and y_1x^r commute. Suppose that this is not the case, that $y_2xy_2^{-1} = x^q$, and that the order, $|x|$, of x is 2^n. G/A is commutative, and so $y_2y_1y_2^{-1} = y_1x^v$ for some integer v. Also,

$$y_2(y_1x^r)y_2^{-1} = y_1x^vx^{rq} \neq y_1x^r$$

so that the congruence $v + rq \equiv r \pmod{2^n}$ or $r(q-1) \equiv -v \pmod{2^n}$ never holds for any choice of r. For this to be so, it is necessary and sufficient that $(q-1)$ involves 2 to a higher power than does v.

Let 2^m be the order of y_2. No power of y_2 less than 2^m commutes with x. It follows that

$$(q^{2^{m-1}} - 1) \not\equiv 0 \pmod{2^n}. \tag{1}$$

It is an easy matter to verify by induction that for any integer i,

$$y_2{}^iy_1y_2{}^{-i} = y_1x^{v(1+q+q^2+\cdots+q^{i-1})} = y_1x^{v(q^i-1)/(q-1)}.$$

Since 2^m is the order of y_2, it follows that

$$v(q^{2^m} - 1)/(q-1) \equiv 0 \pmod{2^n}. \tag{2}$$

This can be rewritten as

$$v(q^{2^{m-1}} - 1)(q^{2^{m-1}} + 1)/(q-1) \equiv 0 \pmod{2^n}. \tag{3}$$

Now $(q^{2^{m-1}} + 1)$ contains exactly one factor of 2 so that for (1) and (3) to hold simultaneously, it is necessary that v contains as many factors of 2 as does $(q - 1)$. This contradiction establishes our claim, and so we may assume that y_1 and y_2 have been chosen so that, in addition to possessing their earlier properties, they also commute.

Let H be the commutative group generated by y_1 and y_2. H has order 2^{s+1} and has trivial intersection with A. Examining the induced representation $\omega_1(A) \uparrow G$, constructed from the earlier coset decomposition of G, we see that the non-identity elements of H are represented by matrices having 0s on the diagonal, and hence 0 character. It follows that

$$\sum_H \chi^\rho(g) = \chi^\rho(1) = 2^{s+1} = |H|$$

and so $\rho(G) \downarrow H$ contains $I(H)$ with multiplicity one. Our claim is therefore confirmed in this case unless $\rho(G)$ is of the second kind. But the real representation $I(H) \uparrow G$ contains $\rho(G)$ with multiplicity one, precluding this possibility [**2**, p. 62].

Finally, suppose that K_1 is a generalized quaternion. We now let $H = \langle y_2 \rangle$ and consider again the representation $\rho(G) \downarrow H$. In this case the character test employed above yields that $\rho(G) \downarrow H$ contains $I(H)$ with multiplicity 2. Case (ii) will thus have been disposed of when we have shown that $\rho(G)$ is of the second kind.

To do this, we consider the irreducible components of the induced representation $\rho(G) \downarrow K_1$. Since $K_1 \lhd G$, these are all in the same family of representations of K_1, by Clifford's theorem, and are all faithful. However, the faithful, irreducible representations of the generalized quaternion groups are all of the second kind, of degree two. Furthermore, since the irreducible components of $\rho(G) \downarrow A$ appear with multiplicity one, by our earlier remarks, it follows that the irreducible components of $\rho(G) \downarrow K_1$ also appear with multiplicity one. (This follows by considering the double restriction $\rho(G) \downarrow K_1 \downarrow A$.) Now the only elements of G to have non-zero character in $\rho(G)$ lie in A, and if λ is an eigenvalue of the matrix for one such element, then λ^{-1} is another, so that the characters for these elements are real. Thus $\rho(G)$ is not of the third kind. If, however, $\rho(G)$ is of the first kind, then the real representation $\rho(G) \downarrow K_1$ of K_1 contains irreducible components of the second kind with multiplicity one, an impossibility. Thus $\rho(G)$ is of the second kind, and Case (ii) is now proved.

Case (iii). We suppose now that p is odd and that A is non-cyclic. We proceed by induction, assuming the theorem for p-groups of order less than that of G.

Denote by $\omega_1(A), \omega_2(A), \ldots, \omega_r(A)$ the irreducible (and possibly repeated) components of $\rho(G) \downarrow A$. Since A is abelian, and non-cyclic, none of these representations is faithful, so that if K_i denotes the kernel of $\omega_i(A)$, K_i is a non-trivial subgroup of A. We take the matrices of $\rho(G)$ so that the matrices for the elements of A appear in diagonal form in such a fashion that all the components $\omega_i(A)$ having kernel K_1 appear before the others in the upper corner. Let us suppose that, counting multiple appearances, there are k such components with kernel K_1. Now let N denote the normalizer of K_1 in G. If h is an element of A, $\omega_i(h)$ the matrix (a root of unity) for h in $\omega_i(A)$, and g an element of G, then g permutes the $\omega_i(A)$ in the sense that, for a fixed g, $\omega_i(ghg^{-1}) = \omega_j(h)$ for each h in A, and some j. We might say that g carries $\omega_i(A)$ to $\omega_j(A)$.

If now g is in N, g permutes the first k components among themselves. If, on the other hand, g is not in N, g carries each of the first k components outside of this set. Using these remarks, and Clifford's theorem, we see that the matrices for the elements of N take the form $\left[\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right]$, where B is a $k \times k$ matrix. If g lies outside of N, these same considerations yield that the matrix for g has all 0s in the upper left $k \times k$ corner. The matrices indicated by B above yield a representation, $\beta(N)$, of N which, in fact is irreducible. To see this last claim, suppose that $\beta(N)$ holds invariant a subspace, V_1, of the carrier space of $\beta(N)$, and consider the subspace generated by all the images of V_1 under the action of $\rho(G)$. Since the matrices for the elements of G outside of N have 0 in the upper $k \times k$ block, it is impossible to obtain all the vectors in the carrier space of $\beta(N)$, a conclusion inconsistent with the irreducibility of $\rho(G)$.

N cannot be all of G, for otherwise the non-trivial subgroup K_1 would lie in the kernel of $\rho(G)$, which is assumed to be trivial. Hence we may apply our inductive hypothesis to assert that there is a subgroup, H, of N such that $I(H)$ appears in $\beta(N) \downarrow H$ with multiplicity one. (p being odd, $\beta(N)$ is necessarily of the third kind.) Since K_1 lies in the kernel of $\beta(N)$, we may assume, without constraint, that K_1 is contained in H. It remains to show that $\rho(G) \downarrow H$ contains $I(H)$ with multiplicity one. Suppose that $\gamma(N)$ is another of the irreducible components of $\rho(G) \downarrow N$, after $\beta(N)$. By our construction, $I(K_1)$ does not appear in $\gamma(N) \downarrow K_1$. But then, since $H \supseteq K_1$, $I(H)$ does not appear in $\gamma(N) \downarrow H$. Thus, since $\rho(G) \downarrow H = \rho(G) \downarrow N \downarrow H$, $I(H)$ appears exactly once in $\rho(G) \downarrow H$. H is seen to be the desired vertex subgroup.

Case (iv). We suppose, finally, that $p = 2$ and that A is non-cyclic. Since $\rho(G)$ is irreducible, the elements in $Z(G)$, the centre of G, are represented by scalar multiples of the identity matrix. $\rho(G)$ may be assumed to be faithful, and so we have that $Z(G)$ is cyclic; thus A is not contained in $Z(G)$, which contains exactly one element of order 2. Now let L be the subgroup of A consisting of all elements of order 2. $|L| = 2^n$, with $n > 1$, and $L \lhd G$. Now [**3**, p. 31, Theorem 6.4], L must have non-trivial intersection with $Z(G)$, since L is normal in G, and so L contains the single second-order element in $Z(G)$. In G, the elements of L fall into conjugacy classes, C_i, with exactly two of these having one element, these elements lying in the centre of G. Treating these two as special cases, we obtain the equation

$$2 + \sum_i |C_i| = 2^n$$

which is possible only if $|C_i| = 2$ for some i, say $i = 1$. Let h_1 and h_2 be the elements in C_1. The centralizer, $C = C_G(h_1)$, of h_1 in G has index 2 in G and so is normal in G. Now let B be the abelian normal subgroup of G generated by h_1 and h_2; $|B| = 4$. The element h_1h_2 is fixed by every element of G, which either exchange h_1 and h_2, or leave them fixed. Hence h_1h_2 lies in $Z(G)$, and has matrix presentation $-I$ in $\rho(G)$. Let h_1 have presentation

$$\begin{bmatrix} I_1 & 0 \\ 0 & -I_2 \end{bmatrix} = M$$

in $\rho(G)$, where I_1 and I_2 are identity matrices of possibly different rank. Clearly, the matrix for h_2 in $\rho(G)$ is $-M$. But h_1 and h_2 are in the same conjugacy class of G, and so have the same character in $\rho(G)$, implying that I_1 and I_2 have the same rank. Observations dealing with Clifford's theorem, and the two representations of B appearing in $\rho(G) \downarrow B$ allow us to assert, as in Case (iii), that the elements in C have matrices of the form $\left[\begin{smallmatrix} E_1 & 0 \\ 0 & F_1 \end{smallmatrix}\right]$ while those outside of C take the form $\left[\begin{smallmatrix} 0 & E_2 \\ F_2 & 0 \end{smallmatrix}\right]$. Here the matrices E_1 and F_1 have the same rank and, as before, give rise to irreducible representations, $\beta_1(C)$ and $\beta_2(C)$, respectively, of C. We show now that $\beta_1(C)$ and $\beta_2(C)$ are both of the first, second, or third kind according as $\rho(G)$ is of the first, second, or third kind, respectively.

We note first that, since $C \lhd G$, $\beta_1(C)$ and $\beta_2(C)$ are in the same family of representations of C, and so are of the same kind.

Suppose first that $\rho(G)$ is of the third kind; that is, χ^ρ is complex. In order for this to happen, at least one (and hence both) of χ^{β_1} or χ^{β_2} must be complex, so that $\beta_1(C)$ is of the third kind.

Next, suppose that $\rho(G)$ is of the second kind. We must have

$$\sum_C \chi^\rho(g)\chi^{\beta_1}(g^{-1}) = |C|.$$

But then, since χ^ρ is real, we may take complex conjugates to obtain

$$\sum_C \chi^\rho(g)\overline{\chi^{\beta_1}}(g^{-1}) = |C|$$

so that $\overline{\beta_1(C)}$ appears in $\rho(G) \downarrow C$. If $\beta_1(C)$ is of the third kind, this implies that $\beta_2(C) = \overline{\beta_1(C)}$, since $\beta_1(C)$ and $\overline{\beta_1(C)}$ are inequivalent. But $\beta_1(C)$ and $\beta_2(C)$ have different kernels, and this is clearly impossible. If, on the other hand, $\beta_1(C)$ is of the first kind, then so too is the induced representation $\beta_1(C) \uparrow G$, which contains $\rho(G)$, and has the same degree. We conclude that $\beta_1(C)$ is of the second kind.

Finally, suppose that $\rho(G)$ is of the first kind. If $\beta_1(C)$ is of the third kind, then we find, as above, that $\beta_2(C) = \overline{\beta_1(C)}$, producing the same contradiction. If $\beta_1(C)$ is of the second kind, then (since $\beta_1(C)$ and $\beta_2(C)$ are inequivalent) $\beta_1(C)$ appears with multiplicity one in the real representation $\rho(G) \downarrow C$, another contradiction. Hence $\beta_1(C)$ is of the first kind in this case.

Our proof is now easily completed by induction. If $\rho(G)$ is of the first or third kind, then we may select a subgroup H (containing h_1) of C fixing a unique one-dimensional subspace of the carrier space of $\beta_1(C)$. Because of the presence of h_1 in H, this subgroup fixes no vector outside of this carrier space, and we conclude that $I(H)$ appears with multiplicity one in $\rho(G) \downarrow H$. A perfectly analogous argument shows that a subgroup H may be selected so that $I(H)$ appears with multiplicity two in $\rho(G) \downarrow H$ in the event that $\rho(G)$ is of the second kind.

Our next, and last, theorem concerns real representations of solvable groups. I speculate that this result holds without the assumption that the group is solvable. (For example, it holds if the degree of the representation is prime, or if the group belongs to the family of alternating or special abelian groups [**4**, p. 89].) However, the assumption that the degree is odd cannot be so easily discarded, as the example of the simple group of order 168 shows. Finally, every solvable group of even order has at least one non-identity representation with the specifications demanded by the theorem, so that the result is not an empty exercise.

THEOREM II. *Suppose that G is a solvable group and that $\rho(G)$ is a real, irreducible representation, of odd degree, of G. Then $\rho(G)$ possesses a vertex subgroup.*

Proof. We observe first that, since any homomorphic image of G is again solvable, we may assume, without loss of generality, that $\rho(G)$ is a faithful representation of G.

Since G is solvable, it possesses a non-trivial subgroup, H, which is elementary abelian, and normal in G [**3**, p. 23, Theorem 4.1]. We claim that H can be chosen to be non-cyclic, and argue as follows. If $|H| = 2$, then the element in H of order 2 belongs to the centre of G, and so is represented by $-I$, which has determinant -1, the degree of $\rho(G)$ being odd. Since $\rho(G)$ is of the first kind, each matrix has determinant 1 or -1, and those that have determinant 1 form a normal subgroup, G_1, of index 2 in G. Since G_1 does not contain H, we easily obtain $G = G_1 \times H$. We can now obtain a subgroup, H_1, of G_1, which is elementary abelian, and normal in G_1 (and hence in G). Further, we cannot have $H_1 \subseteq Z(G_1)$, for then $H_1 \subset Z(G)$ and so $|Z(G)| > 2$, which is impossible if $\rho(G)$ is to be irreducible, of the first kind. Thus we may assume that $|H| \neq 2$.

Suppose that $|H| = p$, p an odd prime, and let $\sigma(H)$ be an irreducible component of $\rho(G) \downarrow H$. $\sigma(H)$ is necessarily complex, of degree 1. Since $\rho(G)$ is real, we have

$$\sum_H \chi^\rho(g)\chi^\sigma(g) = \sum_H \chi^\rho(g)\chi^\sigma(g^{-1})$$

so that $\sigma(H)$ and $\overline{\sigma(H)}$ occur with equal multiplicity in $\rho(G) \downarrow H$. Since all the irreducible components of $\rho(G) \downarrow H$ occur in these complex conjugate pairs, we conclude that $\rho(G)$ has even degree, a contradiction.

We may now choose H to be non-cyclic. Now let $\sigma(H)$ be one of the irreducible components of $\rho(G) \downarrow H$. $\sigma(H)$ is of degree 1, and H is non-cyclic, so that the kernel, K, of $\sigma(H)$ is non-trivial. Let $N = N_G(K)$ be the normalizer of K in G. Let the matrices for $\rho(G) \downarrow H$ be exhibited in diagonal form, with the irreducible components, $\sigma_i(H)$, having kernel K, in the upper corner. The elements of G permute the $\sigma_i(H)$ among themselves, in the sense specified in the proof of Theorem I. Now if g is in N, g permutes the components with kernel K amongst themselves, so that the matrix for g in $\rho(G)$ takes the form $\begin{bmatrix} A(g) & 0 \\ 0 & B(g) \end{bmatrix}$, where the rank of A is equal to the number of irreducible components of $\rho(G) \downarrow H$ having kernel K. (Counting multiple appearances.) The matrices $A(g)$ give rise to a representation, $\beta(N)$, of N, with carrier space V_β.

On the other hand, if g' is in $G - N$, g' takes all the components $\sigma_i(H)$ with kernel K to components having a different kernel so that the matrix for g' in $\rho(G)$ has 0s in the place occupied by the $A(g)$. It follows, as in Theorem I, that $\beta(N)$ is irreducible. Now K is in the kernel of $\beta(N)$, while it is not in the kernels of any of the other irreducible components of $\rho(G) \downarrow N$.

If $\beta(N)$ is of the second kind, it must arise with even multiplicity in $\rho(G) \downarrow N$, since this representation of N is real [**2**, p. 62]. This is impossible, by the previous remark. Further, if $\beta(N)$ is of the third kind, the complex conjugate, $\overline{\beta(N)}$, of $\beta(N)$, appears in $\rho(G) \downarrow N$ as well, since χ^ρ is real. But these have the same kernel, and so this is impossible. Thus $\beta(N)$ is of the first kind.

We can also show that the degree of $\beta(N)$ is odd. To do this, we observe that the same argument used to show that $|H|$ is not an odd prime shows also that $|H|$ is not odd. That is, H is an elementary abelian 2-group, and so every irreducible representation of H is completely determined by its kernel, so that the degree of $\beta(N)$ is equal to the multiplicity of $\sigma(H)$ in $\rho(G) \downarrow H$. By Clifford's theorem, all the irreducible components of $\rho(G) \downarrow H$ have the same multiplicity, which is therefore necessarily odd. N, being a subgroup of G, is solvable, and so $\beta(N)$ satisfies all the assumptions made for $\rho(G)$ (except faithfulness) and so we may assert, by induction on $|G|$, that N has a subgroup, M, which is a vertex subgroup for $\beta(N)$. When we have shown that M is also a vertex subgroup for $\rho(G)$, our proof will be complete.

First, since K is in the kernel of $\beta(N)$, we may assume that $K \subseteq M$. If v is a vector held invariant by all of the matrices in $\rho(G) \downarrow M$, then v is held invariant also by all the matrices in $\rho(G) \downarrow K$. But the identity representation, $I(K)$, of K is obtained only in the upper corner of the diagonal matrices of $\rho(G) \downarrow K$, and so v must belong to V_β, the carrier space of $\beta(N)$. But then v must be fixed under all the matrices of $\beta(N)$ and so belongs to a unique subspace of dimension 1. Thus $I(M)$ appears in $\rho(G) \downarrow M$ with multiplicity one, and Theorem II is proved.

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