REPRESENTATIONS OF INFINITE SOLUBLE GROUPS

by IAN M. MUSSON

(Received 16 June, 1981)

1. Introduction. The purpose of this paper is to study the following two questions.

(1) When does the group algebra of a soluble group have infinite dimensional irreducible modules?

(2) When is the group algebra of a torsion free soluble group primitive?

In relation to the first question, Roseblade [**13**] has proved that if G is a polycyclic group and k an absolute field then all irreducible kG-modules are finite dimensional. Here we prove a converse.

THEOREM A. *Let G be a finitely generated hyperabelian group which is not polycyclic-by-finite. If k is any field, then kG has an infinite dimensional irreducible module.*

We remark that B. A. F. Wehrfritz has recently and independently proved a result similar to Theorem A [**17**].

Recently some progress has been made on the second question. For instance, Roseblade has solved the problem for polycyclic groups in [**14**], while in [**2**] Brown has treated the case of torsion free metanilpotent groups whose Fitting subgroup has infinite rank.

Here we study the finite rank case. In many situations a reduction to the case where the group G has an abelian normal subgroup A with G/A infinite cyclic, seems possible, for example [**4**, Theorem 4.6]. By a result of Passman [**11**, 9.3.2] the group algebra kG will always be primitive if $\Delta(G)=1$, and the field k is large enough. Thus we are concerned to find techniques which will work over any field.

We can adapt the definition of a plinth given on [**11**, p. 547] to this situation. If A is a torsion free abelian subgroup of a group G, and A has finite rank, we say A is a *plinth* in G if there is a subgroup G_0 containing A such that

(i) $A \trianglelefteq G_0$,

(ii) $|G:G_0|<\infty$ and $G_0/C_{G_0}(A)$ is abelian, and

(iii) G_0 and all its subgroups of finite index act rationally irreducibly on A.

If Γ is a group of automorphisms of the abelian group A, we say A is a Γ-plinth if A is a plinth in the split extension $G=A \lhd \Gamma$.

Suppose that A is an $\langle x\rangle$-plinth and $G=\langle A, x\rangle$. The problem of deciding when kG is primitive here splits into two cases. Suppose that A_0 is a finitely generated subgroup of A of maximum rank. If $A_0^{\langle x\rangle}/A_0$ is finite then we may assume $A_0=A_0^{\langle x\rangle}$. In this case G is locally polycyclic and the methods of [**14**] can be applied, at least if k is non-absolute (see Corollary 4.6). We study the contrary case where $A_0^{\langle x\rangle}/A_0$ is infinite.

THEOREM B. *Suppose that A is an $\langle x\rangle$-plinth such that $G=\langle A, x\rangle$ is not locally polycyclic. If k is any field, then kG is primitive.*

To prove Theorem B, we need a variant of Bergman's Theorem [**1**], [**11**, Chapter 9], for these groups. We prove that *if A is an $\langle x\rangle$-plinth and I is a non-zero ideal of kA such that $I^x = I$, then kA/I is algebraic* (Corollary 4.3). We next look for maximal ideals M of kA such that

$$\bigcap_{n\in\mathbb{Z}} M^{x^n} = 0.$$

This is closely related to the Ergodic Conjecture of Farkas and Passman [**4**] and is known to imply primitivity.

We set $M\dagger = A\cap(1+M)$.

THEOREM C. *Suppose that A is an $\langle x\rangle$-plinth such that $A = A_0^{\langle x\rangle}$ for a finitely generated subgroup A_0 of A and suppose $G=\langle A, x\rangle$ is not polycyclic-by-finite. If M is a maximal ideal of kA, then $\bigcap_{n\in\mathbb{Z}} M^{x^n}=0$ if and only if $A/M\dagger$ is infinite.*

If A/A_0 is a p-group and k is an absolute field of characteristic p, then kA has no maximal ideal M with $A/M\dagger$ infinite. However, in this case we deduce Theorem B from a result of Irving, [**8**, Theorem 4.2].

Examples of groups satisfying the hypotheses of Theorems B and C, may be constructed as follows. Let x be an algebraic number and A_0 a $\mathbb{Z}$-lattice in $\mathbb{Q}(x)$, A the $\mathbb{Z}\langle x\rangle$-module generated by A_0, and G the split extension of A by x. If we assume that x^n does not lie in any of the proper subfields of $\mathbb{Q}(x)$ for $n\geq 1$, then A will be an $\langle x\rangle$-plinth. If in addition we ensure that x is not a unit in the ring of algebraic integers of $\mathbb{Q}(x)$ then $G=\langle A, x\rangle$ is not locally polycyclic.

We remark that Irving has shown that for any integer $r>1$ the group algebra of the group presented by $\langle a, x \mid a^x = a^r\rangle$ is primitive over any field, [**8**, Theorem 5.2].

As it turns out, Theorem C provides more information about x-invariant ideals of kA. Given any non-zero x-invariant ideal I of kA, it is easily seen that $\sqrt{I}$ is a semiprime x-invariant ideal of kA, where $\sqrt{I}/I$ is the nil-radical of kA/I. Thus we may assume that I is semiprime. In general, a semiprime ideal of kA is an infinite intersection of prime ideals. However, Theorem C restricts the kinds of maximal ideals M which can contain I to those with $A/M\dagger$ finite.

We prove the following result, which is a sharper version of Corollary 4.3.

THEOREM D. *Let A be an $\langle x\rangle$-plinth and I a non-zero x-invariant semi-prime ideal of kA. Suppose that $A = A_0^{\langle x\rangle}$ where A_0 is finitely generated, and A is not finitely generated. Then $|A:I\dagger|<\infty$, and I is a finite intersection of maximal ideals.*

Theorem A is proved in §3 of this paper; Theorems B–D in §4. In §2, we prove some preliminary results about abelian groups which are needed later.

I am very grateful for the hospitality of the University of Alberta, where this work was carried out.

2. Abelian Groups of Finite Rank. Throughout this section A will denote a torsion free abelian group of finite rank and A_0 a finitely generated subgroup of maximum rank.

LEMMA 2.1. *If M is a maximal ideal of kA then kA/M is algebraic over k.*

Proof. Since kA is generated by elements which are integral over kA_0, kA is integral over kA_0. Hence by [**10**, Theorems 44 and 47], $M_0 = M \cap kA_0$ is a maximal ideal of kA_0. By the Nullstellensatz kA_0/M_0 is finite dimensional and the result follows since kA/M is algebraic over kA_0/M_0.

LEMMA 2.2. *Every maximal ideal M of kA satisfies $|A:M\dagger| < \infty$ if and only if k is an absolute field of characteristic p, and A/A_0 is a finite extension of a p-group.*

Proof. Suppose the stated conditions hold, and that M is a maximal ideal of kA. By Lemma 2.1 kA/M is an absolute field and $A/M\dagger$ is periodic, and so is a finite extension of a p-group. As k has characteristic $p, |A:M\dagger| < \infty$.

To see the necessity of the conditions, let $\bar{k}$ be the algebraic closure of k and $\phi: A \to (\bar{k})^*$ a homomorphism from A to the multiplicative group $(\bar{k})^*$ of $\bar{k}$, and let F be the subfield of $\bar{k}$ generated by k and $\phi(A)$. Then any element of F may be written as a finite sum $\sum_{x \in A} \lambda_x \phi(x)$ where the coefficients λ_x belong to k. We may regard F as a kA-module. Suppose that V is a non-zero submodule of F and $\alpha \in V, \alpha \neq 0$. Then $\alpha^{-1} = \sum_{x \in A} \lambda_x \phi(x) \in F$. For all x in this finite sum $\alpha\phi(x) \in V$. Therefore $1 = \alpha\alpha^{-1} = \sum \lambda_x \alpha\phi(x) \in V$. Hence F is an irreducible kA-module and $F \cong kA/M$ for some maximal ideal M of kA. Clearly $M\dagger = \operatorname{Ker} \phi$.

Now suppose ζ is an element of k which is not a root of unity and let a be a non-trivial element of A. The map $\langle a \rangle \to k^*$ given by $a \to \zeta$ extends to a homomorphism $\phi: A \to (k)^*$, since $(k)^*$ is divisible. As $\operatorname{Ker} \phi \cap \langle a \rangle = 1$, $A/\operatorname{Ker} \phi$ is infinite.

Finally suppose that $A/A_0 \cong P \oplus Q$ where P is a p-group and Q is an infinite p'-group. Since Q has finite rank, we see that either Q involves some $C_{q^\infty}, q \neq p$ or Q involves a direct sum of cyclic groups of prime orders p_i for infinitely many primes p_i. Since $(\bar{k})^*$ contains a copy of each of these groups this yields a homomorphism $\phi: A \to \bar{k}^*$ such that $A/\operatorname{Ker} \phi$ is infinite.

LEMMA 2.3. *Let x be an automorphism of A such that $A = A_0^{\langle x \rangle}$. If C is a subgroup of A with $A/C \cong C_{p^\infty}$, then there is an element a of A_0 whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of C in A.*

Proof. Let $A_0 = \langle a_1, a_2, \ldots, a_n \rangle$. There is no integer s such that $(a_i^{x^r})^{p^s} \in C$ for all $i = 1, \ldots, n$ and $r \in \mathbb{Z}$, since A/C has unbounded exponent, and $A = A_0^{\langle x \rangle}$ is generated by the conjugates $a_i^{x^r}$. Since there are only finitely many a_i, there is an element $a = a_i$ whose conjugates have unbounded order modulo C.

We introduce some notation. Let B be a subgroup of the abelian group A and p a prime. Let $\sqrt{B}$ be the *isolator* of B in A, that is $\sqrt{B}/B$ is the torsion subgroup of A/B. Similary we let $\sqrt[p]{B}/B$ and $\sqrt[p']{B}/B$ denote the p-torsion and p'-torsion subgroups of A/B respectively. We say B is *isolated*, *p-isolated* or *p'-isolated* according as $B = \sqrt{B}$, $B = \sqrt[p]{B}$ or $B = \sqrt[p']{B}$.

LEMMA 2.4. *Let x be an automorphism of A such that $A = A_0^{\langle x \rangle}$. Suppose $\sqrt[p]{A_0}/A_0$ is* *infinite. Then A has a subgroup B containing A_0 such that either*

(1) *$B^x \subsetneq B$ and B/B^x is a finite p-group, or*

(2) *$B \subsetneq B^x$ and B^x/B is a finite p-group.*

If in addition $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x\rangle$-module, then $\bigcap_{n\in\mathbb{Z}} B^{x^n} = 1$.

Proof. Let $A_1 = \sqrt[p']{A_0}$ and $A_2 = \bigcup_{n\geq 0} A_0 A_0^x A_0^{x^2} \ldots A_0^{x^n}$. Then $A_2^x \subseteq A_2$. If $a \in A_1$, then $a^m \in A_0$ for some integer m prime to p, so $(a^x)^m \in A_0^x \subseteq A_2 \subseteq A_1A_2$. Since A_1A_2 is p'-isolated we have $a^x \in A_1A_2$. Therefore $(A_1A_2)^x \subseteq A_1A_2$. If this inclusion is strict, then 1) holds with $B = A_1A_2$.

If $(A_1A_2)^x = A_1A_2$, then since $A_0^{x^{-1}}$ is finitely generated, there is an integer r such that

$$A_0^{x^{-1}} \subset A_1 A_0 A_0^x \ldots A_0^{x^r} = B.$$

Since B is p'-isolated this gives $B^{x^{-1}} \subset B$ as before. The inclusion is strict since $\sqrt[p]{A_0}/A_0$ is infinite, and $B/B^{x^{-1}}$ is a finite p-group, so 2) holds.

Finally, suppose that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x\rangle$-module and let $C = \bigcap_{n\in\mathbb{Z}} B^{x^n}$. Since $C^x = C$, either $C = 1$ or C has rank n.

If C has rank n then $C_0 = C \cap A_0$ has finite index in A_0. But then $A_0^{\langle x\rangle}/C_0^{\langle x\rangle}$ has finite exponent and is finite. This implies A/C is finite and so A/B is finite. This is inconsistent with either (1) or (2).

We remark that if $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x\rangle$-module, the conclusion of Lemma 2.4 is precisely what is required to apply Theorem 4.2 of Irving's paper [**8**]. This shows that kG is primitive, where $G = \langle A, x\rangle$ and k is any field of characteristic p.

3. Infinite Dimensional Irreducible Modules

LEMMA 3.1. *Let G be a polycyclic-by-finite group, k an absolute field and M a finitely generated kG-module of infinite dimension over k. Then there are elements $m \in M$ and $x \in G$ such that mkX is infinite dimensional where $X = \langle x\rangle$.*

Proof. Without loss of generality M is faithful for G and cyclic, and $M \cong kG/I$ where I is a right ideal of kG. Since G must be infinite, it has a torsion free abelian normal subgroup $A = \langle a_1, \ldots, a_n\rangle$. If $I \cap k\langle a_j\rangle = 0$ for some j then we can take $x = a_j$ and $m = 1 + I \in M$. Otherwise for all j, $k\langle a_j\rangle/(I \cap k\langle a_j\rangle)$ has finite dimension, and so $kA/(I \cap kA)$ is finite dimensional. Hence, $T = \{a \in A \mid a - 1 \in I \cap kA\}$ satisfies $|A : T| < \infty$ since k is absolute (see [**11**, 12.3.8]). Therefore, $|A : \mathrm{core}_G(T)| < \infty$ and since M is faithful A is finite, a contradiction.

Proof of Theorem A. Since G is finitely generated, and the class of polycyclic-by-finite groups is finitely presented, we can suppose that G is not polycyclic-by-finite but that every proper homomorphic image of G is polycyclic-by-finite by [**12**, Lemma 6.17]. Let A be a non-trivial abelian normal subgroup of G. Then G/A is polycyclic-by-finite and we may treat A as a $\mathbb{Z}(G/A)$-module.

Case 1. *A has infinite rank.* We can suppose that $A = \langle a\rangle^G$, where $a \in A$, $a \neq 1$. Hence,

A is a cyclic $\mathbb{Z}\,G$-module. Suppose that A is $\mathbb{Z}$-torsion-free. By [**11**, Theorem 12.2.7] there is a free abelian subgroup A_0 of A such that A/A_0 is a π-torsion group where π is a finite set of primes. If $p\notin\pi$, then $A^p\cap A_0=A_0^p$ and so $A_0/A_0^p=A_0/(A^p\cap A_0)\cong(A_0A^p)/A^p\subseteq A/A^p$. Now as A is $\mathbb{Z}$-torsion free and has infinite rank, A_0 also has infinite rank. Therefore, A_0/A_0^p is an infinite elementary abelian p-group and hence so is A/A^p. Therefore, G/A^p is not polycyclic-by-finite. Our assumption on homomorphic images gives $A^p=1$, a contradiction since A is $\mathbb{Z}$-torsion free.

Hence, for some prime p, $A_p=\{a\in A\mid a^p=1\}\neq 1$. By assumption, G/A_p is polycyclic-by-finite. Hence A_p is an infinite elementary abelian p-group. We can regard A_p as an infinite dimensional $\mathbb{F}_pG/A$-module. By Lemma 3.1 there are elements $a\in A_p$ and $x\in G$ such that $a\mathbb{F}_pX$ is infinite dimensional, where $X=\langle x\rangle$. It follows that the elements $\{ax^n\mid n\in\mathbb{Z}\}$ are linearly independent over $\mathbb{F}_p$.

Hence $\langle a\rangle^X\cong\bigoplus_{n\in\mathbb{Z}}a^{x^n}$, and $H=\langle a,x\rangle\cong C_p\sim C_\infty\leq G$. Now by [**11**, Lemma 9.2.8] kH is primitive for any field k. Therefore, kH has an infinite dimensional irreducible module and so does kG.

Case 2. *A has finite rank.* Clearly A is not finitely generated. We first reduce to the case where $G=\langle A,x\rangle$ is abelian-by-(infinite cyclic). Let W be an irreducible $\mathbb{Q}G$-submodule of $A\otimes_{\mathbb{Z}}\mathbb{Q}$ and $B=A\cap W$. Then B is finitely generated as a $\mathbb{Z}G$-module but not as an abelian group since G/B is polycyclic-by-finite. We can suppose that $A=B$, so that A is rationally irreducible. By passing to a subgroup of finite index we can suppose $G/C_G(A)=\langle x_1,\ldots,x_r\rangle$ is abelian. It follows that there is an element a of A and $x=x_i$ such that $\langle a\rangle^{\langle x\rangle}$ is not finitely generated as an abelian group. Now the group $\langle a,x\rangle$ is finitely generated and metabelian but not polycyclic-by-finite so we can suppose that $G=\langle a,x\rangle$ and $A=\langle a\rangle^G$. As before $A\otimes_{\mathbb{Z}}\mathbb{Q}$ is an irreducible $\mathbb{Q}\langle x\rangle$-module. Let A_0 be a finitely generated subgroup of A of maximum rank such that $A=A_0^{\langle x\rangle}$. Let k be a field of characteristic $p\geq 0$. There are now two subcases depending on the structure of A/A_0.

Suppose first that $\sqrt[p]{A_0}/A_0$ is infinite. Then by Lemma 2.4 and [**8**, Theorem 4.2] kG is primitive.

Finally suppose $\sqrt[p]{A_0}/A_0$ is finite. Since A/A_0 involves only finitely many primes, there is a maximal ideal M of kA such that $A/M^\dagger\cong C_{q^\infty}$ where $q\neq p$, by Lemma 2.2. Let $V=kA/M$ and $W=V\otimes_{kA}kG$. We claim that W is an irreducible kG-module. Let $C=M^\dagger$ and using Lemma 2.3 choose an element $a\in C$ whose conjugates $\{a^{x^r}\mid r\in\mathbb{Z}\}$ lie in infinitely many distinct cosets of C in A. Then for all $r\geq 1$, $C^{\langle x^r\rangle}/C$ is infinite and so $C^{\langle x^r\rangle}=A$. Suppose that W' is a non-zero submodule of W and that $w=v_0+v_1x+\ldots+v_rx^r\in W'$ where $v_i\in V$, with $v_0\neq 0$ and $v_r\neq 0$. If $r\geq 1$, then as $C^{\langle x^r\rangle}=A$ there is an element c of C such that either $x^rcx^{-r}\notin C$ or $x^{-r}cx^r\notin C$. In the first case $v_0c=v_0$ and $v_rx^rc\neq v_rx^r$ while in the second $v_0x^{-r}cx^r\neq v_0$ and $v_rx^r(x^{-r}cx^r)=v_rx^r$. A standard 'shortest length' argument shows that $W'\cap V\neq 0$. Hence, $V\subseteq W'$ since V is an irreducible kA-module. However, W is generated by the conjugates of V under x so $W'=W$. This completes the proof of Theorem A.

Remarks. (1) Part of the motivation for Theorem A lies in obtaining a converse to the result of Jategaonkar [**9**] that a finitely generated abelian-by-polycyclic group is

https://doi.org/10.1017/S0017089500005048 Published online by Cambridge University Press

residually finite. For suppose that the group G satisfies the hypothesis of Theorem A, then there is an irreducible $\mathbb{F}_pG$-module A such that the split extension $A \lhd G$ is finitely generated but not residually finite.

(2) We might ask, as in [**3**, Questions 2 and 9] for general conditions under which all irreducible kG-modules are finite dimensional over k. At present only special cases are known. For countable locally finite groups with no elements of order p see [**5**], [**7**] and for soluble groups over $\mathbb{C}$, see [**15**]. If G is a finitely generated linear group, then by [**16**, Theorem 10.16], G is either soluble-by-finite or contains a non-cyclic free subgroup. Since the group algebra of a free group is primitive, it follows from Theorem A that if all the irreducible kG-modules are finite dimensional then G is polycyclic-by-finite.

Finally, we remark that if G is a finitely generated infinite p-group, and k a field of characteristic p, it is not known whether $J(kG) = \omega kG$, see discussion on page 415 of [**11**]. If this is the case then the only irreducible kG-module is the trivial one.

4. Primitive Group Algebras. Throughout this section A will denote a torsion-free abelian group of finite rank and I an ideal of kA, $I \neq kA$. We first outline how the results of Bergman [**1**] may be extended to this situation. With minor changes we can follow the proof given in [**11**, Chapter 9] using log subgroups. A *log subgroup* for I is a subgroup W of A such that for all $\alpha \in I$, $\alpha \neq 0$ there exist $x, y \in \operatorname{Supp} \alpha$ with $xy^{-1} \in W$ and $xy^{-1} \neq 1$. Whenever these subgroups appear as kernels of valuations as in [**11**, Lemma 9.3.5] they are necessarily isolated in A since the value group is torsion-free. Thus we can work with isolated log subgroups.

LEMMA 4.1. *Let A be torsion-free abelian of finite rank and $\mathcal{F} = \{F_\alpha \mid \alpha \in I\}$ a collection of finite non-empty subsets of A. Let $\mathcal{W} = \mathcal{W}(\mathcal{F})$ be the set of all isolated subgroups W of A such that $W \cap F_\alpha \neq \varnothing$ for all α. Then every member of $\mathcal{W}$ contains a minimal member, and $\mathcal{W}$ has only finitely many minimal members.*

Proof. Note that if $\bar{A}$ is any proper torsion-free image of A, then $\bar{A}$ has smaller rank than A. Hence we can use induction on rank(A). If rank $(A) = 0$ then $A = \langle 1 \rangle$ while if rank$(A) = 1$ the only isolated subgroups of A are $\langle 1 \rangle$ and A and the result is trivial in these cases. If $1 \in F_\alpha$ for all α, then $\langle 1 \rangle$ is the unique minimal member of $\mathcal{W}$. Suppose that for some β, $1 \notin F_\beta = \{a_1, a_2, \ldots, a_n\}$. Let $\mathcal{W}_i = \{W \in \mathcal{W} \mid a_i \in W\}$. Then we have $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \ldots \cup \mathcal{W}_n$ and it suffices to show that each $\mathcal{W}_i$ has only finitely many minimal members and that any member of $\mathcal{W}_i$ contains a minimal member.

Consider $\mathcal{W}_1$. Let $A_1 = \sqrt{\langle a_1 \rangle}$ the isolator of $\langle a_1 \rangle$ in A, and let $^-$ denote the natural homomorphism $A \to A/A_1 = \bar{A}$. If $W \in \mathcal{W}_1$ then $W \cap F_\alpha \neq \varnothing$ for all α and so $\bar{W} \cap \bar{F}_\alpha \neq \varnothing$. Moreover, if $W_1, W_2 \in \mathcal{W}_1$ with $\bar{W}_1 = \bar{W}_2$, then since $a_1 \in W_1, W_2$ and the W_i are isolated, $A_1 \subseteq W_1 \cap W_2$ and hence $W_1 = W_2$.

Conversely, if $\bar{W} \in \mathcal{W}(\bar{\mathcal{F}})$ and W denotes the full inverse image of $\bar{W}$ under $^-$ then W is isolated in A, since $\bar{W}$ is isolated in $\bar{A}$. Hence $^-$ gives a one-one correspondence between members of $\mathcal{W}_1$ and $\mathcal{W}(\bar{\mathcal{F}})$ and the result follows by induction.

Let I be an ideal of kA. We define the rank of I by

$$\operatorname{rank}(I) = \operatorname{rank}(A) - \max\{\operatorname{rank}(B) \mid I \cap kB = 0\}.$$

The proof of the next theorem and the corollary may now be adapted from [**11**, Chapter 9].

THEOREM 4.2. *Let A be a torsion-free abelian group of finite rank, and $I \neq kA$ an ideal of kA. Then any isolated log subgroup for I contains a minimal isolated log subgroup, and I has only finitely many minimal isolated log subgroups $W_1, W_2, \ldots, W_m$. Furthermore, $\mathrm{rank}(I) = \min\{\mathrm{rank}(W_i)\}$. Finally if α is any automorphism of A, then $W_1^\alpha, W_2^\alpha, \ldots, W_m^\alpha$ are the minimal isolated log subgroups for I^α.*

COROLLARY 4.3. *Let A be torsion-free abelian of finite rank, and Γ a group of automorphisms of A. Suppose that A is a Γ-plinth. If I is a Γ-invariant ideal of kA, then either $I = 0$ or kA/I is algebraic over k, that is given $a \in A$, there is a polynomial $f(t) \in k[t]$ such that $f(a) \in I$.*

We remark that in the situation of the corollary, any prime ideal of kA containing I must be a maximal ideal.

In proving Theorem B, we may assume that $A = A_0^{\langle x \rangle}$ where A_0 is finitely generated.

LEMMA 4.4. *Let A_0 be a finitely generated subgroup of A of maximum rank. Let x be an automorphism of A such that no power of x centralises A and $A_1 = A_0^{\langle x \rangle}$, $G = A \lhd \langle x \rangle$ and $G_1 = A_1 \lhd \langle x \rangle$.*

(*i*) *Suppose there is a maximal ideal M of kA_1 such that $\bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$, then there is a maximal ideal N of kA such that $\bigcap_{n \in \mathbb{Z}} N^{x^n} = 0$.*

(*ii*) *If kG_1 is primitive then kG is primitive.*

Proof. (*i*) Choose a maximal ideal N of kA containing M, by Zorn's Lemma. Then $M = N \cap kA_1$. If $I = \bigcap_{n \in \mathbb{Z}} N^{x^n} \neq 0$, then $I \cap kA_1 \neq 0$, since kA is a domain which is integral over kA_1. However, $I \cap kA_1 = \bigcap_{n \in \mathbb{Z}} M^{x^n} = 0$. Hence $I = 0$.

(*ii*) This follows by a similar argument using the fact that any non-zero ideal of kG has non-zero intersection with kA, [**11**, Lemma 7.4.9].

As in [**4**] the primitivity problem for torsion free soluble groups of finite rank may often be reduced to the abelian-by-(infinite cyclic) case.

In the next result and the corollary, A will denote the Zalesskii subgroup of the group G (see [**11**, section 9.1]). If I is a non-zero ideal of kG then $I \cap kA \neq 0$.

LEMMA 4.5. *If G is a torsion-free soluble group of finite rank such that $\Delta(G) = 1$, then $\Delta(\langle A, x \rangle) = 1$ for some $x \in G$.*

Proof. This is essentially the same as the proof of [**4**, Lemma 4.5]. We merely indicate the modifications that must be made. First we note that for G torsion-free soluble of finite rank, G is nilpotent-by-finite if and only if G is f.c. hypercentral. The crucial point here is that if $B = \Delta(G)$ and B_0 is a finitely generated subgroup of G such that B/B_0 is torsion, then $G_0 = C_G(B_0)$ has finite index in G and G_0 centralizes B since extraction of roots in B is unique. It now follows as in Lemma 4.2 of [**4**] that $H = G/C_G(A)$ is abelian-by-finite and we may suppose this group is actually abelian. The action of G on A by conjugation induces a representation of H as matrices with rational entries. If the result is false, then by Lemmas 3.2 and 4.4 of [**4**] each matrix in H has a complex eigenvalue which is a root of unity.

let $V_0 = A \otimes_{\mathbb{Z}} \mathbb{Q}$, $V = A \otimes_{\mathbb{Z}} \mathbb{C}$. By Lemma 4.3 of [**4**] there is a finitely generated subgroup H_0 of H with H/H_0 torsion such that H_0 acts trivially on a non-zero subspace W of V. We can assume that

$$W = V^{H_0} = \{v \in V \mid vh = v \quad \text{for all} \quad h \in H_0\}.$$

Then W is a submodule of V. Also by choosing a basis of $\mathbb{C}$ over $\mathbb{Q}$ we see that $W = W_0 \otimes_{\mathbb{Q}} \mathbb{C}$ where $W_0 = V_0^{H_0}$. If $H_1 = C_H(W_0)$ then H/H_1 is isomorphic to a periodic group of matrices over the rational numbers. Hence H/H_1 is finite, see [**12**, Part I, p. 85], so there is an integer $k > 0$ such that $G^k \subseteq C_G(W_0)$. The proof may now be completed as in [**4**, Lemma 4.5].

COROLLARY 4.6. *Let G be a torsion-free soluble group of finite rank with $\Delta(G) = 1$ and suppose that G is locally polycyclic. If k is a non-absolute field, then kG is primitive.*

Proof. By Lemma 4.5 there is an element x in G such that $H = \langle A, x \rangle$ satisfies $\Delta(H) = 1$. It suffices to show that kH is primitive. Since k is non-absolute there is an irreducible kA-module V which is faithful for A. It is easy to see that the induced module $V^H = V \otimes_{kA} kH$ is irreducible since A is self-centralising in H. Let $I = \operatorname{ann}_{kH}(V^H)$. If $I \neq 0$, then $I \cap kA$ is a non-zero x-invariant ideal of kA. Since H is locally polycyclic there is a finitely generated x-invariant subgroup A_0 of A such that A/A_0 is torsion. Then by the argument of Lemma 4.4 $J = I \cap kA_0 \neq 0$ and J is a non-zero x-prime ideal of kA_0 with $J^\dagger = 1$. This contradicts [**14**, Theorem D].

If $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x \rangle$-module we can extend Lemma 2.3.

LEMMA 4.7. *Let x be an automorphism of A such that $A = A_0^{\langle x \rangle}$ for a finitely generated subgroup A_0. Suppose that A itself is not finitely generated and that $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}\langle x \rangle$-module. If B is a subgroup of A with A/B infinite then there is an element a of A_0 whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of B in A.*

Proof. By Lemma 2.3, it is enough to show that A/B has an image isomorphic to C_p^∞ for some prime p. If $\sqrt{B}/B$ is infinite, we observe that $\sqrt{B}/B$ can involve only finitely many primes, so this follows from the structure of Černikov groups, [**6**, Theorem 19.2]. If $\sqrt{B}/B$ is finite we can suppose that $B = \sqrt{B}$. If in addition A/A_0B is infinite the result follows again since A/A_0 is Černikov.

Suppose that A/A_0B is finite. Then A/B is finitely generated and since B is isolated

$$A = B \times C \quad \text{with} \quad C \cong A/B$$

by [**6**, Corollary 25.3]. Since A/B is infinite, $\operatorname{rank}(B) < \operatorname{rank}(A)$, and so $\bigcap_{n \in \mathbb{Z}} B^{x^n} = 1$. In fact, some finite intersection is trivial. To see this note that each finite intersection $B^{x^{n_1}} \cap B^{x^{n_2}} \cap \ldots \cap B^{x^{n_r}}$ is isolated in A and that A has the minimum condition on isolated subgroups. Therefore $B^{x^{n_1}} \cap B^{x^{n_2}} \cap \ldots \cap B^{x^{n_r}} = 1$ for finitely many conjugates of B. Hence A embeds in $A/B^{x^{n_1}} \times A/B^{x^{n_2}} \times \ldots \times A/B^{x^{n_r}}$ which is finitely generated. This contradiction shows that A/A_0B cannot be finite if B is isolated.

Proof of Theorem C. We have $A = A_0^{\langle x\rangle}$ where A_0 is a finitely generated subgroup of A, and A is an $\langle x\rangle$-plinth. Let M be a maximal ideal of kA. We must show that $\bigcap_{n\in\mathbf{Z}} M^{x^n} = 0$ if and only if $A/M\dagger$ is infinite.

If $|A:M\dagger| = n < \infty$, let $B = \{a^n \mid a \in A\}$ then $B \subseteq M\dagger$, and since B is characteristic $\bigcap_{n\in\mathbf{Z}} M^{x^n}$ contains ωkB and so is non-zero. Now suppose that $A/M\dagger$ is infinite and using Lemma 4.7 choose an element a of A whose conjugates $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of $M\dagger$ in A. If $I = \bigcap_{n\in\mathbb{Z}} M^{x^n} \neq 0$, then by Corollary 4.3, there is a non-zero polynomial $f(t) \in k[t]$ such that $f(a) \in I$. Since $I^x = I$ we have $f(a^{x^r}) \in I$ for all $r \in \mathbb{Z}$. However, since the elements $\{a^{x^r} \mid r \in \mathbb{Z}\}$ lie in infinitely many distinct cosets of $M\dagger$, they represent infinitely many distinct elements of the field kA/M. In other words the non-zero polynomial $f(t)$ has infinitely many distinct roots in some extension field. This contradiction shows that $I = 0$.

Proof of Theorem B. We have a group G of the form $G = \langle A, x\rangle$ where A is an $\langle x\rangle$-plinth, and G is not locally polycyclic. We have to show that kG is primitive for all fields k.

By Lemma 4.4, we may assume that $A = A_0^{\langle x\rangle}$ where A_0 is a finitely generated subgroup of A. Let k be a field of characteristic $p \geq 0$. If $\sqrt[p]{A_0}/A_0$ is infinite than kG is primitive by Lemma 2.4 and [8, Theorem 4.2]. On the other hand, if $\sqrt[p]{A_0}/A_0$ is finite, then since A/A_0 is infinite, it is not a finite extension of a p-group. Therefore, by Lemma 2.2 there is a maximal ideal M of kA such that $A/M\dagger$ is infinite. Hence, $\bigcap_{n\in\mathbf{Z}} M^{x^n} = 0$ by Theorem C and again kG is primitive.

Proof of Theorem D. We have an $\langle x\rangle$-plinth A and a non-zero x-invariant semiprime ideal I of kA. We have to show that $A/I\dagger$ is finite and that I is a finite intersection of maximal ideals.

Corollary 4.3 shows that kA/I is algebraic so any prime ideal containing I is maximal. Let a be an element of A and $f(t) \in k[t]$ a polynomial such that $f(a) \in I$. For a maximal ideal M containing I we have $|A:M\dagger| < \infty$ by Theorem C. Let n be the least integer such that $a^n - 1 \in M$, then $\Phi_n(a) \in M$ where $\Phi_n(t)$ is the nth cyclotomic polynomial. Note that n is prime to the characteristic of k. If $g(t)$ is a polynomial of least degree such that $g(a) \in M$, then $g(t)$ is irreducible, and $g(t)$ divides $\Phi_n(t)$ and $f(t)$ since $\Phi_n(a)$ and $f(a)$ belong to I. Therefore, for each such integer $n, \Phi_n(t)$ and $f(t)$ have a common factor. Since the cyclotomic polynomials $\Phi_n(t)$ are relatively prime we obtain only finitely many integers n as M ranges over all maximal ideals containing I. Hence, $a^m - 1 \in I$, for some integer m so that $A/I\dagger$ is periodic. If $I_0 = I\dagger \cap A_0$ then A_0/I_0 is finite, and so $A_0^{\langle x\rangle}/I_0^{\langle x\rangle}$ has finite exponent so is finite. However, $I_0^{\langle x\rangle} \subseteq I\dagger$ since the latter is $\langle x\rangle$-invariant and contains I_0. Therefore, $A/I\dagger$ is finite and $\dim_k kA/I$ is finite. Therefore, there are only finitely many maximal ideals containing I, and I is their intersection.

REFERENCES

1. G. M. Bergman, The logarithmic limit set of an algebraic variety, *Trans. Amer. Math. Soc.* **157** (1971), 459–469.

2. K. A. Brown, Primitive group rings of soluble groups, *Arch. Math.* (*Basel*) **36** (1981), 404–413.

3. D. R. Farkas, Group rings, an annotated questionaire, *Comm. Alg.* **8** (1980), 585–602.

4. D. R. Farkas and D. S. Passman, Primitive Noetherian group rings, *Comm. Alg.* **5** (1978), 301–315.

5. D. R. Farkas and R. L. Snider, Group algebras whose simple modules are injective, *Trans. Amer. Math. Soc.* **194** (1974), 241–248.

6. L. Fuchs, *Abelian groups* (Pergamon, 1960).

7. B. Hartley, Injective modules over group rings, *Quart. J. Math.* **28** (1977), 1–29.

8. R. S. Irving, Some more primitive group rings, *Israel J. Math.* **37** (1980), 331–350.

9. A. V. Jategaonkar, Integral group rings of polycyclic-by-finite groups, *J. Pure and Applied Algebra* **4** (1974), 337–343.

10. I. Kaplansky, *Commutative rings* (University of Chicago Press, 1974).

11. D. S. Passman, *The algebraic structure of group rings* (Wiley-Interscience, 1977).

12. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, parts 1 and 2 (Springer-Verlag, 1972).

13. J. E. Roseblade, Group rings of polycyclic groups, *J. Pure and Applied Algebra* **3** (1974), 307–328.

14. J. E. Roseblade, Prime ideals in group rings of polycyclic groups, *Proc. London Math. Soc.* **36** (1978), 385–447.

15. R. L. Snider, Soluble groups whose representations are finite dimensional, *Abstracts Amer. Math. Soc.* (1980), Abstract 775-A6, page 250.

16. B. A. F. Wehrfritz, *Infinite linear groups* (Springer-Verlag, 1973).

17. B. A. F. Wehrfritz, Groups whose irreducible representations have finite degree I, *Math. Proc. Cambridge Philos. Soc.* **90** (1981), 411–421.

University of Wisconsin-Madison
Madison, WI 53706
USA