CIRCLES OF NUMBERS

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Introduction. Arrange any n integers around a circle. The following procedure can be used to obtain another circle of n integers. For each adjacent pair of the first integers, form the absolute value of their difference and place it between them; then remove the original numbers. This procedure can be repeated over and over. When $n=4$ this always leads eventually to a circle of zeros. On the other hand when $n=3$, unless the original numbers are equal, this never happens. We treat below the general case and related problems, using for convenience a slightly different formulation. Surprisingly there is enough structure to lead to some interesting mathematics.

DEFINITIONS. Let $C=(x_i)_{i=1}^{\infty}$ be an infinite sequence of numbers. Let $\Psi(C)=(x_i')_{i=1}^{\infty}$ where $x_i'=|x_{i+1}-x_i|$ for all i. We call C an *n-cycle* if and only if $x_{i+n}=x_i$ for all i. We shall also represent an n-cycle as an n-tuple $(x_1, x_2, \ldots, x_n)$ when it is convenient.

REMARKS.

1. If C is an n-cycle so is $\Psi(C)$.
2. Any n-cycle is also a kn-cycle (k any natural number).
3. After one application of Ψ all x_i's are non-negative.
4. Let $m(C)=\max_i(x_i)$ (which exists for n-cycles). Then $m(\Psi(C))\leq m(C)$ if all entries in C are non-negative.

DEFINITION. C will be called *repeating* if and only if $\Psi^k(C)=C$ for some k. The smallest such k will be called the *period* of C.

5. The 0-cycle $(0, 0, \ldots, 0)$ is the only repeating cycle with period 1.

6. In view of Remark 4, every n-cycle made up of integers (or rational numbers) must eventually, under repeated application of Ψ, be reduced to a repeating one. This is not always true for cycles made up of irrational numbers (for example, if $C=(1, r, r^2, \ldots, r^n)$ where $r^n=1+r+r^2+\ldots+r^{n-1}$, then $\Psi^t(C)=(r-1)^tC$, $t=1, 2, \ldots$).

7. Observe that $\Psi(F_n, F_{n+1}, F_{n+2})=(F_{n-1}, F_n, F_{n+1})$, where F_n is the n-th Fibonacci number. The 3-cycle (F_n, F_{n+1}, F_{n+2}) requires n applications of Ψ to reach a repeating cycle. Note also that $F_n=5^{-\frac{1}{2}}(t^n-s^n)$ where t and s are respectively the positive and negative roots of $x^2=x+1$ (see Remark 6).

DISCUSSION AND DEFINITIONS. We are concerned primarily with n-cycles made up of integers, and with what happens to such cycles after repeated application of Ψ. In view of Remark 3 we may as well assume that all entries are non-negative (so Remark 4 applies). We are interested in knowing which cycles lead finally to the 0-cycle. These will be called *terminating* cycles, and all others will be called *non-terminating*.

DEFINITION. A cycle made up only of 0's and 1's will be called a *primitive* cycle.

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THEOREM 1. *Every repeating n-cycle is a constant multiple of a primitive n-cycle.*

Proof. By Remark 4, it is clear that $m(C)=m(\Psi^t(C))$, $t=1,2,\ldots$, for any repeating cycle C. Thus C is made up entirely of non-negative numbers $\leq m$, where $m=m(C)$. Furthermore there must be at least one occurrence of m in C. Now let C_1 be C's unique repeating preimage under Ψ ($C_1=\Psi^{k-1}(C)$, where k is the period of C). Since $m(C_1)=m$, the two entries of C_1 whose absolute difference is the entry m of C must be $0, m$ (or $m, 0$). Similarly if C_2 is the repeating preimage of C_1, the three entries of C_2 whose absolute difference give us $0, m$ (or $m, 0$) must be taken from the set $\{0, m\}$. If we continue working backwards in this way through the repeating preimages of C, we get that the n-th preimage C_n of C has a string of n successive entries all taken from $\{0, m\}$ (with m occurring at least once). Since C_n is an n-cycle it has to be a constant multiple of a primitive cycle. The proof is completed by observing that $C=\Psi^n(C_n)$.

THEOREM 2. *Let C be a primitive n-cycle, $n=2^kL$ where L is odd.*

(i) *C is terminating if and only if it is a 2^k-cycle.*

(ii) *C is repeating if and only if it (as an n-tuple vector over $\mathbf{Z}_2$) is orthogonal to every primitive terminating n-cycle.*

Proof. Observe that if $C=(x_i)_{i=1}^{\infty}$ then $|x_{i+1}-x_i|\equiv x_{i+1}+x_i \pmod 2$. Thus Ψ becomes a linear operator on the n-dimensional vector space (over $\mathbf{Z}_2$) of primitive n-cycles. Also notice that $\Psi=I+E$ where $E(x_1, x_2,\ldots)=(x_2, x_3,\ldots)$. Thus $\Psi^r=(I+E)^r=\sum_{j=0}^{r}\binom{r}{j}E^j$ and therefore $\Psi^r(C)=(y_{ri})_{i=1}^{\infty}$ where $y_{ri}=\sum_{j=0}^{r}\binom{r}{j}x_{i+j}$ (addition being over $\mathbf{Z}_2$). So C is terminating if and only if $\sum_{j=0}^{r}\binom{r}{j}x_{i+j}=0$ for some r and all i. We may assume, without loss of generality, that r is a power of 2, in which case $\binom{r}{0}$ and $\binom{r}{r}$ are odd and all other $\binom{r}{j}$ are even ($\equiv 0$ in $\mathbf{Z}_2$). Thus C is terminating if and only if $x_i=x_{i+r}$ for some r (a power of 2) and for all i. This is equivalent to C being a r-cycle which, since C is already a 2^kL-cycle, is equivalent to C being a 2^k-cycle.

For the second part of the theorem notice that $E(u).v=u.E^{-1}(v)$ (for any vectors u and v), so the fact that E leaves invariant the subspace T of primitive n-cycles implies that E leaves invariant its orthogonal complement R (all n-cycles orthogonal to T). Thus $\Psi=I+E$ must also preserve R. But T already contains the kernel of Ψ so Ψ must be a nonsingular linear transformation when restricted to R. Thus R must consist entirely of repeating n-cycles. Since R is the orthogonal complement of T any element not in R is representable (uniquely) as the sum of a non-zero vector of T and an element of R. Since this sum under repeated application of Ψ is reduced to an element of R, all repeating (primitive) n-cycles lie in R and the proof is finished.

Observe that in the proof of Theorem 2 we have also established that any primitive n-cycle can be uniquely represented as the sum of a repeating and a terminating n-cycle.

THEOREM 3. (i) *If n is a power of 2 then every rational n-cycle is terminating.*

(ii) *If n is odd then the only terminating rational n-cycles are the trivial ones (those with all entries equal).*

(iii) *If n is neither a power of 2 nor odd, there are non-terminating rational n-cycles as well as non-trivial terminating rational n-cycles.*

Proof. Every rational n-cycle will eventually be reduced to a repeating one (Remark 6) which must be a constant multiple of a primitive cycle by Theorem 1. If n is a power of 2 then (by Theorem 2) every primitive n-cycle is terminating, hence so is every n-cycle. If n is odd the only primitive terminating n-cycles (by Theorem 2) are $(m, m, \ldots, m)$ where $m=0$ or $m=1$. A simple parity argument shows that $(m, m, \ldots m)$ has no Ψ-preimage for any $m>0$. If n is neither a power of 2 nor odd then $n=2^kL$ where $k\geq 1$ and L is an odd number greater than 1. Then there must be terminating 2^k-cycles (which are n-cycles by Remark 2) and non-terminating L-cycles (which are n-cycles).

DEFINITION AND DISCUSSION. If $n=2$ it is easy to see that each n-cycle terminates in two Ψ-steps. One might conjecture that the terminating n-cycles which (by Theorem 3) exist for every even $n>2$ are all as trivial (vanishing in only a few Ψ-steps). That is not the case. Let the *life-span* of an n-cycle be the minimum number of Ψ-steps required to make it a repeating cycle. It turns out that both the terminating and non-terminating n-cycles whose existence is predicted by Theorem 3 can be specified to have an arbitrary long life-span in every case (except for $n=2$). These remarks are an easy consequence of the following theorem.

THEOREM 4. *Let C be an n-cycle $(n>2)$. Then there exists another n-cycle C_1 and $k>0$ such that $k.\Psi(C)=\Psi^2(C_1)$.*

Note that C_1 has a longer life-span than C but that they are of the same type (either terminating or non-terminating).

Before proving Theorem 4 we shall need a few lemmas and definitions.

LEMMA 1. *Let $C=(x_1, x_2, \ldots, x_n)$ be an n-cycle. Then C has a Ψ-preimage if and only if there exists $A\subseteq\{1,2,\ldots,n\}$ for which $\sum_{i\in A} x_i=\sum_{i\notin A} x_i$.*

Proof. If $C=\Psi(y_1, y_2, \ldots, y_n)$ let $A=\{i\,|\,x_i=y_i-y_{i+1}\}$. Then

$$\sum_{i\in A} x_i=\sum_{i\in A}(y_i-y_{i+1}) \quad \text{and} \quad \sum_{i\notin A} x_i=\sum_{i\notin A}(y_{i+1}-y_i).$$

So

$$\sum_{i\in A} x_i-\sum_{i\notin A} x_i=\sum_{i=1}^{n}(y_i-y_{i+1})=0,$$

since $y_{n+1}=y_1$. Conversely, if there exists such a set A, let $y_1=\sum_{i\in A} x_i$, and $y_{i+1}=y_i+\delta_i x_i$ where $\delta_i=1$ if $i\in A$ and $\delta_i=-1$ if $i\notin A$. Then $C=\Psi(y_1, y_2, \ldots, y_n)$.

DEFINITION. If $C=(x_1, x_2, \ldots, x_n)$ is an n-cycle, a *weak predecessor* of C is an n-cycle C_1 for which $\Psi(C_1)=(ax_1+b, ax_2+b, \ldots, ax_n+b)$, where a, b are integers, $a>0$ (Note that in this case C, C_1 satisfy the equation of Theorem 4).

LEMMA 2. *Let $C=(x_1, x_2, \ldots, x_n)$, $n>2$. Rearrange the x's into a non-increasing list $y_1 \geq y_2 \geq \ldots \geq y_n$ and assume that $y_n=0$. Then C has a weak predecessor if and only if $\sum_{i=1}^{s} y_i \geq \sum_{i=s+1}^{n} y_i$, where $s=[(n-1)/2]$.*

Proof. If C has a weak predecessor then by Lemma 1 there exists $A \subsetneq \{1, 2, \ldots, n\}$, $a>0$ and b such that

$$\sum_{i \in A} (ay_i+b) = \sum_{i \notin A} (ay_i+b). \tag{1}$$

Note that $b \geq 0$ because $y_n=0$ and all ay_i+b are non-negative by Remark 3. We can assume without loss of generality that $|A| \leq n-|A|$ (or replace A by its complement in $\{1, 2, \ldots, n\}$). Let $t=|A|$ and let $B=\{1, 2, \ldots, t\}$. Then

$$\sum_{i \in B} (ay_i+b) \geq \sum_{i \in A} (ay_i+b) = \sum_{i \notin A} (ay_i+b) \geq \sum_{i \notin B} (ay_i+b).$$

Thus $a\left(\sum_{i \in B} y_i\right)+bt \geq a\left(\sum_{i \notin B} y_i\right)+b(n-t)$, and since $t \leq n-t$ and $b \geq 0$ and $a>0$, $\sum_{i \in B} y_i \geq \sum_{i \notin B} y_i$. If $t<n/2$, by taking $s=[(n-1)/2]$ we certainly get $\sum_{i=1}^{s} y_i \geq \sum_{i=s+1}^{n} y_i$. If $t=n/2$ equation (1) reduces to $\sum_{i \in A} y_i = \sum_{i \notin A} y_i$. But since $y_n=0$ we may by dropping n from either A or its complement choose a new A with $t-1$ elements for which this equation still holds, and then proceed as before with $a=1$, $b=0$.

Conversely, if the above equation holds, let $a=n-2s$, $b=\sum_{i=1}^{s} y_i - \sum_{i=s+1}^{n} y_i$. Then $ay_i+b \geq 0$ for all i and $\sum_{i=1}^{s} (ay_i+b) = \sum_{i=s+1}^{n} (ay_i+b)$. By Lemma 1 it follows that C has a weak predecessor. We now restate Theorem 4 in the following way.

THEOREM 4A. *If n is odd every n-cycle has a weak predecessor. If n is even and greater than 2 and $C=(x_1, x_2, \ldots, x_n)$ is an n-cycle, then either C has a weak predecessor or $C'=(m-x_1, m-x_2, \ldots, m-x_n)$ does (here $m=m(C)$).*

Proof. If $t=\min_i (x_i)$ then $C=(x_1, x_2, \ldots, x_n)$ has a weak predecessor if and only if $(x_1-t, x_2-t, \ldots, x_n-t)$ does. Thus the added assumption that $y_n=0$ of Lemma 2 applies. But clearly the inequality in Lemma 2 always holds when n is odd (compare the two sums involving the same number of elements, element by element). When n is even the inequality may sometimes fail to hold. If so consider C' (which will have the same

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Ψ-image as C), so that in effect we replace the y_i by $y_i' = y_1 - y_{n-i+1}$. Then, letting $n = 2k$, we get

$$\sum_{i=1}^{k-1} y_i' - \sum_{i=k}^{n} y_i' = -y_1 + \sum_{i=2}^{k+1} y_i - \sum_{i=k+2}^{n-1} y_i$$

$$= -y_1 + y_k + y_{k+1} + \sum_{i=2}^{k-1} y_i - \sum_{i=k+2}^{n-1} y_i$$

$$\geq -y_1 + y_k + y_{k+1} + \sum_{i=k+2}^{n-1} y_i - \sum_{i=2}^{k-1} y_i$$

$$= \sum_{i=k}^{n-1} y_i - \sum_{i=1}^{k-1} y_i,$$

which is > 0 by the assumption that C does not have a weak predecessor. Thus we have shown that C' does have a weak predecessor.

If n is neither odd nor a power of 2, the problem of distinguishing between terminating and non-terminating n-cycles seems quite difficult. It is clear, for example, that an n-cycle which is congruent mod 2 to a repeating primitive n-cycle must be non-terminating. But the converse is untrue as shown by the following 6-cycle: (18, 25, 34, 19, 0, 13). Almost any kind of alteration in the n-cycle seems capable of changing it from terminating to non-terminating. For example (89, 140, 83, 0, 45, 56) is terminating, but (89, 142, 83, 0, 45, 56) is non-terminating. Similarly (1, 4, 9, 6, 5, 2) is terminating, but with the elements permuted (1, 6, 9, 2, 5, 4) is non-terminating.

The question of determining the life-span of an n-cycle also seems interesting but there appears to be no easy way of tackling it.

Finally, an interesting area to explore is that of n-cycles made up of real but possibly irrational numbers. For instance, for some n's the cycle mentioned in Remark 6 and similar ones seem to be the only cycles having infinite life-span.

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