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NEW LIMITING DISTRIBUTIONS FOR BELLMAN–HARRIS PROCESSES

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Abstract

We present a number of new solutions to an integral equation arising in the limiting theory of Bellman–Harris processes. The argument proceeds via straightforward analysis of Mellin transforms. We also derive a criterion for the analyticity of the Laplace transform of the limiting distribution on $\mathrm{Re}(u) \geq -c$ for some $c > 0$.

Keywords: Mellin transform; branching process; exponential moment

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1. Introduction and main results

Let Z_t, $t \geq 0$, be a supercritical Bellman–Harris process started at $t = 0$ with a single newborn individual. The individual lives for a random time T and then splits into a random number Z_+ of progeny which are identical to their newborn mother. We denote by $G := G(t) := \mathrm{P}(T \leq t)$ their common lifetime distribution. As usual, we assume that G is nonlattice and nondefective, and that $G(0^+) = 0$. Furthermore, we denote by $\pi_k := \mathrm{P}(Z_+ = k)$ the probability that upon division, an individual divides into exactly k progeny, and by $f(s) := \mathrm{E}(\mathrm{e}^{sZ_+}) = \sum_{k=0}^{\infty} \pi_k s^k$ the corresponding generating function. We assume that $1 < \mu := f'(1) < \infty$ and that, effectively, f has radius of convergence larger than 1. The latter implies in particular that $\sigma^2 := \mathrm{E}(Z_+^2) < \infty$, so that if we now define the Malthusian parameter β as the unique $\beta \in (0, \infty)$ for which

$$\mu \int_0^{\infty} \mathrm{e}^{-\beta t} \, \mathrm{d}G(t) = 1, \tag{1}$$

then

$$Z := \lim_{t \to \infty} \mathrm{e}^{-\beta t} Z_t$$

exists in a nondegenerate sense almost surely [8]. Furthermore, the Laplace transform $\varphi(u) := \mathrm{E}(\mathrm{e}^{-uZ})$ of Z satisfies

$$\varphi(u) = \int_0^{\infty} f \circ \varphi(u\mathrm{e}^{-\beta t}) \, \mathrm{d}G(t), \tag{2}$$

and, indeed, we can obtain the Laplace transform of Z as the unique nonconstant solution of this equation [3]. Unfortunately, there are only a few instances where such a solution is known for given f and G in the first place. We turn the problem on its head in the following.

Theorem 1. *Let f be a probability generating function (PGF) with $1 < f'(1) = \mu < \infty$ and radius of convergence larger than* 1*, and let φ be the (unique) nonconstant solution of (2). Then,*

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if φ is analytic as a function of u on $\mathrm{Re}(u) \geq -c_0$ for some $c_0 > 0$, the lifetime distribution G of individuals is necessarily of the form $G(t) = \hat{G}(\beta t)$, where $\hat{G}$ is some probability distribution on $\mathbb{R}^{\geq 0} := \mathbb{R}^+ \cup \{0\}$ whose Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}\hat{G}(t) = \frac{\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s-1}\varphi(u)\,\mathrm{d}u}{\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s-1} f \circ \varphi(u)\,\mathrm{d}u} \tag{3}$$

for $s > 0$.

1.1. Proof of Theorem 1

Fix some c as in the theorem and some arbitrary $s > 1$. We first check that the right-hand side of (3) is well defined. To this end, we use the following result.

Theorem 2. ([6, Theorem 2].) *Let*

$$\mathcal{L}f(u) := \int_0^\infty \mathrm{e}^{-ux} f(x)\,\mathrm{d}x \quad \textit{and} \quad \mathcal{M}f(s) := \int_0^\infty x^{s-1} f(x)\,\mathrm{d}x$$

be the Laplace and Mellin transforms, respectively, of some locally integrable function

$$f\colon \mathbb{R}^{\geq 0} \mapsto \mathbb{C}.$$

Then, if $\mathcal{L}f(u)$ converges absolutely on $\mathrm{Re}(u) \geq -c$ for some $c > 0$, $\mathcal{M}f(s)$ converges absolutely on $\mathrm{Re}(s) > a$ for some $a \leq 1$, and

$$\mathcal{M}f(s) = \frac{\Gamma(s)}{2\pi \mathrm{i}} \int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s} \mathcal{L}f(u)\,\mathrm{d}u$$

at least on $\mathrm{Re}(s) > 1$.

Lew [6] used the Parseval theorem to prove his result, so it is clear that it also holds if $\mathrm{d}f$ is a probability measure on $\mathbb{R}^{\geq 0}$. It is actually quite easy to see this directly: because

$$\frac{1}{2\pi \mathrm{i}} \int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s}\mathrm{e}^{-ut}\,\mathrm{d}u = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{(c+\mathrm{i}v)t}}{(c+\mathrm{i}v)^s}\,\mathrm{d}v = \frac{t^{s-1}}{\Gamma(s)}$$

for $\mathrm{Re}(s) > 1$ (essentially by Laplace's integral [4]), we see at once that Theorem 2 holds for a Dirac mass at t. But then the general case follows immediately, because every probability on $\mathbb{R}^{\geq 0}$ can be approximated uniformly by a combination of discrete probability measures with finite support. In particular, we have

$$\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s}\varphi(u)\,\mathrm{d}u = 0$$

if and only if φ is the Laplace transform of a Dirac mass at 0, which implies that $\varphi(u) = 1$ for arbitrary u, and contradicts our assumption that φ is nonconstant. Since $f'(1) = \mu \neq 0$, we similarly see that the denominator in (3) is nonzero; the finiteness of either integral follows from the analyticity of φ in a neighborhood of 0 (which implies that the function f in Theorem 2 has an exponentially decreasing tail).

Consider then Mellin transform of Z:

$$\mathcal{M}Z(s) := \frac{\Gamma(s)}{2\pi \mathrm{i}} \int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s} \varphi(u)\,\mathrm{d}u$$
$$= \frac{\Gamma(s)}{2\pi \mathrm{i}} \int_{u=-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s} \int_{t=0}^{\infty} f \circ \varphi(u\mathrm{e}^{-\beta t})\,\mathrm{d}G(t)\,\mathrm{d}u.$$

Writing $u = -c + \mathrm{i}v$, we see by Fubini that

$$\mathcal{M}Z(s) = \frac{\Gamma(s)}{2\pi} \int_{\mathbb{R}^{\geq 0} \times \mathbb{R}} \frac{(c + \mathrm{i}v)^s}{(c^2 + v^2)^{s/2}} f \circ \varphi((-c + \mathrm{i}v)\mathrm{e}^{-\beta t})\,\mathrm{d}G(t) \otimes \frac{\mathrm{d}v}{(c^2 + v^2)^{s/2}},$$

because

$$\int_{\mathbb{R}^{\geq 0} \times \mathbb{R}} \mathrm{d}G(t) \otimes \frac{\mathrm{d}v}{(c^2 + v^2)^{s/2}} < \infty$$

if $s > 1$, and the integrand is bounded in absolute value if we choose c such that $\varphi(-c)$ belongs to the disk of convergence of f. But then

$$\mathcal{M}Z(s) = \frac{\Gamma(s)}{2\pi \mathrm{i}} \int_{u=-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s} f \circ \varphi(u)\,\mathrm{d}u \int_{t=0}^{\infty} \mathrm{e}^{-\beta t(s-1)}\,\mathrm{d}G(t)$$

by path independence of the integral with respect to u, and Fubini again. The change of variables $s \to s + 1$ and $t \to t/\beta$ now completes the proof of Theorem 1.

By way of illustration, we prove the following result.

Corollary 1. *Suppose that Z is $\Gamma(\kappa, 1)$-distributed, and that $f(s) = s^m$ for some integer $m \geq 2$. Then, given that $\varphi(u) = \mathrm{E}(\mathrm{e}^{-uZ})$ is a solution of (2), the lifetime distribution G has density*

$$\frac{\mathrm{d}G(t)}{\mathrm{d}t} = \frac{\beta\Gamma(m\kappa)}{\Gamma(\kappa)\Gamma(m\kappa - \kappa)} \mathrm{e}^{-\beta\kappa t}(1 - \mathrm{e}^{-\beta t})^{(m-1)\kappa - 1}$$

for some $\beta \in (0, \infty)$.

Proof. The idea is to first *assume* that a random variable Z with the desired properties exists, and then to use Theorem 1 to check that everything works out. By Theorem 2, we can avoid calculating the respective integrals head-on, and work with the Mellin transform of the corresponding densities instead. This gives

$$\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s-1} \varphi(u)\,\mathrm{d}u = \frac{2\pi \mathrm{i}}{\Gamma(1 + s)\Gamma(\kappa)} \int_0^{\infty} \mathrm{e}^{-x} x^{\kappa-1} x^s\,\mathrm{d}x$$
$$= 2\pi \mathrm{i} \frac{\Gamma(\kappa + s)}{\Gamma(1 + s)\Gamma(\kappa)},$$

and as the m-fold convolution of a $\Gamma(\kappa, 1)$-distributed random variable with itself is $\Gamma(m\kappa, 1)$-distributed, we now find that

$$\int_0^{\infty} \mathrm{e}^{-st}\,\mathrm{d}\hat{G}(t) = \frac{\Gamma(\kappa + s)\Gamma(m\kappa)}{\Gamma(m\kappa + s)\Gamma(\kappa)},$$

or

$$\hat{G}(t) = \frac{\Gamma(m\kappa)}{\Gamma(\kappa)\Gamma(m\kappa - \kappa)} \int_0^t \mathrm{e}^{-\kappa u}(1 - \mathrm{e}^{-u})^{(m-1)\kappa - 1}\,\mathrm{d}u,$$

which proves that the density of the lifetime distribution G is as given in the corollary. We can check this directly: because the Laplace transform of a $\Gamma(\kappa, 1)$-distributed random variable is $\varphi(u) = (1+u)^{-\kappa}$, we have

$$\begin{aligned}
&\int_0^\infty f \circ \varphi(u\mathrm{e}^{-\beta t})\,\mathrm{d}G(t) \\
&\quad = \frac{\beta\Gamma(m\kappa)}{\Gamma(\kappa)\Gamma(m\kappa-\kappa)} \int_0^\infty \left(\frac{1-\mathrm{e}^{-\beta t}}{1+u\mathrm{e}^{-\beta t}}\right)^{m\kappa} \mathrm{e}^{-\beta\kappa t}(1-\mathrm{e}^{-\beta t})^{-\kappa-1}\,\mathrm{d}t \\
&\quad = \frac{\Gamma(m\kappa)}{\Gamma(\kappa)\Gamma(m\kappa-\kappa)} \int_0^1 \left(\frac{1-x}{1+ux}\right)^{m\kappa} x^{\kappa-1}(1-x)^{-\kappa-1}\,\mathrm{d}x
\end{aligned}$$

upon $\mathrm{e}^{-\beta t} =: x$. Upon a second change of variable $(1-x)/(1+ux) =: y$, we obtain

$$\begin{aligned}
&\int_0^\infty f \circ \varphi(u\mathrm{e}^{-\beta t})\,\mathrm{d}G(t) \\
&\quad = \frac{\Gamma(m\kappa)}{\Gamma(\kappa)\Gamma(m\kappa-\kappa)} \int_0^1 y^{m\kappa} \left(\frac{1-y}{1+uy}\right)^{\kappa-1} \left(\frac{y+uy}{1+uy}\right)^{-\kappa-1} \frac{1+u}{(1+uy)^2}\,\mathrm{d}y \\
&\quad = \frac{(1+u)^{-\kappa}\Gamma(m\kappa)}{\Gamma(\kappa)\Gamma(m\kappa-\kappa)} \int_0^1 y^{(m-1)\kappa-1}(1-y)^{\kappa-1}\,\mathrm{d}y \\
&\quad = (1+u)^{-\kappa},
\end{aligned}$$

which once more verifies the corollary.

In the light of this example, we would expect the calculations to yield nice results only for random variables Z whose Laplace *and* Mellin transforms are sufficiently simple. This appears to restrict the analysis to distribution functions which are a linear combination of gamma distributions. An example along these lines is provided by the densities

$$\gamma(z) := \frac{1}{\varkappa\Gamma(1+\varkappa)} \int_{z^{\varkappa}}^\infty \mathrm{e}^{-x^{1/\varkappa}}\,\mathrm{d}x$$

for some $\varkappa \in (0, 1)$, and $f(s) = s^2$. Proceeding in the same way as above, we can employ any standard program for symbolic computation to verify that

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}\hat{G}(t) = \varkappa \frac{2+s}{1+\varkappa+s} \frac{\Gamma(2\varkappa)\Gamma(2+\varkappa+s)}{\Gamma(\varkappa)\Gamma(2+2\varkappa+s)} \\
\times \left(1 - \frac{2\Gamma(2\varkappa)\Gamma(2+\varkappa+s)}{\Gamma(\varkappa)\Gamma(2+2\varkappa+s)}\right)^{-1}.
\end{aligned} \tag{4}$$

The first factor (except for the $\varkappa$) is the Laplace transform of $(1-\varkappa)\mathrm{e}^{-(1+\varkappa)t}$ plus a Dirac mass at 0, and $\Gamma(\varkappa)\Gamma(s)/\Gamma(s+\varkappa)$ is the Laplace transform of the positive function $(1-\mathrm{e}^{-t})^{\varkappa-1}$. This shows that (4) is indeed the Laplace transform of a probability on $\mathbb{R}^{\geq 0}$.

It may be well to point out that the *existence* of a random variable Z with a given distribution arising as the limiting object in a Bellman–Harris process is far from obvious. What our Theorem 1 does, in a modest way, is to reduce the problem of finding a solution to (2) to that of checking whether a given function is the Laplace transform of a probability distribution on $\mathbb{R}^{\geq 0}$ (which may be a nontrivial matter in itself). Here is a precise statement.

Theorem 3. *Suppose that φ is the Laplace transform of a probability distribution on $\mathbb{R}^{\geq 0}$ which is nonconstant and analytic as a function of u on* $\mathrm{Re}(u) \geq -c_0$ *for some $c_0 > 0$. Let f be a*

PGF with $1 < f'(1) = \mu < \infty$ and radius of convergence larger than 1, and suppose that, for any $c \in (0, c_0)$ and any $s > 0$,

$$\frac{\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s-1} \varphi(u)\,\mathrm{d}u}{\int_{-c-\mathrm{i}\infty}^{-c+\mathrm{i}\infty} (-u)^{-s-1} f \circ \varphi(u)\,\mathrm{d}u} =: \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}\hat{G}(t) \tag{5}$$

is the Laplace transform of a probability distribution $\hat{G}$ on the nonnegative reals. Then, for any $\beta > 0$, there exists a Bellman–Harris process Z_t, $t \geq 0$, with lifetime distribution $G(t) := \hat{G}(\beta t)$ and first-generation offspring PGF f such that φ is the Laplace transform of the limiting random variable $Z = \lim_{t\to\infty} \mathrm{e}^{-\beta t} Z_t$.

Proof. Set $s = 1$. Then the integrands on the left-hand side of (5) are analytic in the half-plane $\mathrm{Re}(u) > -c_0$ except for a pole of order 2 at $u = 0$. We close the contour of integration via a semicircle in the right half-plane (which is possible if $\mathrm{Re}(u)$ is not too negative), and obtain

$$\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}\hat{G}(t) = \frac{\varphi'(0)}{f'(\varphi(0))\varphi'(0)} = \frac{1}{\mu}$$

by Cauchy's theorem. Hence,

$$\mu \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}G(t) = \mu \int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}\hat{G}(t) = 1,$$

if we define G as required by the theorem, so that β is in fact the Malthusian parameter of the process. But now we check as in the above that φ satisfies (2) with G as just defined, and since the solution of this equation is essentially unique [3], we are done if the random variable $Z = \lim_{t\to\infty} \mathrm{e}^{-\beta t} Z_t$ is not concentrated at 0. But this follows from the fact that f has radius of convergence larger than 1, and the Kesten–Stigum theorem.

2. Analyticity of φ

The obvious question now relates to the range of applicability of Theorem 1: apart from the fact that the theorem requires a certain Laplace transform to be analytic somewhat into the left half-plane, it also requires a suitably large radius of convergence of the PGF f. It would be nice to know whether we could get one from the other. Part of the answer is given by the following result.

Theorem 4. *Let F_t be the PGF of particle numbers in a Bellman–Harris process at time t, and let f be the PGF of the corresponding first-generation offspring distribution. Say that f has exponential moments up to order $r > 0$ if $f(\mathrm{e}^u) < \infty$ for $u < r$, and let β be the Malthusian parameter as defined in (1). Then F_t has exponential moments at least up to order $r_1 \mathrm{e}^{-\beta t}$ for some suitable constant $r_1 > 0$. In particular, there exists $c > 0$ such that the Laplace transform $\varphi(u) = \mathrm{E}(\mathrm{e}^{-uZ})$ of $Z = \lim_{t\to\infty} \mathrm{e}^{-\beta t} Z_t$ is analytic for $u \geq -c$.*

2.1. Proof of Theorem 4

We will make use of the following result.

Theorem 5. ([7, Theorem 3.1].) *If $\mu G(0) < 1$ and there exists a $u_1 > 0$ such that $\mathrm{E}(\mathrm{e}^{uZ_+}) < \infty$, $0 < u \leq u_1$, then, for all $t > 0$, there exists a $u_0 := u_0(t) > 0$ such that $\mathrm{E}(\mathrm{e}^{uZ_t}) < \infty$, $0 < u \leq u_0$.*

See [7] for the proof.

What we need to make sure of is that $u_0(t)$ does not become too small in comparison with $\mathrm{e}^{-\beta t}$. The following might qualify as a 'natural' proof of this fact: by Theorem 5, we can write

$$\mathrm{e}^{u_2} := \mathrm{E}(\mathrm{e}^{u_0 Z_t}) = F_t(\mathrm{e}^{u_0}) = F_t \circ \exp(\mathrm{e}^{\beta t} \log F_{-t}(\mathrm{e}^{u_2})\mathrm{e}^{-\beta t}) < \infty$$

for some $u_2 > 0$. (We use F_{-t} to denote the inverse of F_t.) Hence, if we can prove that

$$\liminf_{t\to\infty} \mathrm{e}^{\beta t} \log F_{-t}(\mathrm{e}^{u}) > 0 \tag{6}$$

for some $u > 0$, Theorem 4 will be an immediate consequence of Fatou's lemma and the nondegeneracy of Z. Suppose, by way of contradiction, that (6) *does not* hold. Now write s instead of e^u, and recall that

$$F_t(s) = (1 - G(t))s + \int_0^t f \circ F_{t-u}(s)\,\mathrm{d}G(u). \tag{7}$$

This equation is generally presented with the caveat that $|s| \le 1$, but given its probabilistic content (and proof [5, pp. 130–131]), there is nothing about it which requires a lot more than that $F_t(s)$ be finite and $F_{t-u}(s)$ belong to the region of convergence of f for $u \in [0, t]$. In view of Theorem 5, we can certainly assume that $F_t(s) < \infty$, but we still need to know to which extent such an estimate can be uniform in t. We clarify this point in the following result.

Lemma 1. *For $s \ge 1$ and $y \ge 0$, $F_{t+y}(s) \ge F_t(s)$.*

Proof. Let $Z_t[x]$ be the number of particles at time t in a Bellman–Harris process started at $t = 0$ with a particle aged x. Then

$$Z_{t+u} = Z_{t+u}[0] = \sum_{i=1}^{Z_t} Z_u[x_i],$$

where the $x_i =: x_i(t)$ are the ages of individuals at time t, and the $Z_u[x_i]$ are mutually independent by the branching property. Since $\mathrm{E}(Z_u[x_i])$ is bounded on every finite u-interval [8], we now can write

$$\mathrm{E}(s^{Z_{t+u}} \mid Z_t, x_1, \dots, x_{Z_t}) = \prod_{i=1}^{Z_t} \mathrm{E}(s^{Z_u[x_i]}) \ge \prod_{i=1}^{Z_t} s^{\mathrm{E}(Z_u[x_i])} \tag{8}$$

by Jensen's inequality. But $\mathrm{E}(Z_u[0])$ is nondecreasing in u [5, p. 141] (hence greater than or equal to 1), which by Equation (4) of [8] implies that $\mathrm{E}(Z_u[x_i]) \ge 1$ as well. Hence, the left-hand side of (8) is at least as large as s^{Z_t} for $s \ge 1$, which after taking expectations yields the desired result.

We now define

$$h(s) := \frac{f(s) - 1}{s - 1}$$

and

$$\mathcal{X}_t(s) := \mathrm{e}^{-\beta t} \frac{F_t(s) - 1}{s - 1}.$$

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Equation (7) then implies that

$$\mathcal{X}_t(s) = \mathrm{e}^{-\beta t}(1 - G(t)) + \int_0^t h \circ F_{t-u}(s)\mathrm{e}^{-\beta(t-u)} \frac{F_{t-u}(s) - 1}{s - 1}\mathrm{e}^{-\beta u}\,\mathrm{d}G(u)$$
$$= \mathrm{e}^{-\beta t}(1 - G(t)) + \int_0^t \left(\frac{h \circ F_u(s)}{\mu} - 1\right)\mathcal{X}_u(s)\,\mathrm{d}G_\beta(t-u) + \int_0^t \mathcal{X}_u(s)\,\mathrm{d}G_\beta(t-u), \tag{9}$$

where $G_\beta := G_\beta(t)$ denotes the measure

$$G_\beta(t) = \mu \int_0^t \mathrm{e}^{-\beta u}\,\mathrm{d}G(u).$$

(See [1] for a similar line of reasoning.) We then subtract $\mathrm{e}^{-\beta t}(1 - G(t))$ from both sides of (9) and convolve with G_β:

$$\int_0^t \mathcal{X}_u(s)\,\mathrm{d}G_\beta(t-u) - \int_0^t \mathrm{e}^{-\beta u}(1 - G(u))\,\mathrm{d}G_\beta(t-u)$$
$$= \int_{u=0}^t \int_{v=0}^u \frac{h \circ F_v(s)}{\mu}\mathcal{X}_v(s)\,\mathrm{d}G_\beta(u-v)\,\mathrm{d}G_\beta(t-u)$$
$$= \int_{v=0}^t \frac{h \circ F_v(s)}{\mu}\mathcal{X}_v(s)\int_{u=v}^t \mathrm{d}G_\beta(u-v)\,\mathrm{d}G_\beta(t-u)$$
$$= \int_{v=0}^t \frac{h \circ F_v(s)}{\mu}\mathcal{X}_v(s)\,\mathrm{d}G_\beta^{*2}(t-v).$$

Here G_β^{*2} denotes the convolution of G_β with itself. If we use this to replace the final term in (9), we find that

$$\mathcal{X}_t(s) = \mathrm{e}^{-\beta t}(1 - G(t)) + \int_0^t \mathrm{e}^{-\beta u}(1 - G(u))\,\mathrm{d}G_\beta(t-u)$$
$$+ \int_0^t \left(\frac{h \circ F_u(s)}{\mu} - 1\right)\mathcal{X}_u(s)\,\mathrm{d}(G_\beta(t-u) + G_\beta^{*2}(t-u))$$
$$+ \int_0^t \mathcal{X}_u(s)\,\mathrm{d}G_\beta^{*2}(t-u).$$

The idea is now to proceed by induction: by Theorem 5 and Lemma 1, $\mathcal{X}_u(s) < \infty$ for every $u \in [0, t]$, provided that s is sufficiently small. Moreover, $G_\beta^{*n} \to 0$ on bounded intervals [2, p. 144], so that if now we define

$$U_\beta(t) := \sum_{i=1}^{\infty} G_\beta^{*i}(t)$$

(which is just the renewal measure for G_β without the Dirac mass at 0), we obtain

$$\mathcal{X}_t(s) = \mathrm{e}^{-\beta t}(1 - G(t)) + \int_0^t \mathrm{e}^{-\beta u}(1 - G(u))\,\mathrm{d}U_\beta(t-u)$$
$$+ \int_0^t \left(\frac{h \circ F_u(s)}{\mu} - 1\right)\mathcal{X}_u(s)\,\mathrm{d}U_\beta(t-u),$$

or

$$\mathcal{X}_t(s) - \mathcal{X}_t(1) = \int_0^t \left(\frac{h \circ F_{t-u}(s)}{\mu} - 1\right) \mathcal{X}_{t-u}(s) \,\mathrm{d}U_\beta(u), \tag{10}$$

where $\mathcal{X}_t(1) = \mathrm{e}^{-\beta t} F_t'(1)$. Since

$$\mathrm{e}^{-\beta t} F_t'(1) \to \frac{\mu - 1}{\beta\mu \int_0^\infty u \,\mathrm{d}G_\beta(u)} \tag{11}$$

as t tends to ∞ [2, Theorem 3A], $\mathcal{X}_t(1)$ is a finite number for every $t \in \mathbb{R}^{\geq 0}$. We now multiply (10) by $\mathrm{e}^{\beta t}(s-1)$ and take the result at $F_{-t}(s)$:

$$\begin{aligned} s - 1 - \mathcal{X}_t(1)\mathrm{e}^{\beta t}(F_{-t}(s) - 1) \\ = \int_0^t \left(\frac{h \circ F_{t-u} \circ F_{-t}(s)}{\mu} - 1\right) \mathrm{e}^{\beta u}(F_{t-u} \circ F_{-t}(s) - 1) \,\mathrm{d}U_\beta(u). \end{aligned} \tag{12}$$

Since we are operating under the assumption that $\mathrm{e}^{\beta t} \log F_{-t}(s) \to 0$, we can pick a t such that $\mathcal{X}_t(1)\mathrm{e}^{\beta t}(F_{-t}(s) - 1) < (s-1)/2$ and

$$\mathrm{e}^{\beta t} \log F_{-t}(s) \leq \mathrm{e}^{\beta(t-u)} \log F_{u-t}(s)$$

for every $u \in [0, t]$. But then

$$\begin{aligned} F_{t-u} \circ F_{-t}(s) &= \mathrm{E}(\exp(\log F_{-t}(s) Z_{t-u})) \\ &\leq \mathrm{E}(\exp(\mathrm{e}^{-\beta u} \log F_{u-t}(s) Z_{t-u})) \\ &\leq s^{\mathrm{e}^{-\beta u}} \end{aligned}$$

by Jensen's inequality again (the function $x \mapsto x^{\mathrm{e}^{-\beta u}}$ is strictly concave on $\mathbb{R}^{\geq 0}$ if βu is larger than 0), and we deduce from (12) that

$$s - 1 \leq \frac{2h'(\xi)}{\mu} \int_0^t \mathrm{e}^{\beta u}(s^{\mathrm{e}^{-\beta u}} - 1)^2 \,\mathrm{d}U_\beta(u) \tag{13}$$

for some $\xi \in (1, s)$. Now $2h'(\xi) \to \sigma^2 + \mu^2 - \mu$ as $\xi \to 1$, and the integrand is of order $\mathrm{e}^{-\beta u} \log^2 s$, which is integrable with respect to $U_\beta(u)$: in fact, U_β is just the Lebesgue measure (on $\mathbb{R}^{\geq 0}$) plus an error term with an exponentially decreasing tail [9]. But then (13) implies that $s - 1 \leq K \log^2 s$ for some $K > 0$, which is a contradiction for s sufficiently close to 1. This proves (6) and Theorem 3.

Our proof shows that the constant r_1 in the statement of Theorem 4 should in general be close to a supremum over the lim inf's in (6). This, in turn, might be close to the reciprocal of $\mathcal{X}_t(1)$ in (11), but it may be too soon to really state this as a conjecture. It might also be interesting to see how the above proof works out for the Galton–Watson case; we can do slightly better and verify our central equation (10) with G a Dirac mass at $t = 1$. In this case, we have $\beta = \log \mu$ and $G_\beta = G$, and that F_t is the $\lfloor t \rfloor$th iterate of f, which we write as $F_t = f_n$ for $\lfloor t \rfloor = n$. The function $\mathcal{X}_t$ now equals $\mu^{-n}(f_n(s) - 1)/(s-1)$, except for a factor $\mu^{\lfloor t \rfloor - t}$, which, because U_β is now concentrated on the positive integers, is the same on both sides of (10). This gives

$$\begin{aligned} \mathcal{X}_n(s) - 1 &= \sum_{k=0}^{n-1} \left(\frac{h \circ f_k(s)}{\mu} - 1\right) \mathcal{X}_k(s) \\ &= \sum_{k=0}^{n-1} \mu^{-k-1} \frac{f \circ f_k(s) - 1}{f_k(s) - 1} \frac{f_k(s) - 1}{s - 1} - \mu^{-k} \frac{f_k(s) - 1}{s - 1}, \end{aligned}$$

or

$$\mathcal{X}_n(s) - 1 = \mu^{-n} \frac{f_n(s) - 1}{s - 1} - 1,$$

as it must be, because the sum telescopes. Proceeding now in the same way as above, we obtain from the previous equation

$$s - 1 - \mu^n (f_{-n}(s) - 1) \leq \frac{h' \circ f_{-1}(s)}{\mu} \sum_{k=1}^{n} \mu^k (f_{-k}(s) - 1)^2.$$

By concavity of the inverse function, the sum on the right-hand side is no larger than $(s - 1)^2/(\mu - 1)$, which yields the desired contradiction to $\liminf_{n\to\infty} \mu^n (f_{-n}(s) - 1) = 0$ for s sufficiently close to 1.

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References

[1] Angerer, W. (2007). On Seneta's constants for the supercritical Bellman–Harris process with $E(Z_+ \log Z_+) = \infty$. *Sankhyā* **69,** 256–264.
[2] Athreya, K. B. and Ney, P. E. (1972). *Branching Processes*. Springer, New York.
[3] Athreya, K. B. and Schuh, H.-J. (2003). On the supercritical Bellman–Harris process with finite mean. *Sankhyā* **65,** 229–248.
[4] De Azevedo Pribitkin, W. (2002). Laplace's integral, the gamma function, and beyond. *Amer. Math. Monthly* **109,** 235–245.
[5] Harris, T. E. (1963). *The Theory of Branching Processes*. Springer, Berlin.
[6] Lew, J. S. (1975). On some relations between the Laplace and Mellin transforms. *IBM J. Res. Develop.* **19,** 582–586.
[7] Nakayama, M. K., Shahabuddin, P. and Sigman, K. (2004). On finite exponential moments for branching processes and busy periods for queues. In *Stochastic Methods and Their Applications* (J. Appl. Prob. Spec. Vol. **41A**), eds J. Gani and E. Seneta, Applied Probability Trust, Sheffield, pp. 273–280.
[8] Schuh, H.-J. (1982). Seneta constants for the supercritical Bellman–Harris process. *Adv. Appl. Prob.* **14,** 732–51.
[9] Stone, C. (1965). On moment generating functions and renewal theory. *Ann. Math. Statist.* **36** 1298–1301.