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A GENERALISATION TO SEVERAL DIMENSIONS OF THE NEUBERG-PEDOE INEQUALITY, WITH APPLICATIONS

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A well-known inequality relating the areas and squares of the sides of two triangles is generalised to higher-dimensional euclidean spaces. Extension of the results to non-euclidean spaces is also considered.

0. Notation and main results

Let Σ_A, Σ_B be two non-degenerate simplices in the n-dimensional euclidean space E^n, with vertices $a_1, a_2, \ldots, a_{n+1}$ and $b_1, b_2, \ldots, b_{n+1}$, respectively. Let the lengths of their edges be $a_{ij} = |a_i a_j|$, $b_{ij} = |b_i b_j|$ $(i, j = 1, 2, \ldots, n+1)$, and their volumes be $V(A)$, $V(B)$, respectively. Denote the determinants of the following bordered matrices

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$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & -\tfrac{1}{2}a_{ij}^2 & \\ 1 & & & \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & -\tfrac{1}{2}b_{ij}^2 & \\ 1 & & & \end{pmatrix}$$

by A, B, and their cofactors of $-\frac{1}{2}a_{ij}^2$, $-\frac{1}{2}b_{ij}^2$ by A_{ij}, B_{ij} $(i, j = 1, 2, \ldots, n+1)$, respectively.

We shall prove the following assertion.

THEOREM 1. *If Σ_A, Σ_B are two non-degenerate simplices in E^n, then*

$$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}^2 B_{ij} \geq 2n \cdot (n!)^2 V(A)^{2/n} V(B)^{2-(2/n)}, \tag{1.1}$$

and equality holds if, and only if, the two simplices are similar.

Note that if we take $n = 2$, then Theorem 1 gives as a special case the well-known Neuberg-Pedoe inequality

$$a'^2\left(-a^2+b^2+c^2\right) + b'^2\left(a^2-b^2+c^2\right) + c'^2\left(a^2+b^2-c^2\right) \geq 16\Delta\Delta' . \tag{*}$$

Here Δ, Δ' denote the areas of two triangles, whose sides have lengths a, b, c and a', b', c', respectively. A necessary and sufficient condition for equality to hold is that the two triangles are similar.

Thus Theorem 1 is an extension of the Neuberg-Pedoe inequality to higher-dimensional spaces. We consider the following theorem as one of its interesting applications.

THEOREM 2. *If Σ_A, Σ_B are two non-degenerate simplices in E^n such that*

1° $a_{ij} \leq b_{ij}$ $(i, j = 1, 2, \ldots, n+1)$,

2° *Σ_B is not obtuse (that is to say, none of the interior angles formed by the $(n-1)$-dimensional faces of Σ_B is obtuse), then $V(A) \leq V(B)$.*

This theorem can be written in the following equivalent form, where it is not necessary to assume $a_{ij} \le b_{ij}$.

THEOREM 2*. *If* Σ_B *is not obtuse, then*

$$\frac{V(A)}{V(B)} \le \left(\max_{i,j} \frac{a_{ij}}{b_{ij}}\right)^n .$$

1. Proof of Theorem 1

For the proof we introduce the following notations:

$$(1.2) \qquad \begin{cases} q_{ij} = \tfrac{1}{2}\left(a_{i,n+1}^2 + a_{j,n+1}^2 - a_{ij}^2\right) , \\ \\ r_{ij} = \tfrac{1}{2}\left(b_{i,n+1}^2 + b_{j,n+1}^2 - b_{ij}^2\right) , \quad (i,\ j = 1,\ 2,\ \ldots,\ n+1) ; \end{cases}$$

$$(1.3) \qquad Q = \left(q_{ij}\right) , \quad R = \left(r_{ij}\right) , \quad (i,\ j = 1,\ 2,\ \ldots,\ n) ,$$

so that Q, R are $n \times n$ matrices;

$$(1.4) \qquad s_{ij}(\lambda) = q_{ij} + \lambda r_{ij} ,$$

$$S(\lambda) = \left(s_{ij}(\lambda)\right) , \quad (i,\ j = 1,\ 2,\ \ldots,\ n) ,$$

so that $S(\lambda) = Q + \lambda R$ is also an $n \times n$ matrix;

$$(1.5) \qquad f_{ij}(\lambda) = -\tfrac{1}{2}\left(a_{ij}^2 + \lambda b_{ij}^2\right) ,$$

$$F(\lambda) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & f_{ij}(\lambda) & \\ 1 & & & \end{pmatrix} , \quad (i,\ j = 1,\ 2,\ \ldots,\ n+1) ,$$

so that $F(\lambda)$ is an $(n+2) \times (n+2)$ matrix.

First we investigate the roots of the equation

$$\det F(\lambda) = 0 .$$

Adding $-f_{i,n+1}(\lambda)$ times the first row to the ith row and $-f_{n+1,j}(\lambda)$ times the first column to the jth column, we get

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$$\det F(\lambda) = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & f_{ij}(\lambda) & \\ 1 & & & \end{vmatrix} = \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & 0 \\ \vdots & & s_{ij}(\lambda) & \\ 1 & 0 & \cdots & 0 \end{vmatrix}$$

$$= -\det S(\lambda) = -\det(Q+\lambda R) .$$

Put

(1.6) $$-\det F(\lambda) = \det(Q+\lambda R) = c_0\lambda^n + c_1\lambda^{n-1} + \ldots + c_n .$$

Observe that both Q and R are real symmetric positive definite matrices; hence all the coefficients $c_0, c_1, \ldots, c_n$ are non-negative, and the roots of the equation are real and non-positive. By Maclaurin's Inequality ([7], Theorem 52, or [2], pp. 10-11) we obtain

(1.7) $$\frac{1}{n}\frac{c_1}{c_0} \geq \left(\frac{2}{n(n-1)}\frac{c_2}{c_0}\right)^{1/2} \geq \left(\frac{6}{n(n-1)(n-2)}\frac{c_3}{c_0}\right)^{1/3} \geq \ldots \geq \left(\frac{c_n}{c_0}\right)^{1/n} .$$

Hence

(1.8) $$c_1 \geq n c_0^{1-(1/n)} c_n^{1/n} .$$

On the other hand, expanding the polynomial (1.6) directly, we obtain

(1.9) $$\begin{cases} c_0 = -\det \mathcal{B} = -B , \\ c_n = -\det \mathcal{A} = -A , \\ c_1 = \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}^2 B_{ij} . \end{cases}$$

Using the well-known formula for the volume of a simplex (see, for example, [8] or [3]),

(1.10) $$V(A)^2 = -(n!)^{-2}A , \quad V(B)^2 = -(n!)^{-2}B ,$$

and substituting (1.9), (1.10) into (1.8), we get

(1.1) $$\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}^2 B_{ij} \geq 2n(n!)^2 V(A)^{2/n} V(B)^{2-(2/n)} .$$

We shall now prove the sufficiency and then the necessity of the stated condition for equality. First suppose the two simplices Σ_A, Σ_B are similar sequentially (that is to say, there is a similarity transformation mapping a_i on b_i for $i = 1, 2, \ldots, n+1$); define μ_0 by

(1.11) $a_{ij} = \mu_0 b_{ij}$, where $\mu_0 > 0$ and $i, j = 1, 2, \ldots, n+1$.

From (1.2) and (1.3), we have

(1.12) $$q_{ij} = \mu_0^2 r_{ij} \text{ , that is } Q = \mu_0^2 R \text{ .}$$

By a linear transformation, R is transformed to the unit matrix:

$$TRT' = E \text{ ,}$$

and then

(1.13) $$T(Q+\lambda R)T' = TQT' + \lambda E \text{ .}$$

Putting $\lambda = -\mu_0^2$, we obtain

$$Q + \lambda R = Q - \mu_0^2 R = 0 \text{ ,}$$

$$T(Q+\lambda R)T' = 0 \text{ .}$$

It follows from (1.13) that

(1.14) $$TQT' - \mu_0^2 E = 0 \text{ .}$$

Equation (1.14) means that μ_0^2 is n-fold eigenvalue of the matrix TQT' , that is to say, it is the n-fold root of the equation

$$\det(TQT'-\mu E) = 0 \text{ .}$$

Then, from (1.13), $-\mu_0^2$ is the n-fold root of the equation

$$\det\left(T(Q+\lambda R)T'\right) = (\det T)^2 \det(Q+\lambda R) = 0 \text{ ,}$$

that is to say, $-\mu_0^2$ is the n-fold root of the equation

$$\det(Q+\lambda R) = -\det F(\lambda) = 0 \text{ .}$$

Therefore equality holds in Maclaurin's Inequality:

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(1.15)
$$\frac{1}{n}\frac{c_1}{c_0} = \left(\frac{c_n}{c_0}\right)^{1/n} .$$

Thus equality holds also in (1.1), and the sufficiency is proved.

Conversely, suppose equality holds in (1.1). This means that $\det(Q+\lambda R) = 0$ has an n-fold root. Denote this root by $-\mu_1^2$. Then

$$TQT' - \mu_1^2 E = 0 .$$

From (1.13) we obtain $Q - \mu_1^2 R = 0$, or $Q = \mu_1^2 R$, that is to say, $q_{ij} = \mu_1^2 r_{ij}$. Finally, from (1.2) we obtain

$$a_{ij} = |\mu_1| b_{ij} \quad (i, j = 1, 2, \ldots, n+1) ,$$

which means that Σ_A and Σ_B are similar. This proves the necessity.

2. Proof of Theorem 2

Let $\theta_{ij}(B)$ be the interior angle formed by the two $(n-1)$-dimensional faces opposite the vertices b_i, b_j of the simplex Σ_B . In order to prove Theorem 2, we first state the following lemma.

LEMMA. *With the same notation as before,*

(2.1)
$$\cos \theta_{ij}(B) = \frac{B_{ij}}{\sqrt{B_{ii}B_{jj}}} \quad (i, j = 1, 2, \ldots, n+1) .$$

This lemma, which is the higher dimensional cosine law, can be found in [13].

Let $S_i(B)$ and $S_j(B)$ be the $(n-1)$-dimensional volumes of the $(n-1)$-dimensional faces opposite the vertices b_i and b_j , respectively. By the well-known formula quoted above, we obtain

$$(2.2)\qquad \begin{cases} S_i(B)^2 = -\left((n-1)!\right)^{-2} B_{ii} , \\ \\ S_j(B)^2 = -\left((n-1)!\right)^{-2} B_{jj} . \end{cases}$$

Substituting (2.1) and (2.2) into (1.1), we have

$$(1.1^*)\quad \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{ij}^2 S_i(B) S_j(B) \cos\theta_{ij}(B) \ge 2n^3 V(A)^{2/n} V(B)^{2-(2/n)} .$$

This inequality we call our Theorem 1*. Now we go on to prove Theorem 2. It is very easy to check the following equality:

$$(2.3)\qquad \tfrac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} b_{ij}^2 B_{ij} = -nB = n(n!)^2 V(B)^2 .$$

Next, by the hypothesis 2° of Theorem 2, the simplex Σ_B is non-obtuse, that is $\cos\theta_{ij}(B) \ge 0$, hence

$$(2.4)\qquad B_{ij} \ge 0 , \quad (i, j = 1, 2, \ldots, n+1) .$$

Moreover, since, by condition 1° of Theorem 2, $a_{ij} \le b_{ij}$, we obtain, from (2.3),

$$(2.5)\qquad \begin{aligned} V^2(B) &= \frac{1}{2n(n!)^2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} b_{ij}^2 B_{ij} \\ &\ge \frac{1}{2n(n!)^2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a_{ij}^2 B_{ij} \\ &\ge V(A)^{2/n} V(B)^{2-(2/n)} . \end{aligned}$$

Hence

$$V(A) \le V(B) ,$$

and Theorem 2 is proved.

3. Non-euclidean cases

In this section we consider the possibility of extending Theorem 2 to non-euclidean spaces with constant curvature. Theorem 2 is not valid, in general, in hyperbolic space. We can find counter-examples even in 2 dimensions.

THEOREM 3.1. *There are two triangles* Δ_A, Δ_B *in the hyperbolic (or Lobachevsky) plane such that*

1° *the length of each side of* Δ_A *does not exceed the length of the corresponding side of* Δ_B ,

2° Δ_B *is non-obtuse,*

but

$$\text{area } \Delta_A > \text{area } \Delta_B .$$

Proof. Let Δ_A be an equilateral triangle with each of its angles equal to $\pi/6$; and let Δ_B be a right isosceles triangle, with the two sides adjacent to the right angle of the same length as a side of Δ_A .

First of all, as we know, in hyperbolic spaces the hypotenuse of a right triangle is the longest side, so we have

1° the length of each side of Δ_A does not exceed the length of the corresponding side of Δ_B .

Next, since Δ_B is a right triangle, and because the sum of the interior angles is less than π in any triangle, Δ_B can not have an obtuse angle, that is to say

2° Δ_B is non-obtuse.

Finally, compare the areas of Δ_A and Δ_B . As the defect of Δ_A is equal to $\pi/2$, we have

$$\text{area } \Delta_A = \frac{1}{|K|} \frac{\pi}{2} ,$$

where K is the curvature of the hyperbolic plane. On the other hand, the sum of the interior angles of Δ_B is greater than $\pi/2$, hence its defect is less than $\pi/2$, and we have

$$\text{area } \Delta_B < \frac{1}{|K|} \frac{\pi}{2} .$$

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Comparing these, we obtain

$$\text{area } \Delta_A > \text{area } \Delta_B ,$$

and the proof of Theorem 3.1 is completed.

Theorem 2 is, however, valid in 2-dimensional elliptic geometry, or, equivalently, in spherical trigonometry. We prove:

THEOREM 3.2. *If* Δ_A *and* Δ_B *are two spherical triangles on the same (2-dimensional) spherical surface, such that*

1° *the length of each side of* Δ_A *does not exceed the length of the corresponding side of* Δ_B ,

2° Δ_B *is non-obtuse,*

then

$$\text{area } \Delta_A \le \text{area } \Delta_B .$$

Proof. Without loss of generality we take the radius of the sphere to be unity; and we denote the corresponding side-lengths of Δ_A and Δ_B by a_{ij} and b_{ij} $(i, j = 1, 2, 3)$, and the corresponding interior angles by α_{ij} and β_{ij} , respectively. Let

$$A^* = \begin{pmatrix} 1 & \cos a_{12} & \cos a_{13} \\ \cos a_{21} & 1 & \cos a_{23} \\ \cos a_{31} & \cos a_{32} & 1 \end{pmatrix} ,$$

$$B^* = \begin{pmatrix} 1 & \cos b_{12} & \cos b_{13} \\ \cos b_{21} & 1 & \cos b_{23} \\ \cos b_{31} & \cos b_{32} & 1 \end{pmatrix} ,$$

$$\det A^* = A^* , \quad \det B^* = B^* .$$

Because A^* and B^* are positive definite, the equation

$$\det(A^*+\lambda B^*) = 0 \tag{3.1}$$

has only negative real roots. Expanding (3.1), we obtain

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$$(3.2)\quad B^*\lambda^3 + \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \cos a_{ij}B^*_{ij}\right)\lambda^2 + \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \cos b_{ij}A^*_{ij}\right)\lambda + A = 0 .$$

Using the arithmetic-geometric inequality, we have, from (3.2),

$$(3.3)\quad \frac{1}{3B^*}\sum_{i=1}^{3}\sum_{j=1}^{3} \cos a_{ij}B^*_{ij} \geq \left(\frac{A^*}{B^*}\right)^{1/3} ,$$

or

$$(3.4)\quad \sum_{i=1}^{3}\sum_{j=1}^{3} \cos a_{ij}B^*_{ij} \geq 3A^{*1/3}B^{*2/3} .$$

On the other hand, by the spherical cosine law we have

$$(3.5)\quad B^*_{ij} = -\sqrt{B^*_{ii}}\,\sqrt{B^*_{jj}} \cos \beta_{ij} \quad (i \neq j) .$$

Then (3.4) can be written as

$$(3.6)\quad \sum_{k=1}^{3} B^*_{kk} - \sum_{i\neq j} \cos a_{ij}\sqrt{B^*_{ii}}\,\sqrt{B^*_{jj}} \cos \beta_{ij} \geq 3A^{*1/3}B^{*2/3} .$$

Now $a_{ij} \leq b_{ij}$, by 1°; hence $\cos a_{ij} \geq \cos b_{ij}$. Furthermore, by 2°, Δ_B is non-obtuse, so $\cos \beta_{ij} \geq 0$. Thus the value of the left-hand side of (3.6) does not decrease if we replace $\cos a_{ij}$ by $\cos b_{ij}$, that is to say,

$$\sum_{k=1}^{3} B^*_{kk} - \sum_{i\neq j} \cos b_{ij}\sqrt{B^*_{ii}}\,\sqrt{B^*_{jj}} \cos \beta_{ij} \geq 3A^{*1/3}B^{*2/3} .$$

Hence, using (3.5),

$$(3.7)\quad \sum_{i=1}^{3}\sum_{j=1}^{3} \cos b_{ij}B^*_{ij} \geq 3A^{*1/3}B^{*2/3} .$$

On the other hand, expanding B^* directly, we have

$$(3.8)\quad \sum_{i=1}^{3}\sum_{j=1}^{3} \cos b_{ij}B^*_{ij} = 3B^* .$$

Comparing (3.7) with (3.8), one gets then

$$(3.9)\quad A^* \leq B^* .$$

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Again as a consequence of 1°, $\cos(a_{ij}/2) \geq \cos(b_{ij}/2)$, and we obtain

$$(3.10)\quad \frac{\sqrt{A^*}}{\cos(a_{23}/2)\cos(a_{31}/2)\cos(a_{12}/2)} \leq \frac{\sqrt{B^*}}{\cos(b_{23}/2)\cos(b_{31}/2)\cos(b_{12}/2)} .$$

By the area formula for spherical triangles

$$\sin\left(\tfrac{1}{2} \text{ area } \Delta_A\right) = \frac{\sqrt{A^*}}{4\cos(a_{23}/2)\cos(a_{31}/2)\cos(a_{12}/2)} ,$$

$$\sin\left(\tfrac{1}{2} \text{ area } \Delta_B\right) = \frac{\sqrt{B^*}}{4\cos(b_{23}/2)\cos(b_{31}/2)\cos(b_{12}/2)} .$$

Thus finally, we get

$$\sin\left(\tfrac{1}{2} \text{ area } \Delta_A\right) \leq \sin\left(\tfrac{1}{2} \text{ area } \Delta_B\right) ,$$

and thus

$$\text{area } \Delta_A \leq \text{area } \Delta_B ,$$

and the proof of Theorem 3.2 is completed.

The method used in this paper can not extend Theorem 2 to higher-dimensional elliptic spaces. We propose the following conjecture and hope the reader either proves or disproves it.

CONJECTURE. *Let* Σ_A^*, Σ_B^* *be two simplices in an* n-*dimensional elliptical space (or on an* n-*sphere) such that*

1° $a_{ij} \leq b_{ij}$ $(i, j = 1, 2, \ldots, n+1)$,

2° Σ_B^* *is non-obtuse;*

then

$$V(A^*) \leq V(B^*) .$$

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