BULL. AUSTRAL. MATH. SOC. 46A07
VOL. 22 (1980), 99-112.

ON THE STABILITY OF BARRELLED TOPOLOGIES, III

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Let E be a barrelled space with dual $F \neq E^*$. It is shown that F has uncountable codimension in E^* . If M is a vector subspace of E^* of countable dimension with $M \cap F = \{o\}$, the topology $\tau(E, F+M)$ is called a countable enlargement of $\tau(E, F)$. The results of the two previous papers are extended: it is proved that a non-barrelled countable enlargement always exists, and sufficient conditions for the existence of a barrelled countable enlargement are established, to include cases where the bounded sets may all be finite dimensional. An example of this case is given, derived from Amemiya and Kōmura; some specific and general classes of spaces containing a dense barrelled vector subspace of codimension greater than or equal to c are discussed.

1. Completeness, codimension in E^* and non-barrelled countable enlargements

We use the notation and terminology of [7], [12] and the abstract above. All the spaces are supposed locally convex and Hausdorff.

THEOREM 1. *Let E be a barrelled space with dual $F \neq E^*$ and let M be a vector subspace of E^* , with $M \cap F = \{o\}$, of finite or countable dimension. Then E with the topology $\tau(E, F+M)$ is not complete.*

Proof. Let H be a hyperplane in $F + M$, containing F . Then H

Received 19 December 1979. The second author would like to thank the University of Western Australia for some financial support enabling him to visit the University.

https://doi.org/10.1017/S0004972700006377 Published online by Cambridge University Press

is dense, and if E with $\tau(E, F+M)$ is complete, there is an absolutely convex $\sigma(F+M, E)$-compact set B such that $H \cap B$ is not compact. By the lemma in [12], $B \subseteq A + C$, where A is $\sigma(F, E)$-compact and C finite dimensional and compact. Since B is closed in $A + C$, $H \cap B$ is closed in $H \cap (A+C)$. But $H \cap (A+C) = A + H \cap C$ since $A \subseteq F \subseteq H$, and this is a sum of compact sets, and so is compact. Thus $H \cap B$ is compact, a contradiction.

In [7] (Section 3) it was pointed out that the property of being barrelled may or may not be stable under the removal of one dimension from the dual. We may look at Theorem 1 from this point of view.

COROLLARY. *Let* E *be a complete space and let* F *be a dense vector subspace of the dual of* E, *of finite or countable codimension. Then* $\tau(E, F)$ *is not barrelled.*

Proof. If E is complete under a topology with dual $F + M$, then E is complete under $\tau(E, F+M)$.

Although we make no use of it here, we observe a simple counterpart to this corollary: if E is B-complete (or B_r-complete) and F is a proper dense vector subspace of its dual, then $\tau(E, F)$ is not barrelled. For if F is not closed in the dual G of E, there is an absolutely convex $\sigma(G, E)$-compact set B such that $F \cap B$ is not $\sigma(G, E)$-compact. Then $F \cap B$ is $\sigma(F, E)$-bounded but not $\tau(E, F)$-equicontinuous. (This result is also a consequence of the closed graph theorem.)

Theorem 1 may also be proved by applying Proposition 2.2 of [2], using the lemma of [12] and the remarks in [7], Section 6.

The most striking case of Theorem 1 is also our most useful.

THEOREM 2. *The dual of a barrelled space* E *is either* E^* *or has uncountable codimension in* E^*.

(For E with $\tau(E, E^*)$ is complete.)

We turn to the problem of stability of a barrelled topology; we can now remove the restriction in Theorem 2 of [7] that E should have an infinite dimensional bounded set, and answer the question of existence of a non-barrelled countable enlargement completely.

THEOREM 3. *Let* E *be a barrelled space with dual* $F \neq E^*$. *Then* *there exists a vector subspace* M *of* E^* *, of countable dimension, with* $M \cap F = \{o\}$ *, such that* $\tau(E, F+M)$ *is not barrelled.*

Proof. Let H be a hyperplane in E^* containing F . Then $\tau(E, H)$ is not barrelled, by Theorem 2, and so there is a $\sigma(H, E)$-bounded set B not $\tau(E, H)$-equicontinuous. Since $\tau(E, F)$ is barrelled, so is $\tau(E, F+N)$ for any finite dimensional $N \subseteq H$ ([7], Theorem 1) and so $B \nsubseteq F + N$. Hence there is a linearly independent countable subset A of B such that $F \cap \text{span } A = \{o\}$. Let $M = \text{span } A$. Then A is $\sigma(F+M, E)$-bounded but does not lie in any $F + N$, and so ([12], Theorem 2) $\tau(E, F+M)$ is not barrelled.

2. An example of Amemiya and Kōmura

In ([1], Theorem 1) a general result was given on the existence of barrelled countable enlargements for spaces with suitable infinite dimensional bounded sets. We now consider a specific example of a barrelled topology which has a barrelled countable enlargement although the bounded sets are finite dimensional. Further examples will be given in Section 4.

Amemiya and Kōmura construct in ([1], Section 4) a dual pair (E, F) and a countable dimensional subspace H of E^* such that

(i) the bounded sets for the topologies $\sigma(E, F)$, $\sigma(F, E)$ and $\sigma(F+H, E)$ are finite dimensional,

(ii) $F + H$ is separable under $\sigma(F+H, E)$.

There is in fact an error in the proof of their lemma which has been corrected in [5]. Referring to their proof, the error is that their space $G = G_0 \times \omega(N_2)$ has codimension 1 and not c in ω . The first part of the construction in Section 8 of [7], applied to $\omega(N_2)$, or alternatively Corollary 2 of Lemma 3 here, yield a simpler correction of this error, since $\omega(N_2)$ thereby has a dense barrelled vector subspace L of codimension c containing the e_n $(n \in N_2)$. The original proof works if we redefine G to be $G_0 \times L$.

By (i), the space E is barrelled under $\sigma(E, F)$ and $\sigma(E, F+H)$. It remains to show that F has infinite codimension in $F + H$, for then

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any supplement M of F in $F + H$ will be a countable dimensional subspace of E^* such that $F \cap M = \{o\}$ and $\tau(E, F+M) = \sigma(E, F+M)$ is barrelled. We establish first:

LEMMA 1. *F is not $\sigma(F, E)$-separable.*

Proof. We refer to [1] for notation and definitions. Let (f_n) be any sequence in F. Since each f_n is a finite linear combination of elements of $\cup\{F_\alpha : \alpha < \Omega\}$, we can find $\alpha_n < \Omega$ such that f_n vanishes on $E\ (> \alpha_n)$ $(n \in \mathbb{N})$. Then if $\alpha = \sup_n \alpha_n$ we have $\alpha < \Omega$ and each f_n vanishes on $E\ (> \alpha)$. This allows us to find a non-zero element of $E = E\ (< \alpha) \times E\ (\alpha) \times E\ (> \alpha)$ which annihilates each f_n, and so $\{f_n : n \in \mathbb{N}\}$ cannot be $\sigma(F, E)$-dense in F.

Our assertion now follows from (ii) and the following general result, which is probably well-known. (See Note Added in Proof, p. 110.)

LEMMA 2. *Let E be a separable Hausdorff locally convex space and let L be a vector subspace of E of finite codimension. Then L is separable in the induced topology.*

Proof. Suppose that L has codimension 1 in E and let f be an element of E^* such that $L = f^{-1}(0)$. Denoting the dual of E by F as usual, let G be the linear span of $\{f\} \cup F$. By the lemma in [14], E is $\sigma(E, G)$-separable. Since L is $\sigma(E, G)$-closed and of finite codimension, E under $\sigma(E, G)$ is the topological direct sum of L and any one of its supplements. Thus L is $\sigma(E, G)$-separable and therefore $\sigma(E, F)$-separable since the two topologies coincide on L. Since the same convex subsets are separable for all topologies of a given dual pair, we deduce that L is separable for the original topology of E.

The general case follows by induction.

Let E, F, H be as in the above example and let E_1 be any Hausdorff barrelled space over the same field with dual F_1. Put $E_0 = E \times E_1$, $F_0 = F \oplus F_1$ and $H_0 = \{(h, o) : h \in H\} \subseteq E^* \oplus E_1^*$. We see easily that

(a) F_0 is a countable codimensional vector subspace of $F_0 + H_0$,

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(b) $\tau(E_0, F_0)$ and $\tau(E_0, F_0+H_0)$ are barrelled topologies,

(c) the $\sigma(E_0, F_0)$-bounded sets have the same dimensions as the $\sigma(E_1, F_1)$-bounded sets.

Thus examples exist of barrelled spaces with barrelled countable enlargements and with bounded sets of any dimensions possible.

In conclusion it is perhaps of interest to note that if a Hausdorff locally convex space E has an infinite dimensional $\sigma(E, F)$-bounded set, then it has an infinite dimensional $\sigma(E, F+M)$-bounded set for any countable dimensional vector subspace M of E^*. For if every $\sigma(E, F+M)$-bounded set is finite dimensional, then $\sigma(F+M, E)$ is barrelled. Hence $\sigma(F, E)$ is barrelled (since F has at most countable codimension in $F + M$) and so the $\sigma(E, F)$-bounded sets are finite dimensional. (This may also be proved by direct elementary means.) In particular, if F has countable codimension in E^*, every $\sigma(E, F)$-bounded set is finite dimensional.

3. Existence of barrelled countable enlargements

In this section we establish some general sufficient conditions for the existence of barrelled countable enlargements, and discuss spaces having dense barrelled vector subspaces of uncountable codimension.

Let (E, F) be a dual pair. We shall say that F has *plenty of bounded sequences* if and only if every sequence in F contains an infinite subsequence spanned by a $\sigma(F, E)$-bounded set. Equivalently, for every sequence (f_n) in F, there exist scalars (λ_n), with $\lambda_{n_k} \neq 0$ for some sequence (n_k), such that $\lambda_n f_n \to o$ in $\sigma(F, E)$.

For example, if F is metrisable under some topology finer than $\sigma(F, E)$, then F has plenty of bounded sequences. But if F is the strict inductive limit of a sequence of Fréchet spaces and has dual E, then F does not have plenty of bounded sequences. Also if $F = \mathbb{R}^\Lambda$, then F has plenty of bounded sequences if and only if Λ is countable or finite.

THEOREM 4. *Let E be a barrelled space with dual $F \neq E^*$ and*

$\dim E \geq c$. *Suppose that* G *is a vector subspace of* E *of dimension* c *such that the dual of* G *has plenty of bounded sequences. Then there exists a barrelled countable enlargement.*

Proof. Since $\dim G = c$, $G \cong \mathbb{R}^{\mathbb{N}}$ (or $\mathbb{C}^{\mathbb{N}}$) algebraically. Let H be the dual of G under the product topology, so that $H \cong \mathbb{R}^{(\mathbb{N})}$ (or $\mathbb{C}^{(\mathbb{N})}$), and let K be the dual of G under the topology induced by $\tau(E, F)$. Then $H \cap K$ cannot be infinite dimensional. For if (h_n) is a linearly independent sequence in $H \cap K$, then, since K has plenty of bounded sequences, some subsequence lies in the span of a $\sigma(K, G)$-bounded set. Since $\sigma(H, G)$-bounded sets are finite dimensional, this subsequence, and hence (h_n) , cannot be linearly independent.

Let N be an algebraic supplement of $H \cap K$ in H and let M be the set of extensions to E of elements of N , defined to be zero on some algebraic supplement of G in E . Then M is a vector subspace of E^* of countable dimension and $M \cap F = \{o\}$ since $N \cap K = \{o\}$.

Let B be absolutely convex $\sigma(F+M, E)$-bounded and C the set of restrictions to G of the elements of B ; then C is $\sigma(K+N, G)$-bounded. If the natural projection of C on N is not finite dimensional, there is a sequence $(g_n) \subseteq C$ such that $g_n = f_n + h_n$, $f_n \in K$, $h_n \in N$ and (h_n) linearly independent. Since K has plenty of bounded sequences, there are scalars λ_n such that $\lambda_n f_n \to o$ in $\sigma(K, G)$ and $\lambda_n \neq 0$ for an infinity of n . We may suppose $|\lambda_n| \leq 1$. Then $(\lambda_n g_n) \subseteq C$ and so is bounded on G , $(\lambda_n f_n)$ is bounded on G , and therefore $(\lambda_n h_n)$ is $\sigma(H, G)$-bounded. Hence some infinite subsequence of (h_n) is finite dimensional, contradicting the linear independence of (h_n) . Thus the projection of C on N is finite dimensional, so also is the projection of B on M , and by Theorem 2 of [12], $\tau(E, F+M)$ is barrelled.

This theorem is a natural generalisation of Theorem 1 of [12]. The hypothesis there that E contains a bounded set of c-dimensional span G ensures that the dual of G (under the topology induced by $\tau(E, F)$)

is a subspace of a normed space with dual containing G , and so has plenty of bounded sequences.

In [7], Theorem 5 shows that if M° is dense and barrelled, and of countable codimension, then $\tau(E, F+M)$ is not barrelled, and the associated barrelled topology is identified. By contrast we have the next result (*cf.* [7], Theorem 3, [12], Theorem 1 and also [2], Proposition 3.2 and Example 3.5).

THEOREM 5. *Suppose that* E *is barrelled with dual* $F \neq E^*$, *and that* E *has a dense barrelled vector subspace of codimension greater than or equal to* c . *Then there exists a barrelled countable enlargement.*

Proof. Let L be a vector subspace of E , of codimension c , containing the given subspace and so itself both dense and barrelled. Let G be an algebraic supplement of L in E ; then $G \cong \mathbb{R}^{\mathbb{N}}$ (or $\mathbb{C}^{\mathbb{N}}$) algebraically. Let H be the dual of G under the product topology, and let M be the set of extensions to E of elements of H , defined to be zero on L . Then M has countable dimension, $M \cap F = \{o\}$ since L is dense, and $M^{\circ} = L$.

Let B be $\sigma(F+M, E)$-bounded. Then the projection X of B on F is $\sigma(F, M^{\circ})$-bounded, and so $\tau(E, F)$-equicontinuous (by [7], lemma in Section 6) since M° is dense and barrelled. Hence the projection Y of B on M is bounded, and, since $\sigma(H, G)$-bounded sets are finite dimensional, so is Y . Thus $B \subseteq X + Y$ is $\tau(E, F+M)$ equicontinuous.

We are therefore led to search for dense barrelled vector subspaces of codimension greater than or equal to c . The next result is perhaps of independent interest.

THEOREM 6. *Suppose that* E *is barrelled and that in the dual* F *of* E , *every sequence has an infinite subsequence lying in a* $\sigma(E^*, E)$-*complete vector subspace of* F . *Then every dense vector subspace of* E *is barrelled.*

Proof. Let L be a dense vector subspace of E ; then $\sigma(F, L)$ is Hausdorff. Let B be $\sigma(F, L)$-bounded; if B is not $\sigma(F, E)$-bounded, there is some x in E and a sequence (f_n) in B with $|f_n(x)| \to \infty$. Some subsequence of (f_n) lies in a $\sigma(E^*, E)$-complete vector subspace N

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of F, by hypothesis; it is $\sigma(F, L)$-bounded but not $\sigma(F, E)$-bounded. Now N is of minimal type under the topology induced by $\sigma(F, E)$ ([10], page 191) and $\sigma(F, L)$ is a coarser Hausdorff topology. Hence these topologies must coincide on N, which gives a contradiction. Thus B is $\sigma(F, E)$-bounded, equicontinuous on E since E is barrelled, and so certainly on L.

(See also Section 4 and [4], 1.9.)

Next, we note two general examples of a barrelled space with a dense barrelled vector subspace of codimension greater than or equal to c. To avoid repetition, we shall call such a subspace *satisfactory*, for the rest of this section.

NOTE 3.1. *Let G be a barrelled space and H a dense hyperplane in G. If Λ is an index set with $|\Lambda| \geq c$, put $G_\lambda = G$ and $H_\lambda = H$ for each $\lambda \in \Lambda$, and take E to be the direct sum $\sum \{G_\lambda : \lambda \in \Lambda\}$. Then $L = \sum \{H_\lambda : \lambda \in \Lambda\}$ is satisfactory.*

NOTE 3.2. *Let E be a barrelled space and E_1 any barrelled vector subspace of E. Then if E_1 has a satisfactory subspace, so has E.*

Proof. Let G be an algebraic supplement of E_1 in E, L_1 a satisfactory subspace of E_1, and put $L = L_1 + G$. Then L is dense. Let B be a barrel in L. Then $B \cap L_1$ is a barrel, and so a neighbourhood of o, in L_1; since L_1 is dense in E_1, $E_1 \subseteq \text{span } \overline{B \cap L_1} \subseteq \text{span } \overline{B}$ (closures in E). Thus $\overline{B}$ is a barrel, and so a neighbourhood of o, in E, and $B = \overline{B} \cap L$ is a neighbourhood of o in L.

COROLLARY. *If E is a strict inductive limit of a sequence (E_n) of barrelled spaces and some E_n has a satisfactory subspace, then so has E.*

Finally, we collect together some conditions of Baire type. It is convenient to state here the following:

LEMMA 3. *Suppose that* $\dim E \geq c$. *We may write* $E = \cup E_n$, *where*

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(E_n) *is an increasing sequence of vector subspaces of* E *and* codim $E_n \geq c$ *for each* n .

Proof. If $\{x_\lambda : \lambda \in \Lambda\}$ is a basis of E , write $\Lambda = \cup\Lambda_n$, where (Λ_n) is an increasing sequence and $|\Lambda_n| = |\Lambda\backslash\Lambda_n| = |\Lambda|$ for each n . Then let $E_n = \mathrm{span}\{x_\lambda : \lambda \in \Lambda_n\}$. Each E_n has codimension $|\Lambda| \geq c$.

(This method is essentially that of Saxon and Levin [9], Example A, used in [7], Section 8 and in [2], Example 3.6, Remark (ii).)

COROLLARY 1. *If* E *is barrelled,* dim $E \geq c$ *and its completion* $\hat{E}$ *is a Baire space, then* E *has a dense vector subspace of codimension greater than or equal to* c .

Proof. Writing $E = \cup E_n$ as in Lemma 3, some E_n is dense by Theorem 4 of Valdivia [13].

COROLLARY 2. *If* E *is a Baire space and* dim $E \geq c$, *then* E *has a satisfactory subspace.*

Proof. By Corollary 1, we may write $E = \cup\{E_n : n \geq m\}$ where E_m , and so each E_n , is dense. Then some E_n is non-meagre and so barrelled.

COROLLARY 3. *The strict inductive limit of a sequence of Fréchet spaces has a satisfactory subspace.*

(By Corollary 2 and the corollary to Note 3.2.)

Clearly, if dim $E \geq c$, E has a satisfactory subspace if E possesses the following property.

(db) *If* E *is the union of an increasing sequence* (E_n) *of vector subspaces, then some* E_n *is dense and barrelled.*

In [8], Saxon defines Baire-like spaces, and Todd and Saxon [11] point out that the theorem of Valdivia referred to above shows that if E is barrelled and $\hat{E}$ is a Baire space, then E is Baire-like. They use a stronger property to define unordered Baire-like spaces, and prove ([11], 2.2) its equivalence to the more demanding condition than (db), obtained by omitting the word "increasing". A proof similar to part of their Theorem 2.2 shows that (db) implies that E is Baire-like. Thus we have:

Baire $\Rightarrow$ unordered Baire-like $\Rightarrow$ (db) $\Rightarrow$ Baire-like $\Rightarrow$ barrelled and (trivially)

COROLLARY 4. *If* E *is unordered Baire-like and* $\dim E \geq c$, *then* E *has a satisfactory subspace.*

We point out incidentally that, for the proof of the closed graph theorem cited by Todd and Saxon ([6], Theorem 2), the property (db) is a sufficient condition on the domain in the usual case, when the range space is the inductive limit of an increasing sequence of subspaces.

4. *GM*- and *GN*-spaces

The *GM*-spaces introduced by Eberhardt and Roelcke in [4] provide a class of barrelled spaces whose bounded sets are finite dimensional and whose duals have the sequential property required in Theorem 6 ([4], 1.1, 2.1). Thus we can apply Theorem 5 to a *GM*-space provided that it has a dense vector subspace of codimension greater than or equal to c . This is true for a specific subclass of *GM*-spaces.

THEOREM 7. *Let* E *be an* $\aleph_0$*-product* ([4], 3.4) *with at least* c *nonzero factors. Then* E *has a dense vector subspace of codimension greater than or equal to* c .

Proof. We have $E = \prod \{E_\mu : \mu \in I\}$ with the projective limit topology defined by the natural projections

$$p_J : E \to \prod \{E_\mu : \mu \in J\}$$

where J runs through all countable subsets of I and $\prod \{E_\mu : \mu \in J\}$ has its finest locally convex topology ([4], 3.5, 3.7). The vector subspace $G = \{(x_\mu) : |\{\mu : x_\mu \neq o\}| \leq \aleph_0\}$ is clearly dense in E . The result will follow if we show that G has codimension at least c in E . By hypothesis we may choose subsets I_r $(r \in \mathbb{R})$ of I such that for all r ,

$$|I_r| = c \ , \quad E_\mu \neq \{o\} \ \text{ if } \ \mu \in I_r \ \text{ and } \ I_r \cap I_s = \emptyset \quad (r \neq s) \ .$$

Then let x_r be any element of E with I_r as support $(r \in \mathbb{R})$. The linear span of $\{x_r : r \in \mathbb{R}\}$ has dimension c and intersects G only in o.

We do not know if there is a GM-space (not having its finest locally convex topology) for which no barrelled countable enlargement exists. However we can answer the corresponding question about the GM-property. We begin with two lemmas which are of independent interest.

LEMMA 4. *Let E be a barrelled space with $F \neq E^*$ and let M be a countable dimensional vector subspace of E^* such that $M \cap F = \{o\}$. Then $M^{oo} \nsubseteq F + M$ (bipolar in E^*).*

Proof. Suppose $M^{oo} \subseteq F + M$ and let $P = F \cap M^{oo}$. Then P is $\sigma(F, E)$-closed and $M^{oo} = P + M$. Consider E/P^o. This is barrelled under the quotient topology defined by $\tau(E, F)$, its dual is P and its algebraic dual is P^{oo} (bipolar in E^*), which is a vector subspace of M^{oo}. Since P therefore has at most countable codimension in P^{oo}, it follows by Theorem 2 that $P = P^{oo}$.

Now M^{oo} is barrelled under the topology induced by $\sigma(E^*, E)$ (being the algebraic dual of E/M^o) and P is a closed countable codimensional subspace of M^{oo}. Thus by ([9], p. 92, Proposition) $\sigma(E^*, E)$ induces the finest locally convex topology on M, which is certainly false.

The next lemma is concerned with the condition of Theorem 6.

LEMMA 5. *Suppose that E is a barrelled space and has a barrelled countable enlargement $\tau(E, F+M)$. Then no infinite dimensional sequence in M lies in a $\sigma(F+M, E)$-complete vector subspace.*

Proof. If this is false we can find an infinite dimensional vector subspace P of M such that $P^{oo} \subseteq F + M$ (bipolar in E^*). Let Q be a supplement of P in M and put $G = F + Q$. It follows easily from the results of ([12], Section 3) that $\tau(E, G)$ is barrelled. Thus by Lemma 4,

$$P^{oo} \nsubseteq G + P = F + M$$

which is a contradiction.

Recalling our introductory comments on GM-spaces we may deduce immediately:

THEOREM 8. *Let E be a Hausdorff GM-space with dual $F \neq E^*$. Then E cannot be a GM-space under any countable enlargement $\tau(E, F+M)$.*

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NOTE 4.1. It is easily seen that the *GM*-property is preserved under finite dimensional enlargements of the dual space.

NOTE 4.2. Theorem 8 and ([4], 1.4) show that the *GM*-property is an example of a property which is always inherited by countable codimensional vector subspaces but is never preserved by countable dimensional enlargements of the dual.

NOTE 4.3. It is clear from the dual characterisation of *GM*-spaces ([4], 1.1) that the dual F of an infinite dimensional *GM*-space E must have infinite dimensional $\sigma(F, E)$-bounded sets. Thus the space E of Amemiya and Kōmura discussed in Section 2 is not a *GM*-space.

Eberhardt's *GN*-spaces [3], which are also barrelled, exhibit rather different behaviour.

THEOREM 9. *Let E be a Hausdorff GN-space with dual $F \neq E^*$ and suppose that M is a vector subspace of E^*, of countable dimension, such that $F \cap M = \{o\}$. Then E with $\tau(E, F+M)$ is a GN-space if and only if it is barrelled.*

Proof. The necessity is clear. Suppose that $\tau(E, F+M)$ is barrelled and let B be a $\sigma(F+M, E)$-bounded set. We know from ([12], Section 3) that $B \subseteq X + Y$ where X is a $\sigma(F, E)$-bounded set and Y is a finite dimensional bounded subset of M. By hypothesis the $\sigma(F, E)$-closed linear span of X is $\sigma(F, E)$-complete and therefore $\sigma(E^*, E)$-closed ([3], 2.3). Since the linear span of Y is finite dimensional, it follows that the $\sigma(E^*, E)$-closed linear span of $X + Y$ is contained in $F + M$. We then have immediately that the $\sigma(F+M, E)$-closed linear span of B is $\sigma(F+M, E)$-complete. Thus E with $\tau(E, F+M)$ is a *GN*-space.

Since each *GM*-space is a *GN*-space we have immediately:

COROLLARY. *Let E be a Hausdorff GM-space with a barrelled countable enlargement $\tau(E, F+M)$. Then E with $\tau(E, F+M)$ is a GN-space* (*cf.* Theorems 7, 8).

Note Added in Proof. Lemma 2 has been derived by Lech Drewnowski and Robert H. Lohman as Corollary 3 in their paper "On the number of separable locally convex spaces", *Proc. Amer. Math. Soc.* 58 (1976), 185-188.

We should like to thank the Editor, Dr S.A. Morris, and Professor S.A. Saxon, from whom he obtained this reference.

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