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RADICAL FORMULA AND WEAKLY PRIME SUBMODULES

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Abstract. Let B be a submodule of an R-module M. The intersection of all prime (resp. weakly prime) submodules of M containing B is denoted by $\mathrm{rad}(B)$ (resp. $\mathrm{wrad}(B)$). A generalisation of $\langle E(B)\rangle$ denoted by $UE(B)$ of M will be introduced. The inclusions $\langle E(B)\rangle \subseteq UE(B) \subseteq \mathrm{wrad}(B) \subseteq \mathrm{rad}(B)$ are motivations for studying the equalities $UE(B) = \mathrm{wrad}(B)$ and $UE(B) = \mathrm{rad}(B)$ in this paper. It is proved that if R is an arithmetical ring, then $UE(B) = \mathrm{wrad}(B)$. In Theorem 2.5, a generalisation of the main result of [**11**] is given.

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1. Introduction. Throughout this paper all rings are commutative with identity, and all modules are unitary. Also we consider R to be a ring, M a unitary R-module, B a submodule of M and $\mathbb{N}$ the set of positive integers.

Let N be a proper submodule of M. It is said that N is a *prime* submodule of M if the conditions $ra \in N$, $r \in R$ and $a \in M$ imply that $a \in N$ or $rM \subseteq N$. In this case, if $P = (N : M) = \{t \in R |\ tM \subseteq N\}$, we say that N is a P-*prime* submodule of M, and it is easy to see that P is a prime ideal of R. Prime submodules have been studied in several papers such as [**2–8**, **11–13**].

A proper submodule N of M is called a *weakly prime* submodule if for each $x \in M$ and $a, b \in R$, the condition $abx \in N$ implies that $ax \in N$ or $bx \in N$.

Weakly prime submodules have been studied in [**2**, **4**, **6**, **7**]. If we consider R as an R-module, then prime submodules and weakly prime submodules are exactly prime ideals of R. For every R-module, it is easy to see that any prime submodule is a weakly prime submodule, but the converse is not always true. For example let R be a ring with $\dim R \neq 0$ and $P \subset Q$ a chain of prime ideals of R. Then it is easy to see that for the free R-module $R \oplus R$, the submodule $P \oplus Q$ is a weakly prime submodule which is not a prime submodule.

Recall that for an ideal I of a ring R, the *radical* of I denoted by $\sqrt{I}$ is defined to be $\sqrt{I} = \{r \in R |\ r^n \in I,\ \text{for some } n \in \mathbb{N}\}$.

For any subset B of M, the *envelope* of B, denoted by $E(B)$ is defined to be

$$E(B) = \{x |\ x = ra,\ r^n a \in B,\ \text{for some } r \in R,\ a \in M,\ n \in \mathbb{N}\}.$$

$\langle E(B)\rangle$ is the submodule generated by $E(B)$. Then $\langle E(B)\rangle$ is a module version of the radical of ideals, and obviously $B \subseteq \langle E(B)\rangle$.

For an ideal I of a ring R, $\sqrt{\sqrt{I}} = \sqrt{I}$. But for its generalisation to modules, it is not true (see [**5**, Example 1]). So for a submodule B of M, we consider

$E_0(B) = B$, $E_1(B) = E(B)$, $E_2(B) = E(\langle E(B)\rangle)$, and for any positive integer n, it is defined $E_{n+1}(B) = E(\langle E_n(B)\rangle)$ inductively. We will call $E_n(B)$ the nth envelope of B.

Recall that for an ideal I of R, $\sqrt{I} = \bigcap_{\substack{P \text{ prime ideal} \\ I \subseteq P}} P.$ $(*)$

The intersection of all prime (resp. weakly prime) submodules of M containing B is denoted by rad(B) (resp. wrad(B)). If there does not exist any prime (resp. weakly prime) submodule of M containing B, then we say rad$(B) = M$ (resp. wrad$(B) = M$). Obviously wrad$(B) \subseteq$ rad(B).

As a generalisation of the equality $(*)$, it is said that a module M *satisfies the radical formula (s.t.r.f.)* if for every submodule B of M, $\langle E(B)\rangle =$ rad(B). It is said that a ring R s.t.r.f. if every R-module s.t.r.f. (see for example [**5**, **8**, **11**, **13**]).

For every submodule B of M, we consider

$$UE(B) = \bigcup_{n \in \mathbb{N}} \langle E_n(B)\rangle;$$

$UE(B)$ will be called the *union of envelopes* of B. One can easily see that $B \subseteq \langle E_n(B)\rangle \subseteq UE(B) \subseteq$ wrad$(B) \subseteq$ rad(B), for any $n \in \mathbb{N}$. Therefore $UE(B)$ is a submodule of M containing B and $UE(B) = \varinjlim \langle E_n(B)\rangle$. If we consider R as an R-module, then obviously for each ideal I of R, $UE(I)$ still is $\sqrt{I}$. Hence the equality $(*)$ provokes us to study the equalities $UE(B) =$ wrad(B) and $UE(B) =$ rad(B) in this paper.

Definition . Let $n \in \mathbb{N} \cup \{0\}$. If (resp. $\langle E_n(B)\rangle = wrad(B)$) $\langle E_n(B)\rangle = rad(B)$ for every submodule B of M, we will say that M (resp. weakly) s.t.r.f. of degree n. It will be said that the ring R (resp. weakly) s.t.r.f. of degree n, if every R-module (resp. weakly) s.t.r.f. of degree n.

Definition . Let M be an R-module. If (resp. $UE(B) = wrad(B)$) $UE(B) = rad(B)$, for every submodule B of M, we will say that the (resp. weakly) radical formula holds for M. It will be said that the (resp. weakly) radical formula holds for a ring R if the (resp. weakly) radical formula holds for every R-module.

Recall that a ring R is said to be an *arithmetical* ring if for all ideals I, J and K of R we have $I + (J \cap K) = (I + J) \cap (I + K)$ (see [**9**, **10**]).

According to [**9**, Theorem 1] a ring R is arithmetical if and only if for each maximal ideal P of R every two ideals of the ring R_P are comparable (R_P is a chained ring).

Then obviously Prüfer domains, valuation rings and Dedekind domains are arithmetical.

Note. Let B be a submodule of an R-module M.

(1) Dedekind domains or more generally ZPI-rings, or zero dimensional rings s.t.r.f. (see [8, Theorem 1, 13, Theorems 2.8, 2.10]). Hence in this case $UE(B) = wrad(B) = rad(B)$.
(2) If R is a Prüfer domain with dim R = n, then M s.t.r.f. of degree n (see [**5**, Theorem 2.4]). Consequently $UE(B) = wrad(B) = rad(B)$.
(3) If R is an arithmetical ring with DCC on prime ideals, or M has DCC on cyclic submodules, then the radical formula holds for M (see [**5**, Corollaries 2.5 and 2.7]). So $UE(B) = wrad(B) = rad(B)$

(4) If M is a multiplication module or has DCC on cyclic submodules or if M is a divisible module over an integral domain, then every weakly prime submodule of M is a prime submodule (see [**4**, Theorem 2.7]). Then obviously for these modules, $wrad(B) = rad(B)$.

The equality $\mathrm{wrad}(B) = \mathrm{rad}(B)$ has been studied in [**2**, Section 4].

2. Radical formula.

THEOREM 2.1. *The weakly radical formula holds for every arithmetical ring.*

Proof. Let B be a submodule of an R-module M, where R is an arithmetical ring. We will prove that $UE(B) = \mathrm{wrad}(B)$. We consider two cases.

Case 1. $B = 0$. We will show that for any maximal ideal P of R, $(UE(0))_P = (\mathrm{wrad}(0))_P$. Hence $(\frac{\mathrm{wrad}(0)}{UE(0)})_P = 0$, for any maximal ideal P of R, which implies that $UE(0) = \mathrm{wrad}(0)$, by [**1**, Proposition 3.8].

It is easy to see that

$$UE(0_P) = (UE(0))_P \subseteq (\mathrm{wrad}(0))_P \subseteq \mathrm{wrad}(0_P).$$

So it suffices to show that $\mathrm{wrad}(0_P) \subseteq UE(0_P)$. Therefore by localisation we may assume that every two ideals of the ring R are comparable. In this case we will show that $UE(0)$ is a weakly prime submodule of M.

Suppose that $r_1r_2x \in UE(0)$, where $r_1, r_2 \in R$ and $x \in M$. Every two ideals of the ring R are comparable; then assume that $r_2 = r_1t_1$, where $t_1 \in R$. Now we have $r_1^2(t_1x) \in UE(0) = \bigcup_{n\in\mathbb{N}}\langle E_n(0)\rangle$. Let $r_1^2(t_1x) \in \langle E_m(0)\rangle$, where $m \in \mathbb{N}$. Then $r_2x = r_1(t_1x) \in E\langle E_m(0)\rangle = E_{m+1}(0) \subseteq UE(0)$. This completes the proof.

Case 2. $0 \neq B$. Consider $\bar{M} = \frac{M}{B}$. By case 1, for the R-module $\bar{M}$ we have $UE(\frac{B}{B}) = \mathrm{wrad}(\frac{B}{B})$. One can easily check that $\frac{UE(B)}{B} = UE(\frac{B}{B})$ and $\mathrm{wrad}(\frac{B}{B}) = \frac{\mathrm{wrad}(B)}{B}$, which completes the proof. □

Recall that a ring R is said to be an *Archimedean* ring if $\cap_{n\in\mathbb{N}} Rr^n = 0$, for each non-unit $r \in R$.

In [**5**, Corollary 2.7] it is shown that if R is an arithmetical ring, then the radical formula holds for every R-module with DCC on cyclic submodules. In (iii) of the theorem given below, we will generalise this result and prove that every module with DCC on cyclic submodules over any arbitrary ring s.t.r.f.

LEMMA 2.2. *Let n be a non-negative integer.*

(i) *If for every maximal ideal P of R, R_P s.t.r.f. of degree n, then R s.t.r.f. of degree n.*

(ii) *Let R be a Noetherian ring. The radical formula holds for R if and only if the radical formula holds for R_P, for every maximal ideal P of R.*

Proof. Let B be a proper submodule of an R-module M.

(i) The proof follows from the following inclusions, which can be shown easily:

$$\langle E_n(B_P)\rangle = (\langle E_n(B)\rangle)_P \subseteq (\mathrm{rad}(B))_P \subseteq \mathrm{rad}(B_P).$$

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(ii) By [**11**, Corollary 2.3], $(\mathrm{rad}(B))_P = \mathrm{rad}(B_P)$. Consequently,

$$UE(B_P) = (UE(B))_P \subseteq (\mathrm{rad}(B))_P = \mathrm{rad}(B_P). \qquad \square$$

THEOREM 2.3. *Let R be a ring.*
(i) *If R is a local archimedean arithmetical ring, then R s.t.r.f.*
(ii) *If R is a local arithmetical ring with ACC on principal ideals, then R s.t.r.f.*
(iii) *Every module with DCC on cyclic submodules s.t.r.f.*
(iv) *If R is a ring with DCC on principal ideals (R is perfect), then R s.t.r.f.*

Proof. (i) Suppose that M is an R-module and m the maximal ideal of R. Similar to the proof of Theorem 2.1, it suffices to show that $\langle E(0)\rangle = \mathrm{rad}(0)$. Consider $x \in \mathrm{rad}(0)$. We will prove that $x \in E(0)$.

Since m is a maximal ideal of R, mM is a prime submodule of M or $mM = M$. Thus $x \in mM$. Then $x = \sum_{i=1}^{l} r_i a_i$ such that for each i, $1 \le i \le l$, $r_i \in m$ and $a_i \in M$. Every two ideals of R are comparable; then $\{Rr_i, i = 1, 2, 3, \dots, l\}$ is a chain of ideals of R. Without loss of generality we may suppose that Rr_1 is the maximal element of this chain. So $x = r_1 x_1$, for some $x_1 \in M$. We show that $0 \in S = \{r_1^n x_1 \mid n \in \mathbb{N}\}$.

Suppose that $0 \notin S$. Now we define the set T as follows:

$$T = \{K \mid K \text{ is a submodule of } M,\ K \cap S = \emptyset\}.$$

Since $0 \in T$, $T \neq \emptyset$. By Zorn's Lemma, T has a maximal element. Let N be a maximal element of T. We show that N is a prime submodule of M. Suppose $ay \in N$, where $a \in R$ and $y \in M \setminus N$. We have one of the following two cases:

Case 1. $Ra \subseteq Rr_1^n$, for every $n \in \mathbb{N}$.

Case 2. $Ra \nsubseteq Rr_1^d$, for some $d \in \mathbb{N}$.

If Case 1 holds, then $a \in Ra \subseteq \cap_{n\in\mathbb{N}} Rr_1^n = 0 \subseteq (N : M)$; so we have the proof.

If Case 2 is satisfied, since every two ideals of R are comparable, we have, $Rr_1^d \subseteq Ra$. Let $r_1^d = ba$, where $b \in R$. Note that $y \notin N$; then since N is a maximal element of T, we have $(N + Ry) \cap S \neq \emptyset$. Consider $r_1^t x_1 \in (N + Ry) \cap S$, where $t \in \mathbb{N}$. Then $r_1^t x_1 = n' + cy$, where $n' \in N$ and $c \in R$. Now $r_1^{d+t} x_1 = r_1^t b a x_1 = ban' + cbay \in N$, that is $N \cap S \neq \emptyset$, which is a contradiction. Therefore N is a prime submodule of M, and since $N \in T$, we have $N \cap S = \emptyset$, which is a contradiction with the fact that $r_1 x_1 = x \in S \cap \mathrm{rad}(0) \subseteq S \cap N$.

Consequently $0 \in S$; that is for some $n_0 \in \mathbb{N}$, $r_1^{n_0} x_1 = 0$. Thus $x = r_1 x_1 \in E(0)$.

(ii) We will show that R is a Noetherian ring.

Let $I_1 \subset I_2 \subset I_3 \subset \dots$ be a chain of ideals of R. For each $j \ge 2$, consider $x_j \in I_j \setminus I_{j-1}$. Every two ideals of R are comparable, and if $Rx_{j+1} \subseteq Rx_j$, $x_{j+1} \in I_j$, which is impossible, so $Rx_j \subset Rx_{j+1}$. Since the chain $Rx_2 \subset Rx_3 \subset Rx_4 \subset \dots$ stops, the chain $I_1 \subset I_2 \subset I_3 \subset \dots$ must stop.

Now by the Krull intersection theorem, $\cap_{n\in\mathbb{N}} Rr^n = 0$, for each non-unit $r \in R$. Hence by part (i), R s.t.r.f.

(iii) Suppose that M is an R-module with DCC on cyclic submodules and B is an arbitrary submodule of M. Then every localisation of M also has DCC on cyclic submodules, by [**5**, Lemma 2.6(ii)]. So the localisation technique helps us to consider R to be a local ring and M an R-module with DCC on cyclic submodules. We will show that $\langle E(B)\rangle$ is a prime submodule of M or $\langle E(B)\rangle = M$.

Let m be the maximal ideal of R, and consider $r \in m$ and $x \in M$. Since the chain $\dots \subseteq Rr^3x \subseteq Rr^2x \subseteq Rrx$ stops, for some $n_0 \in \mathbb{N}$, we have $Rr^{n_0+1}x = Rr^{n_0}x$. Hence for some $t \in R$, $r^{n_0}(1-rt)x = 0$. Note that $1-rt$ is a unit element of R, so $r^{n_0}x = 0$, which implies that $rx \in \langle E(B)\rangle$. Therefore $mM \subseteq \langle E(B)\rangle$, that is $m \subseteq (\langle E(B)\rangle : M)$. So if $\langle E(B)\rangle \neq M$, then $m = (\langle E(B)\rangle : M)$, and consequently $\langle E(B)\rangle$ is a prime submodule of M. Thus $\mathrm{rad}(B) \subseteq \langle E(B)\rangle \subseteq \mathrm{rad}(B)$.

(iv) First consider R as an R-module. According to the proof of part (iii), for each maximal ideal m of R, $r \in m$ and $x \in R_m \setminus m_m$, there exists $n_0 \in \mathbb{N}$, with $r^{n_0}x = 0$. So $r^{n_0} = 0$.

Now let M be an arbitrary R-module. Use the localisation, and suppose that m is the maximal ideal of R. Consider $r \in m$ and $x \in M$. For some $n_0 \in \mathbb{N}$, $r^{n_0} = 0$, so $r^{n_0}x = 0$. Now follow the proof of part (iii). □

In the main theorem of [**11**], the author proved that if R is a Noetherian domain of Krull dimension one, then R s.t.r.f. if and only if R is a Dedekind domain. We will generalise this result in Theorem 2.5 and Corollary 2.6.

LEMMA 2.4. *Let R be a local domain of Krull dimension one, and let m be its maximal ideal. Consider $M = R \oplus R$ as an R-module. If there exists $(x, y) \in M$ such that $x \in m \setminus Ry$ and $y \in m \setminus Rx$, then*

(i) *$rad(R(x, y)) = \{(r_1, r_2) \in R \oplus R : r_1 y = r_2 x\}$;*

(ii) *$UE(R(x, y)) \subseteq (m^2 + Rx) \oplus (m^2 + Ry)$.*

Proof. (i) See [**11**, Proposition 3.1].

(ii) It is sufficient to show that $\langle E_n(R(x, y))\rangle \subseteq (m^2 + Rx) \oplus (m^2 + Ry)$, for each non-negative integer n. We will prove the result by induction on n.

If $n = 0$, then obviously $\langle E_0(R(x, y))\rangle = R(x, y) \subseteq (m^2 + Rx) \oplus (m^2 + Ry)$.

Now let $z \in E_n(R(x, y))$, and assume the validity of the result for $n-1$. There exist $r \in R$, $(c, d) \in M$ and $n_0 \in \mathbb{N}$ such that $z = r(c, d)$ and $r^{n_0}(c, d) \in \langle E_{n-1}(R(x, y))\rangle$. So by the induction hypothesis, we have $r^{n_0}(c, d) \in (m^2 + Rx) \oplus (m^2 + Ry)$.

Note that $m \subseteq \sqrt{((m^2 + Rx) \oplus (m^2 + Ry) : M)}$, and m is a maximal ideal of R. Then $(m^2 + Rx) \oplus (m^2 + Ry)$ is an m-primary submodule of M. Therefore $r^{n_0}(c, d) \in (m^2 + Rx) \oplus (m^2 + Ry)$ implies that $(c, d) \in (m^2 + Rx) \oplus (m^2 + Ry)$ or $r \in m$. If $(c, d) \in (m^2 + Rx) \oplus (m^2 + Ry)$, then obviously $z = r(c, d) \in (m^2 + Rx) \oplus (m^2 + Ry)$.

Now consider the case $r \in m$. By part (i), $z = (rc, rd) \in E_n(R(x, y)) \subseteq \mathrm{rad}(R(x, y)) = \{(r_1, r_2) \in R \oplus R : r_1 y = r_2 x\}$. So $rcy = rdx$. If $r = 0$, then evidently $z = 0 \in (m^2 + Rx) \oplus (m^2 + Ry)$, otherwise $cy = dx$. Since $x \in m \setminus Ry$, and $y \in m \setminus Rx$, we have $c, d \in m$. Thus in this case $z = r(c, d) \in m^2 \oplus m^2 \subseteq (m^2 + Rx) \oplus (m^2 + Ry)$. □

It is easy to see that if the radical formula holds for a ring R, then for each ideal I of R, the radical formula holds for the ring $\frac{R}{I}$.

In [**11**, Theorem 3.3], S. H. Man proves the following result: Let R be a Noetherian domain of dimension one. Suppose that for any submodule B of any R-module M, $< E_1(B) >= \mathrm{rad}(B)$; then R is a Dedekind domain.

The following theorem is a generalisation of [**11**, Theorem 3.3]. One implication of the following theorem is as follows: Let R be a Noetherian domain of dimension one. Suppose that for any submodule B of any R-module M, there exists a non-negative integer n_B (related to B) such that $< E_{n_B}(B) >= \mathrm{rad}(B)$; then even in this case R is a Dedekind domain.

THEOREM 2.5. *Let R be a Noetherian domain of dimension one. Then the following are equivalent:*

(i) *the radical formula holds for R;*

(ii) *The radical formula holds for the R-module $R \oplus R$;*
(iii) *R is a Dedekind domain.*

Proof. (ii) $\Longrightarrow$ (iii) We will show that for every maximal ideal P of R, R_P is a discrete valuation ring. By Lemma 2.2(ii) the radical formula holds for the R_P-module $R_P \oplus R_P$. Hence by localisation we may suppose that R is a Noetherian local domain of dimension one, that m is its maximal ideal and that the radical formula holds for the R-module $R \oplus R$.

If $m = m^2$, then by Nakayama's lemma, $m = 0$. That is R is a field. So let $m \neq m^2$. Choose $x \in m \setminus m^2$. We will show that $m = Rx$. Once it is shown that $m = Rx + m^2$, then by Nakayama's lemma $m = Rx$.

Suppose that r is an arbitrary element of m. If $r \in Rx$, then $r \in Rx + m^2$. Now assume that $r \notin Rx$. Note that $x \notin m^2$; then $0 \neq Rx$. As $\dim R = 1$, the only prime ideal of R containing Rx is m. So $\sqrt{Rx} = m$. Then $r \in \sqrt{Rx}$. Let n be the smallest positive integer such that $r^n \in Rx$. Note that $r \notin Rx$, thus $n \geq 2$. So if we put $y = r^{n-1}$, $y \notin Rx$. Also $x \notin Ry$; otherwise $x = ty$, for some $t \in R$. Since $y \notin Rx$, t is not a unit element of R; then $t \in m$. Now $x = ty \in m^2$, which is a contradiction. We have $r^n \in Rx$; then there exists $r' \in R$ with $ry = r^n = r'x$.

Put $T = \{(r_1, r_2) \in R \oplus R : r_1 y = r_2 x\}$. Note that $ry = r'x$, so according to Lemma 2.4, we have

$$(r, r') \in T = \operatorname{rad}(R(x, y)) = UE(R(x, y)) \subseteq (m^2 + Rx) \oplus (m^2 + Ry).$$

Therefore, $r \in m^2 + Rx$, that is $m \subseteq m^2 + Rx$. So $m = m^2 + Rx$.

(iii) $\Longrightarrow$ (i) One may follow the assertion from Theorem 2.3(ii). $\square$

COROLLARY 2.6. *Let R be a Noetherian domain of dimension one and $n \in \mathbb{N}$. The following are equivalent:*

(i) *R s.t.r.f. of degree n;*
(ii) *The R-module $R \oplus R$ s.t.r.f. of degree n;*
(iii) *R is a Dedekind domain.*

Proof. (ii) $\Longrightarrow$ (iii) Let B be a submodule of the R-module $R \oplus R$. Then $\operatorname{rad}(B) = <E_n(B)> \subseteq \bigcup_{m \in \mathbb{N}} \langle E_m(B) \rangle = UE(B) \subseteq \operatorname{rad}(B)$, and so $UE(B) = \operatorname{rad}(B)$. Thus the radical formula holds for $R \oplus R$. $\square$

Recall that we say a module M (resp. weakly) s.t.r.f. of degree zero if (resp. $B = \operatorname{wrad}(B)$) $B = \operatorname{rad}(B)$, for any submodule B of M. In the rest of this section, we will discuss the modules and rings which (resp. weakly) satisfy the radical formula of degree zero (the case $n = 0$ of Corollary 2.6).

LEMMA 2.7. *Let M be an R-module, W a weakly prime submodule of M and $x, y \in M$. If $rx \in W$, where $r \in R$, then $W = (W + Rx) \cap (W + Rry)$.*

Proof. See [**6**, Corollary 2.4(i)]. $\square$

PROPOSITION 2.8. *Let M be an R-module. Consider the following statements:*

(i) *M s.t.r.f. of degree zero;*
(ii) *M weakly s.t.r.f. of degree zero;*
(iii) *For any maximal ideal $\mathcal{M}$ of R, $(\mathcal{M}M)_{\mathcal{M}} = 0$;*
(iv) *For every non-weakly prime submodule B of M, $B = rad(B)$;*

(v) *For any maximal ideal $\mathcal{M}$ of R, there exists $s \in R \setminus \mathcal{M}$ with $s\mathcal{M}M = 0$.*

Then (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii), (i) $\Longrightarrow$ (iv) and (v) $\Longrightarrow$ (iii), and if M is a Noetherian module, all of the above statements are equivalent.

Proof. (i) $\Longrightarrow$ (ii) The proof is evident.

(ii) $\Longrightarrow$ (iii) Consider $r \in \mathcal{M}$ and $x \in M$. Note that $r^2x \in Rr^2x$; then $rx \in E(Rr^2x) \subseteq \mathrm{wrad}(Rr^2x) = Rr^2x$. Consequently, there exists $t \in R$ with $r(1-tr)x = 0$. Now since in the module $M_{\mathcal{M}}$, $\frac{r}{1}(\frac{1-tr}{1})\frac{x}{1} = 0$ and $\frac{1-tr}{1}$ is a unit element of $R_{\mathcal{M}}$, $\frac{rx}{1} = 0$. Hence $(\mathcal{M}M)_{\mathcal{M}} = 0$.

(iii) $\Longrightarrow$ (i) Let B be an arbitrary submodule of M and $\mathcal{M}$ a maximal ideal of R. Since $\mathcal{M}_{\mathcal{M}}M_{\mathcal{M}} = 0$, $\mathcal{M}_{\mathcal{M}}M_{\mathcal{M}} \subseteq B_{\mathcal{M}}$, which implies that $(B_{\mathcal{M}} : M_{\mathcal{M}}) = \mathcal{M}_{\mathcal{M}}$ or $(B_{\mathcal{M}} : M_{\mathcal{M}}) = R_{\mathcal{M}}$. Hence $B_{\mathcal{M}}$ is a prime submodule of $M_{\mathcal{M}}$ or $B_{\mathcal{M}} = M_{\mathcal{M}}$, and in both cases $B_{\mathcal{M}} = \mathrm{rad}(B_{\mathcal{M}})$. Now $B_{\mathcal{M}} \subseteq (\mathrm{rad}(B))_{\mathcal{M}} \subseteq \mathrm{rad}(B_{\mathcal{M}}) = B_{\mathcal{M}}$, i.e. $B_{\mathcal{M}} = (\mathrm{rad}(B))_{\mathcal{M}}$. Therefore $B = \mathrm{rad}(B)$.

(i) $\Longrightarrow$ (iv) The proof is evident.

(v) $\Longrightarrow$ (iii) The proof is obvious.

Now suppose M is a Noetherian module.

(iv) $\Longrightarrow$ (i) Consider the following set:

$$T = \{B \mid B \text{ is a submodule of } M, \text{ and } B \neq \mathrm{rad}(B)\}.$$

If $T \neq \emptyset$, then let W be a maximal element of T. By our hypothesis W is a weakly prime submodule of M. We will show that W is a prime submodule of M, and it is a contradiction with the fact that $W \in T$.

Let $ra \in W$, where $r \in R$, and $a \in M \setminus W$. If $r \notin (W : M)$; then there exists $y \in M$ such that $ry \notin W$. By Lemma 2.7, $W = (W + Ra) \cap (W + Rry)$. As W is a maximal element of T, each of the submodules $W + Ra$ and $W + Rry$ is an intersection of prime submodules. Hence W must be an intersection of prime submodules, which is a contradiction.

(iii) $\Longrightarrow$ (v) Let $\mathcal{M}M$ be generated by $y_1, y_2, y_3, \ldots, y_k$. For each i, there exists $s_i \in R \setminus \mathcal{M}$ such that $s_iy_i = 0$. Evidently, $s\mathcal{M}M = 0$, where $s = \prod_{i=1}^{k} s_i$. □

Recall that a ring R is said to be an *absolutely flat* (or a *von Neumann regular*) ring if every R-module is a flat module (see [**1**, p. 35, Exercise 27]). According to [**1**, p. 44, Exercise 10], R is absolutely flat if and only if for each maximal ideal m of R, R_m is a field.

Corollary 2.9. *Let R be a ring. The following are equivalent:*

(i) *R s.t.r.f. of degree zero;*

(ii) *R weakly s.t.r.f. of degree zero;*

(iii) *The R-module $R \oplus R$ s.t.r.f. of degree zero;*

(iv) *The R-module $R \oplus R$ weakly s.t.r.f. of degree zero;*

(v) *R is an absolutely flat ring.*

Proof. (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (iv) The proof is obvious, by Proposition 2.8.

(i) $\Longrightarrow$ (iii) The proof is evident.

(iii) $\Longrightarrow$ (v) Put $M = R \oplus R$. Let $\mathcal{M}$ be a maximal ideal of R. According to Proposition 2.8, $\mathcal{M}_{\mathcal{M}}(R_{\mathcal{M}} \oplus R_{\mathcal{M}}) = (\mathcal{M}M)_{\mathcal{M}} = 0$; then $\mathcal{M}_{\mathcal{M}} = 0$; that is $R_{\mathcal{M}}$ is a field. So R is an absolutely flat ring.

(v) $\Longrightarrow$ (i) Let M be an R-module. Since R is an absolutely flat ring, $R_{\mathcal{M}}$ is a filed for every maximal ideal $\mathcal{M}$ of R. So $\mathcal{M}_{\mathcal{M}} = 0$, and then $(\mathcal{M}M)_{\mathcal{M}} = 0$. Now the proof follows from Proposition 2.8. □

COROLLARY 2.10. *An integral domain R weakly s.t.r.f. of degree zero if and only if R is a field.*

Proof. Note that any absolutely flat domain is a field (see [**1**, p. 34, Exercise 27]). □

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