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Solution of a linear difference equation

A. Brown

A solution is given for u_{n+1} in terms of u_1 and u_0, where the elements of the sequence $\{u_n\}$ satisfy the linear difference equation

$$H(n)u_{n+1} + G(n)u_n + F(n)u_{n-1} = 0 , \quad (n = 1, 2, \ldots) .$$

Two linearly independent solutions of the equation are written as determinants and relations are given which can be used to check the evaluation of these determinants.

1. Solution of a particular case

To illustrate the procedure that will be used for the general case, consider the equation

$$u_{n+1} + n^{1/2}u_n + u_{n-1} = 0 , \quad (n = 1, 2, \ldots) . \tag{1.1}$$

If u_0 and u_1 are specified, then $u_2, u_3, u_4, \ldots$ can in turn be calculated in terms of u_0 and u_1, although after seven or eight steps the expressions for u_n become messy and it is hard to see how the pattern of the solution will generalise. To obtain a general expression for the

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solution we introduce the determinant

$$(1.2) \qquad A_m^n = \begin{vmatrix} m^{1/2} & 1 & 0 & \cdots & . & . \\ 1 & (m+1)^{1/2} & 1 & \cdots & . & . \\ 0 & 1 & (m+2)^{1/2} & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & (n-1)^{1/2} & 1 \\ . & . & . & \cdots & 1 & n^{1/2} \end{vmatrix} ,$$

where the diagonal elements are $m^{1/2}$, $(m+1)^{1/2}$, ..., $n^{1/2}$ and the diagonal is bordered with 1's , with all other elements zero. (We shall take m and n as positive integers, with $n \geq m$.) If the determinant is expanded in terms of its last row and column, then

$$(1.3) \qquad A_m^n = \left(n^{1/2}\right)A_m^{n-1} - A_m^{n-2} , \quad (n \geq m+2) ,$$

and in the same way, expanding the determinant in terms of its first row and column gives

$$(1.4) \qquad A_m^n = \left(m^{1/2}\right)A_{m+1}^n - A_{m+2}^n , \quad (n \geq m+2) .$$

In terms of these determinants, the solution for u_{n+1} is given by

$$(1.5) \qquad u_{n+1} = (-1)^n\left\{u_0A_2^n+u_1A_1^n\right\} , \quad (n = 2, 3, \ldots) .$$

This can be proved by induction, using equation (1.3). If we assume that equation (1.5) holds for $n \leq N-1$, then

$$\begin{aligned} u_{N+1} &= -\left(N^{1/2}\right)u_N - u_{N-1} \\ &= (-1)^N u_0\left\{\left(N^{1/2}\right)A_2^{N-1}-A_2^{N-2}\right\} + (-1)^N u_1\left\{\left(N^{1/2}\right)A_1^{N-1}-A_1^{N-2}\right\} \\ &= (-1)^N\left\{u_0A_2^N+u_1A_1^N\right\} . \end{aligned}$$

It is easy to verify that equation (1.5) holds for $n = 2$ and $n = 3$ and the induction argument extends the result to the general case.

2. Comments on the above solution

Taking $u_0 = 0$ and $u_1 = 1$ gives $(-1)^n A_1^n$ as a particular solution of equation (1.1), with $(-1)^n A_2^n$ as a linearly independent solution (corresponding to the initial conditions $u_0 = 1$, $u_1 = 0$). There is a relation between these linearly independent solutions, namely,

$$A_1^n A_2^{n+1} - A_1^{n+1} A_2^n = 1 \, , \quad (n = 2, 3, \ldots) \, , \tag{2.1}$$

and this relation can be used as a check in evaluating them numerically. Equation (2.1) can be obtained by using equation (1.3) to show that

$$A_1^n A_2^{n+1} - A_1^{n+1} A_2^n = A_1^{n-1} A_2^n - A_1^n A_2^{n-1} \, , \quad (n = 3, 4, \ldots) \, . \tag{2.2}$$

By successive use of this reduction formula

$$A_1 A_2^{n+1} - A_1^{n+1} A_2^n = A_1^2 A_2^3 - A_1^3 A_2^2 = 1 \, ,$$

since

$$A_1^2 = -1 + \sqrt{2} \, , \quad A_2^3 = -1 + \sqrt{6} \, , \quad A_1^3 = -1 + \sqrt{3}(-1+\sqrt{2}) \, , \quad A_2^2 = \sqrt{2} \, .$$

The solution (1.5) arises from writing the set of equations (1.1) in matrix form. If we use the first four of these equations to illustrate the procedure, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \sqrt{2} & 1 & 0 & 0 \\ 1 & \sqrt{3} & 1 & 0 \\ 0 & 1 & \sqrt{4} & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} -u_1 - u_0 \\ -u_1 \\ 0 \\ 0 \end{bmatrix}$$

This gives a set of equations for u_2, u_3, u_4, u_5 in terms of u_0 and u_1 and the solution for u_5, say, can be written down from Cramer's rule as

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$$(2.4)\qquad u_5 = \begin{vmatrix} 1 & 0 & 0 & -u_1-u_0 \\ \sqrt{2} & 1 & 0 & -u_1 \\ 1 & \sqrt{3} & 1 & 0 \\ 0 & 1 & \sqrt{4} & 0 \end{vmatrix}$$

$$= (u_0+u_1)A_2^4 - u_1A_3^4 = u_0A_2^4 + u_1A_1^4 ,$$

since $A_2^4 - A_3^4 = A_1^4$ from equation (1.4). In equation (2.3) the matrix of coefficients on the left-hand side is triangular, with determinant 1, and this makes it easier to apply Cramer's rule.

3. Extension to general case

For the difference equation

$$(3.1)\qquad u_{n+1} + f(n)u_n + u_{n-1} = 0 , \quad (n = 1, 2, \ldots) ,$$

a similar form of solution can be used. If we introduce determinants B_m^n of the same type as A_m^n but with $f(m)$, $f(m+1)$, ..., $f(n)$ as the diagonal elements instead of $m^{1/2}$, $(m+1)^{1/2}$, ..., $n^{1/2}$, then

$$(3.2)\qquad u_{n+1} = (-1)^n\left(u_0B_2^n+u_1B_1^n\right) , \quad (n = 2, 3, \ldots) .$$

As before, $(-1)^nB_1^n$ and $(-1)^nB_2^n$ are linearly independent solutions, with

$$(3.3)\qquad B_1^nB_2^{n+1} - B_1^{n+1}B_2^n = 1 , \quad (n = 2, 3, \ldots) ,$$

and the determinants satisfy recurrence relations

$$\left.\begin{aligned} &(3.4)\qquad B_m^n = f(n)B_m^{n-1} - B_m^{n-2} \\ &(3.5)\qquad B_m^n = f(m)B_{m+1}^n - B_{m+2}^n \end{aligned}\right\} , \quad (n \geq m+2)$$

If we now move to the equation

$$(3.6)\qquad H(n)u_{n+1} + G(n)u_n + F(n)u_{n-1} = 0 , \quad (n = 1, 2, \ldots) ,$$

we can assume that $H(n) \neq 0$ for all n, otherwise the step-by-step determination of $\{u_n\}$ will break down. Also, if we are to obtain a solution in terms of u_0 and u_1, $F(1)$ must be non-zero, since u_0 only comes into the set of equations through the term $F(1)u_0$ in the equation corresponding to $n = 1$. As before, we can write the solution in terms of tri-diagonal determinants C_m^n, defined for $m = 1, 2, \ldots$, $n = 1, 2, \ldots$, and $n \geq m$ by

$$(3.7) \qquad C_m^n = \begin{vmatrix} G(m) & H(m) & 0 & \cdots & \cdot & \cdot \\ F(m+1) & G(m+1) & H(m+1) & \cdots & \cdot & \cdot \\ 0 & F(m+2) & G(m+2) & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & G(n-1) & H(n-1) \\ \cdot & \cdot & \cdot & \cdots & F(n) & G(n) \end{vmatrix}$$

As for A_m^n and B_m^n, all the elements of C_m^n are zero except on the principal diagonal and the two bordering diagonals. These determinants satisfy recurrence relations (obtained in the same way as before), namely, for $n \geq m+2$,

$$(3.8) \qquad C_m^n = G(n)C_m^{n-1} - F(n)H(n-1)C_m^{n-2} ,$$

$$(3.9) \qquad C_m^n = G(m)C_{m+1}^n - F(m+1)H(m)C_{m+2}^n .$$

If we put $L_n = \prod_{r=1}^{n} H(r)$, then the solution for u_{n+1} can be written in the form

$$(3.10) \qquad u_{n+1} = (-1)^n\left\{u_1C_1^n+u_0F(1)C_2^n\right\}/L_n , \quad (n = 2, 3, \ldots) .$$

As in Section 1, this can be proved by induction, using equation (3.8) to obtain the solution for u_{N+1} from the solution for u_N and u_{N-1} and establishing the result for u_3 and u_4 by direct methods. (Cramer's rule can again be used without difficulty.) Since u_0 and u_1 can be given arbitrary values, $\left\{(-1)^nC_1^n\right\}/L_n$ and $\left\{(-1)^nC_2^n\right\}/L_n$ are linearly independent solutions of equation (3.6).

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In evaluating C_1^n and C_2^n , relationships which can be used as checks are

(3.11) $$C_1^n C_2^{n+1} - C_1^{n+1} C_2^n = H(n)F(n+1)\left\{C_1^{n-1}C_2^n - C_1^n C_2^{n-1}\right\} ,$$

(3.12) $$C_1^{n-1} C_2^{n+1} - C_1^{n+1} C_2^{n-1} = G(n+1)\left\{C_1^{n-1}C_2^n - C_1^n C_2^{n-1}\right\} .$$

These relationships, which hold for $n = 3, 4, \ldots$, can be verified by using equation (3.8) to express C_1^{n+1} and C_2^{n+1} in terms of C_1^n, C_2^n, C_1^{n-1} and C_2^{n-1} In place of equation (3.11), an alternative form can be obtained by using this equation repeatedly, as a reduction formula, to give

(3.13) $$F(1)\left\{C_1^n C_2^{n+1} - C_1^{n+1} C_2^n\right\}$$
$$= F(1)\{H(n)H(n-1) \ldots H(3)\}\{F(n+1)F(n) \ldots F(4)\}\left\{C_1^2 C_2^3 - C_1^3 C_2^2\right\} .$$

Now

$$C_1^2 = G(1)G(2) - F(2)H(1) ,$$

$$C_2^2 = G(2) ,$$

$$C_2^3 = G(2)G(3) - F(3)H(2) ,$$

$$C_1^3 = G(3)C_1^2 - G(1)H(2)F(3) ,$$

so

$$C_1^2 C_2^3 - C_1^3 C_2^2 = F(3)F(2)H(2)H(1) .$$

Equation (3.13) now gives

(3.14) $$F(1)\left\{C_1^n C_2^{n+1} - C_1^{n+1} C_2^n\right\} = J_{n+1} L_n , \quad (n = 2, 3, \ldots) ,$$

where

(3.15) $$J_n = \prod_{r=1}^{n} F(r) .$$

Combining equation (3.12) and equation (3.14) gives

$$(3.16)\quad F(1)\left\{C_1^{n-1}C_2^{n+1}-C_1^{n+1}C_2^{n-1}\right\} = G(n+1)J_nL_{n-1}\ , \quad (n = 3, 4, \ldots)\ .$$

If it should happen that $F(k+1) = 0$ for a particular value of k, equation (3.14) gives

$$C_1^{n+1}/C_1^n = C_2^{n+1}/C_2^n\ , \quad \text{for} \quad n \geq k\ .$$

Department of Applied Mathematics,
Faculty of Arts,
Australian National University,
Canberra, ACT.

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