AN EXAMPLE FOR HOMOTOPY COMMUTATIVITY OF H-SPACES

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In this note we give an example of a homotopy commutative H-space \( X \) that is not dominated by any homotopy associative, homotopy commutative H-space. In particular, \( X \) is not dominated by \( \Omega^2 S^2 X \).

We recall that a space \( X \) is dominated by a space \( Y \) provided that there exists maps \( i: X \to Y \) and \( r: Y \to X \) such that \( r \circ i \) is homotopic to the identity map of \( X \) (written \( r \circ i \sim 1_X \)). We let \( \Omega \) and \( S \) denote the loop space and suspension functors, respectively.

If \( P \) denotes a homotopy property, then we call a functor \( T \) a universal example for \( P \) provided that a space \( X \) possesses property \( P \) if and only if \( X \) is dominated by \( T(X) \). For example, \( X \) is an H-space if and only if it is dominated by \( \Omega S X \); hence \( \Omega S \) is a universal example for the property of being an H-space. For another example, let \( X \) be an H-space. The projective plane of \( X \), \( P_2(X) \), is defined to be the mapping cone of the Hopf construction \( X \ast X \to SX \). Then \( X \) is homotopy associative if and only if it is dominated by \( \Omega P_2(X) \), since in that case the Hopf construction extends to a fibration over \( P_2(X) \), Stasheff (1963). It is conjectured in Stasheff (1970) that an H-space \( X \) is homotopy commutative if and only if it is dominated by \( \Omega^2 S^2 X \). We shall show that this conjecture is false by showing that there is no functor to the category of homotopy commutative, homotopy associative H-spaces that serves as universal example for homotopy commutativity. Our example is based on the following fact.

**Theorem.** Suppose that a space \( X \) is dominated by a homotopy associative, homotopy commutative H-space \( Y \). Then \( X \) is dominated by \( \Omega(SX)_2 \). (Here \( (SX)_2 \) denotes \( SX \times SX \) modulo the relation \( \sim \) given by \( (*,w) \sim (w,*), w \in SX \), \(* \) the basepoint of \( SX \).)

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309
PROOF. If \( Y \) is as hypothesized, then it was shown in Williams (1971) that \( Y \) is dominated by \( \Omega(SY)_2 \). Let \( i : X \to Y \), \( r : Y \to X \) be such that \( r \circ i \sim 1_X \) and \( j : Y \to \Omega(SY)_2 \), \( d : \Omega(SY)_2 \to Y \) be such that \( d \circ j \sim 1_Y \). Let \( j' : X \to \Omega(SX)_2 \) denote the inclusion. Then from the diagram

\[
\begin{array}{ccc}
\Omega(SX)_2 & \xrightarrow{\Omega(Si)_2} & \Omega(SY)_2 \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{j'} & \omega \\
\downarrow i & & \downarrow 1_Y \\
Y & \xrightarrow{j} & Y \\
\downarrow r & \downarrow d & \downarrow r \circ i \\
X & & X
\end{array}
\]

We see that

\[(r \circ d \circ \Omega(Si)_2) \circ j' \sim r \circ d \circ j \circ i \sim r \circ i \sim 1_X.\]

REMARK 1. It was shown in Williams (1971) that \( \Omega(SY)_2 \) does serve as universal example for homotopy commutativity for homotopy associative \( H \)-spaces (This fact also follows from Husseini (1963) and from Stasheff (1961).)

REMARK 2. For a space \( X \) to be a homotopy commutative \( H \)-space, it is sufficient that it be dominated by \( \Omega(SX)_2 \), Williams (1971).

EXAMPLE. Our example is based on ideas in Adams (1961). Let \( m : S^3 \times S^3 \to S^1 \) be an \( H \)-space multiplication whose separation element for homotopy commutativity is of order 2 or 4 in \( \pi_6(S^3) \), James (1955). (Such a multiplication is no homotopy associative.) Let \( X \) be the space obtained by attaching cells to \( S^3 \) to kill the 2-component of \( \pi_k(X) \) in dimensions \( \geq 6 \). The multiplication on \( S^3 \) extends to a homotopy commutative multiplication on \( X \), since the obstruction lie in vanishing cohomology groups. Observe that \( \pi_9(X) = \pi_9(S^3) \approx Z_3 \). Further the 3-component of \( \pi_9(\Omega^2S^2X) \) is isomorphic to the 3-component of \( \pi_9(\Omega^2S^5) \) which is trivial, since \( \pi_9(\Omega^2S^5) = \pi_{11}(S^5) = Z_2 \). Since \( \Omega^2S^2X = \Omega(\Omega S(SX)) \) and \( \Omega S(SX) \) is obtained from \( (SX)_2 \) by attaching cells in dimensions greater than eleven, James (1955), \( \pi_9(\Omega S(SX))_2 \approx \pi_9(\Omega^2S^2X) \). Since \( Z_3 \) cannot be a subgroup of a trivial group, \( X \) cannot be dominated by \( \Omega(SX)_2 \).

In conclusion, we remark that \( X \) provides an example similar to that of Adams (1961) of a homotopy commutative, non-homotopy associative \( H \)-space

References

J. F. Adams (1961), 'The sphere, considered an as \( H \)-space mod \( p \)', *Quart. J. Math.*, Oxford (1), 12, 52–60.


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