Proceedings of the Edinburgh Mathematical Society (1996) 39, 371–376 ©

DIRECT FINITENESS OF CERTAIN MONOID ALGEBRAS II

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(Received 23rd September 1994)

Let S be a nontrivial monoid with zero and let F be a field. A sufficient condition, on the 0-simple principal factors of S and the characteristic of F, is given for the contracted monoid algebra of S over F to be directly finite.

1991 *Mathematics subject classification*: 16S36, 20M25.

A ring R with unity 1 is said to be *directly finite* (or *von Neumann finite*) if and only if, for all $a,b\in R, ab=1$ implies $ba=1$. It is shown here that the contracted monoid algebra $F_0[S]$ of a monoid $S=S^0$ over a field F of characteristic zero is directly finite if every 0-simple principal factor of S is completely 0-simple with either finitely many left ideals or finitely many right ideals. A similar result holds if we replace the requirement that F has characteristic zero by the restriction that each subgroup of S be abelian.

1. Preliminary remarks

We denote the ring [algebra] of all $n\times n$ matrices over a ring [algebra] R by $M_n(R)$ ($n\in\mathbb{N}$). Otherwise the notation is that of [**1**], where all the key semigroup concepts are introduced.

Let S be a semigroup. As in [**1**], we write '$S=S^0$' to indicate that S has a zero and at least one other element. The semigroup algebra of S over a field F is denoted by $F[S]$ and, in the case where $S=S^0$, the contracted semigroup algebra of S over F is denoted by $F_0[S]$ (see [**1**, §5.2]). Clearly, every semigroup algebra can be regarded as a contracted semigroup algebra (simply adjoin a zero to the semigroup!); but the converse is false—for example, $M_2(F)$ is a contracted semigroup algebra that cannot be expressed as a semigroup algebra over F. In the case where $S=S^0$ is a monoid (a semigroup with identity) we call $F_0[S]$ the *contracted monoid algebra* of S over F.

It is shown in [**1**, Lemma 2.39] that each nonzero principal factor of a semigroup $S=S^0$ is either null or 0-simple. The proposition below gives a necessary condition for the direct finiteness of a contracted monoid algebra in terms of the principal factors of the monoid. In the proof we make use of a concept introduced in [**4**]: a ring R is said to be *quasidirectly finite* if and only if, for all $a,b\in R$, $ab=a+b$ implies $ab=ba$. If R has a unity then quasidirect finiteness and direct finiteness are equivalent properties.

Proposition. *Let $S=S^0$ be a monoid and let F be a field. If $F_0[S]$ is directly finite then each 0-simple principal factor of S that contains a nonzero idempotent is completely 0-simple.*

Proof. Assume that $F_0[S]$ is directly finite. Let Q be a 0-simple principal factor of S containing a nonzero idempotent e. Then e is an idempotent in S itself. Now $F_0[S]$ is quasidirectly finite and so its subring $F_0[eSe]$ is quasidirectly finite. Since this subring has unity e, it is directly finite. Hence if $x, y \in eSe$ are such that $xy=e$ then $yx=e$. Thus S contains no bicyclic subsemigroup with identity e and therefore the same is true of Q. Consequently, by [**1**, Theorem 2.54], Q is completely 0-simple. □

Recall that a semigroup $S=S^0$ is termed *completely semisimple* if and only if each of its nonzero principal factors is completely 0-simple.

Corollary. *Let $S=S^0$ be a regular monoid and let F be a field. If $F_0[S]$ is directly finite then S is completely semisimple.*

In the proposition above, the necessary condition for direct finiteness involves 0-simple principal factors that contain nonzero idempotents. The following example shows that it is possible for the contracted algebra of a monoid $S=S^0$ to be directly finite when S has a 0-simple principal factor with no nonzero idempotent.

Example 1. Let $A := \left\{\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \in M_2(\mathbb{R}) : a>0 \text{ and } b>0\right\}$ and let $S := A \cup \{O, I\}$, where O and I are the zero and unity of $M_2(\mathbb{R})$. Under matrix multiplication S is a monoid. It can be verified that A is a simple subsemigroup of S ('Andersen's semigroup'; see [**1**, §2.1, Exercise 9]). Consequently, $A \cup \{O\}$ is a 0-simple principal factor of S and it is easily seen to contain no nonzero idempotent. On the other hand, $A \cup \{I\}$ is a submonoid of the multiplicative group G of all real 2×2 nonsingular matrices. Now, for a field F of characteristic zero, $F[G]$ is directly finite, by Kaplansky's theorem [**2**; **5**, Corollary 2.1.9]. Hence $F[A \cup \{I\}]$ is directly finite; that is, $F_0[S]$ is directly finite.

2. A sufficient condition for direct finiteness

We begin with a definition.

Definition. Let F be a field and let A be an F-algebra. By a *countably infinite family of matrix units* in A we mean a family (e_{ij}) of nonzero elements of A, indexed by the set $\mathbb{N} \times \mathbb{N}$, such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for all choices of i, j, k, l.

The two lemmas below concern the nonexistence of such families in certain matrix algebras.

Lemma 1. *Let F be a field of characteristic zero, let G be a group and let $n \in \mathbb{N}$. Then the F-algebra $M_n(F[G])$ contains no countably infinite family of matrix units.*

Proof. Suppose that such a family (e_{ij}) does exist in $M_n(F[G])$. Let K denote the subfield of F generated by the coefficients of the entries of the e_{ij} $(i, j \in \mathbb{N})$. Then K is countable. Hence K can be embedded in $\mathbb{C}$ and so we can regard (e_{ij}) as a countably infinite family of matrix units in the $\mathbb{C}$-algebra $M_n(\mathbb{C}[G])$. By Kaplansky's trace theorem for matrix algebras over a complex group ring ([**2**]; see also [**3**] and [**5**]), there exists a linear functional $\mathrm{tr}: M_n(\mathbb{C}[G]) \to \mathbb{C}$ such that

(1) $(\forall a, b \in M_n(\mathbb{C}[G])\ \mathrm{tr}(ab) = \mathrm{tr}(ba)$,

(2) $(\forall e = e^2 \in M_n(\mathbb{C}[G]) \setminus \{0\})\ \mathrm{tr}(e)$ is real and $0 < \mathrm{tr}(e) \leqq n$.

For all $i \in \mathbb{N}$, $\mathrm{tr}(e_{ii}) = \mathrm{tr}(e_{i1}e_{1i}) = \mathrm{tr}(e_{1i}e_{i1}) = \mathrm{tr}(e_{11})$, by (1). Write $\alpha := \mathrm{tr}(e_{11})$. By (2), α is real and $0 < \alpha \leqq n$. Choose $k \in \mathbb{N}$ such that $k\alpha > n$ and take $f := e_{11} + e_{22} + \cdots + e_{kk}$. Then $f = f^2 \neq 0$ in $M_n(\mathbb{C}[G])$ and $\mathrm{tr}(f) = \Sigma_{i=1}^{k} \mathrm{tr}(e_{ii}) = k\alpha > n$, contradicting (2). Thus no such family (e_{ij}) exists. □

Lemma 2. *Let F be an arbitrary field, let A be a commutative F-algebra and let $n \in \mathbb{N}$. Then the F-algebra $M_n(A)$ contains no countably infinite family of matrix units.*

Proof. For each $r \in \mathbb{N}$, let f_r be the polynomial over F in the noncommuting indeterminates $x_1, x_2, \ldots, x_r$ defined by

$$f_r(x_1, x_2, \ldots, x_r) := \sum_{\sigma \in S_r} (\mathrm{sgn}\,\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(r)},$$

where S_r denotes the symmetric group on $\{1, 2, \ldots, r\}$ and, for $\sigma \in S_r$, $\mathrm{sgn}\,\sigma$ is $+1$ or -1 according as σ is even or odd. Clearly $M_n(A) = A \otimes_F M_n(F)$. From the theorem of Amitsur and Levitski [**5**, Theorem 5.1.9] $f_{2n}(b_1, b_2, \ldots, b_{2n}) = 0$ for all choices of $b_1, b_2, \ldots, b_{2n}$ in $M_n(F)$. Hence, since A is commutative and f_{2n} is homogeneous and multilinear,

$$f_{2n}(a_1, a_2, \ldots, a_{2n}) = 0 \tag{1}$$

for all choices of $a_1, a_2, \ldots, a_{2n}$ in $M_n(A)$.

Now suppose that $M_n(A)$ contains a countably infinite family (e_{ij}) of matrix units. Then, in particular, from (1) we have that

$$0 = f_{2n}(e_{12}, e_{23}, \ldots, e_{2n\,2n+1}) = e_{1\,2n+1}.$$

But this is false, since each e_{ij} is nonzero. The result follows. □

From Lemmas 1 and 2 we derive

Lemma 3. *Let F be a field and let S be a completely 0-simple semigroup with finitely many left ideals or finitely many right ideals. If either (a) F has characteristic zero or (b) every subgroup of S is abelian then $F_0[S]$ contains no countably infinite family of matrix units.*

Proof. Since S is completely 0-simple, it is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ [1, Theorem 3.5]. Here G is a nonzero maximal subgroup of S, I and Λ are nonempty sets, respectively indexing the 0-minimal right ideals and the 0-minimal left ideals of S, and P is a $\Lambda \times I$ matrix over G^0 with at least one nonzero entry in each row and column. Following the argument in [1, Lemma 5.17], we may assume that the algebra $F_0[S]$ consists of all $I \times \Lambda$ matrices over $F[G]$ with at most finitely many nonzero entries. Addition and scalar multiplication in $F_0[S]$ are just the usual matrix operations, while multiplication $\circ$ is given in terms of ordinary matrix multiplication by

$$X \circ Y = XPY \quad (X, Y \in F_0[S]).$$

Assume, without loss of generality, that S has finitely many right ideals. This is equivalent to the assumption that I is finite. Write $n := |I|$. Now suppose that $F_0[S]$ contains a countably infinite family of matrix units (E_{ij}). Thus $E_{ij}PE_{kl} = \delta_{jk}E_{il}$ and so $(E_{ij}P)(E_{kl}P) = \delta_{jk}(E_{il}P)$ for all $i, j, k, l \in \mathbb{N}$. For all $i, j \in \mathbb{N}$ write $e_{ij} := E_{ij}P$. Then $e_{ij} \neq 0$; for otherwise we would have that $E_{ij} = e_{ij}E_{jj} = 0$, which is false. Hence (e_{ij}) is a countably infinite family of matrix units in $M_n(F[G])$. But if F has characteristic zero this contradicts Lemma 1, while if G is abelian it contradicts Lemma 2. □

We now come to the main result.

Theorem. *Let F be a field and let $S = S^0$ be a monoid in which each 0-simple principal factor of S is completely 0-simple with finitely many left ideals or finitely many right ideals. If either (a) F has characteristic zero or (b) every subgroup of S is abelian then $F_0[S]$ is directly finite.*

Proof. Suppose that there exist $a, b \in F_0[S]$ such that $ab = 1$ and $ba \neq 1$, where 1 is the identity of S. Write $e := 1 - ba$. Then $e = e^2 \neq 0$. Clearly

$$ae = 0 = eb. \tag{1}$$

Now define a family (f_{ij}) of elements of $F_0[S]$, with index set $\mathbb{N} \times \mathbb{N}$, by writing

$$(\forall i, j \in \mathbb{N}) \quad f_{ij} := b^i e a^j.$$

Since $ab = 1$, we have that

$$(\forall i, j \in \mathbb{N}) \quad a^i f_{ij} b^j = e \tag{2}$$

and so each f_{ij} is nonzero. Also, for all $i, j, k, l \in \mathbb{N}$, $f_{ij}f_{kl} = b^i e b^{k-j} e a^l$, $b^i e a^l$ or $b^i e a^{j-k} e a^l$ according as $j<k$, $j=k$ or $j>k$; hence, from (1),

$$(\forall i, j, k, l \in \mathbb{N}) \quad f_{ij}f_{kl} = \delta_{jk} f_{il}. \tag{3}$$

Let S^* denote the set of nonzero elements of S. For $a \in F_0[S]$ we have that $a = \sum_{x \in S^*} \alpha_x x$ for some elements $\alpha_x \in F$ and we write $\operatorname{supp} a := \{x \in S^* : \alpha_x \neq 0\}$. If $a \neq 0$ this is a nonempty finite subset of S^*. Now choose J to be maximal (under the usual partial ordering [**1**, §2.1]) in the finite collection of $\mathscr{J}$-classes of S that have nonempty intersection with $\operatorname{supp} e$. Let M denote the union of all $\mathscr{J}$-classes J' of S such that $J' \ngeq J$. It is easy to verify that M is an ideal of S. Next, let $\theta: F_0[S] \to F_0[S/M]$ denote the homomorphism that extends, by linearity, the natural homomorphism from S to the Rees quotient S/M. For all $i, j \in \mathbb{N}$, write $e_{ij} := f_{ij}\theta$. We show that (e_{ij}) is a countably infinite family of matrix units in $F_0[(J \cup M)/M]$.

Let $i, j, \in \mathbb{N}$. Since $f_{ij} = b^i e a^j$, it follows that for each $x \in \operatorname{supp} f_{ij}$ there exists $y \in \operatorname{supp} e$ such that $J_x \leqq J_y$. Thus either $J_x = J$ or $J_x \subseteq M$. Hence $\operatorname{supp} f_{ij} \subseteq J \cup M$ (an ideal of S) and so $e_{ij} \in F_0[(J \cup M)/M]$. Also, from (2), $(a^i\theta)e_{ij}(b^j\theta) = e\theta$, which is nonzero by the choice of J, and so $e_{ij} \neq 0$. Further, from (3),

$$(\forall i, j, k, l \in \mathbb{N}) \quad e_{ij}e_{kl} = \delta_{jk} e_{il}. \tag{4}$$

Now $(J \cup M)/M$ is isomorphic to the principal factor of S corresponding to J; furthermore, from (4), this principal factor cannot be null. Hence, by hypothesis, $(J \cup M)/M$ is completely 0-simple, with either finitely many left ideals or finitely many right ideals. But (4) shows that $F_0[(J \cup M)/M]$ contains a countably infinite family of matrix units—which, by Lemma 3, is impossible if either (a) F has characteristic zero or (b) every subgroup of S is abelian. Thus if either (a) or (b) holds then $F_0[S]$ is directly finite. □

Corollary. *Let F be a field and let $S = S^0$ be a regular monoid in which each $\mathscr{J}$-class contains finitely many idempotents. If either (a) F has characteristic zero or (b) every subgroup of S is abelian then $F_0[S]$ is directly finite.*

Note that a nonzero $\mathscr{J}$-class of a regular monoid $S = S^0$ contains finitely many idempotents if and only if the corresponding principal factor is completely 0-simple with finitely many left ideals *and* finitely many right ideals. In fact, the corollary remains valid if we replace '$\mathscr{J}$-class' in the statement by '$\mathscr{D}$-class', since the hypothesis still ensures that S is completely semisimple and so $\mathscr{J} = \mathscr{D}$ on S.

From the corollary we deduce, in particular, that the algebra of a free inverse monoid over an arbitrary field is directly finite; for, by [**6**, Lemma 1.3], each such monoid has trivial subgroups and finite $\mathscr{J}$-classes. A more direct proof of this result may be outlined as follows. Let F be a field, let S be a free inverse monoid and let Q be a principal factor of S. Then $F_0[Q] \cong M_n(F)$, where n is the number of nonzero idempotents in Q; and, since $F_0[Q]$ has a unity, there is a surjective homomorphism from $F[S]$ to $F_0[Q]$.

We observe next that $F[S]$ is isomorphic to a subdirect product of the family of all such algebras $F_0[Q]$. It follows easily that, since each $F_0[Q]$ is directly finite, so also is $F[S]$.

Finally, we remark that, for a monoid $S=S^0$ and a field F, $F_0[S]$ is directly finite if and only if $F_0[T]$ is directly finite for every *finitely generated* submonoid T of S. We conclude with an example of a finitely generated completely semisimple monoid $S=S^0$ (see §1) that has a principal factor with infinitely many left and right ideals and whose algebra over an arbitrary field is directly finite.

Example 2. Let G be an infinite cyclic group with generator a and let $S:=G\cup(\mathbb{Z}\times\mathbb{Z})\cup\{0\}$, with multiplication extending that in G and satisfying the conditions

$$a^n(i,j)=(i+n,j), \quad (i,j)a^n=(i,j-n),$$

$$(i,j)(k,l)=\begin{cases}(i,l) & \text{if } j=k\\ 0 & \text{if } j\neq k,\end{cases} \qquad (i,j,k,l,n\in\mathbb{Z})$$

$$0a^n=a^n0=0(i,j)=(i,j)0=0^2=0.$$

It is readily verified that $S(=S^0)$ is an inverse monoid, with nonzero $\mathscr{J}$-classes G and $\mathbb{Z}\times\mathbb{Z}$. Further, since $a^i(0,0)a^{-j}=(i,j)$ for all $i,j\in\mathbb{Z}$, it follows that S is generated by the three elements a, a^{-1} and $(0,0)$. Let M denote the principal factor of S corresponding to $\mathbb{Z}\times\mathbb{Z}$. Then M is a completely 0-simple semigroup with infinitely many left and right ideals.

Now let F be a field. Clearly, $F_0[S]=F[G]\oplus F_0[M]$; also $F[G]$ is a subring and $F_0[M]$ an ideal of $F_0[S]$. But, by [**4**, Lemma 4 and subsequent Remark], $F_0[M]$ is quasidirectly finite. Hence, since $F[G]$ is commutative, it follows from [**4**, Lemma 1] that $F_0[S]$ is directly finite.

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