FUNCTIONAL PEARLS

On removing duplicates

RICHARD S. BIRD

Programming Research Group, Oxford University, UK

1 Introduction

The function remdup (also called mkset in some functional languages) removes
duplicates from a given list. The following definition of remdup is standard and leads
to a quadratic time algorithm

\[
\text{remdup}([]) = [] \\
\text{remdup}([a] ++ x) = [a] ++ \text{remdup}(x - [a]).
\]

The operator (\(-\)) used in the last expression subtracts one list from another; its
definition is

\[x - y = [a | a \leftarrow x; a \notin y].\]

Defined in this way remdup \(x\) returns the list of distinct elements of \(x\) in the order in
which they first appear in \(x\). In other words, the position of remdup \(x\) as a subsequence
of \(x\) is lexically the smallest among all possible solutions.

Now let us change the problem and ask that remdup simply return the lexically least
solution. Note the subtle difference between this version and the previous one: before,
it was the position of the subsequence that was lexically the least, now it is the
subsequence itself. To make the distinction clear, consider \(x = [1, 4, 2, 4, 3]\). With the
original definition remdup \(x\) returns \([1, 4, 2, 3]\); the lexically least subsequence, however,
is \([1, 2, 4, 3]\).

The question we are interested in is this: can we also find a quadratic time
algorithm for the new problem? The answer turns out to be yes, but justifying it
requires a bit of work.

2 Specification

Formally, remdup \(x\) is the lexically least sequence \(y\) satisfying the following properties

1. \(y\) is a subsequence of \(x\); in symbols, \(y \subseteq \text{subs } x\).
2. \(y\) contains all the elements of \(x\); in symbols, \(\text{set } y = \text{set } x\), where \(\text{set } z\) is the set
   of elements in the list \(z\).
3. \(y\) does not contain duplicates.

To express these conditions in a form suitable for manipulation, let us define the
predicate \(\text{ok } x y\) by

\[\text{ok } x y = (\text{set } x = \text{set } y) \land \text{nodup } y,\]
where \( \text{nodup} \ y \) is the condition that \( y \) does not contain duplicates. Also, let \( \sqcap \) denote the operation that selects the lexically smaller of its two arguments. Then we have

\[
\text{remdup} \ x = \sqcap / \text{ok} \ x \triangleleft \text{subs} \ x,
\]

where \( \sqcap / \) distributes \( \sqcap \) over the elements in a list

\[
\sqcap / [x_1, x_2, \ldots, x_n] = x_1 \sqcap x_2 \ldots \sqcap x_n,
\]

and \( p \triangleleft x \) is an abbreviation for \( \text{filter} \ p \ x \). Thus (1) reads: the lexically smallest subsequence of \( x \) satisfying the predicate \( \text{ok} \ x \).

The above notations for reduce and filter, together with the abbreviation \( f \star x \) for \( \text{map} \ f \ x \), are introduced elsewhere (Bird, 1987, 1989). They satisfy a number of useful algebraic laws, and are the basic components of a calculus for deriving programs. It would take up too much space to discuss all the laws here, and the reader is referred to the cited references for full details. We shall try and get away with just using the ones we need, appending a brief explanation when necessary.

### 3 First derivation

The specification of \( \text{remdup} \) given in (1) is incomplete because definitions of the functions \( \text{set} \), \( \text{nodup} \), \( \text{subs} \), and \( \sqcap \) are missing. These functions can reasonably be defined in a number of ways, and the best method depends in part on the kind of result we are looking for. For our first derivation we shall head for a standard decomposition of the form

\[
\begin{align*}
\text{remdup} \ [] & = \ldots \\
\text{remdup} \ ([a] \uplus x) & = \ldots,
\end{align*}
\]

so a similar style of definition is appropriate for the missing functions

\[
\begin{align*}
\text{subs} \ [] & = [\{} \\
\text{subs} \ ([a] \uplus x) & = \text{subs} \ x \cup ([a] \uplus) \star \text{subs} \ x \\
\text{set} \ [] & = \{} \\
\text{set} \ ([a] \uplus x) & = \{a\} \cup \text{set} \ x \\
\text{nodup} \ [] & = \text{True} \\
\text{nodup} \ ([a] \uplus x) & = \text{nodup} \ x \land a \notin x.
\end{align*}
\]

In order to define the lexicographic ordering (which we denote by \( \preceq \)), it is convenient to introduce first the partial order \( \sqsubseteq \) defined by

\[
x \sqsubseteq y = (\exists k: \text{take} \ k \ x = \text{take} \ k \ y \land x \cdot k < y \cdot k),
\]

where \( x \cdot k \) denotes the \( k \)th element of \( x \) (counting from 0). Then we have

\[
x \preceq y = x \in \text{inits} \ y \lor x \sqsubseteq y,
\]

where \( \text{inits} \ y \) denotes the set of initial segments of \( y \). The operator \( \sqcap \) is now defined to return the smaller under \( \preceq \) of its arguments.
An important property of the lexicographic ordering is that
\[ y \leq z \Rightarrow x \rhd y \leq x \rhd z, \]

for all \( x, y \) and \( z \). In fact, this is the only property of \( \leq \) we use in the first derivation. It is equivalent to the assertion that \( \rhd \) distributes forward through \( \cap \)

\[ (x \rhd y) \cap (x \rhd z) = x \rhd (y \cap z). \tag{3} \]

The same law can also be phrased as a property of \( \cap / \)

\[ \cap / (x \rhd \bullet) \ast xs = x \rhd \cap / xs, \]

for all \( x \) and set of sequences \( xs \), provided \( xs \) is not empty. (The restriction that \( xs \) be non-empty can be lifted provided we introduce a fictitious identity element \( \omega = \cap / \emptyset \) of \( \cap \), and make \( \omega \) the zero element of \( \rhd \).)

Although \( x \leq y \) does not imply \( x \rhd z \leq y \rhd z \), we do have

\[ x \leq y \Rightarrow x \rhd z_1 \leq y \rhd z_2, \]

for all \( z_1, z_2 \), and this is why the subordinate partial order \( \preceq \) is useful in its own right.

Now for the derivation. It is carried out in some detail to show the kind of manipulations that arise in our functional calculus. There are two cases to consider, the first being

\[
\text{remdup} []
\]

\[ = \{1\} \]

\[ \cap / (ok[]) \preceq \text{subs} [] \]

\[ = \{\text{given characterisation of subs}\} \]

\[ \cap / (ok[]) \preceq \{[]\} \]

\[ = \{\text{claim: ok[]} = \text{True}\} \]

\[ \cap / [[]] \]

\[ = \{\text{one-point rule: } \oplus / (a) = a\} \]

\[ [\]. \]

Hence, we have our first equation

\[
\text{remdup} [] = [.].
\]

For the second case we argue

\[
\text{remdup} ([a] \rhd x)
\]

\[ = \{1\} \]

\[ \cap / (ok [(a] \rhd x) \preceq \text{subs} ([a] \rhd x) \]

\[ = \{\text{characterisation of subs}\} \]

\[ \cap / (ok [(a] \rhd x) \preceq (\text{subs} x \cup ([a] \rhd \bullet) \ast xs) \]

\[ = \{\preceq \text{distributes over } \cup\} \]

\[ \cap / (ok [(a] \rhd x) \preceq \text{subs} x \cup \cap / (ok [(a] \rhd x) \preceq ([a] \rhd \bullet) \ast xs) \]

\[ = \{\text{promoting } \cap / \text{ over } \cup\} \]

\[ (\cap / (ok [(a] \rhd x) \preceq \text{subs} x) \cap (\cap / (ok [(a] \rhd x) \preceq ([a] \rhd \bullet) \ast subs x) \]
The two terms in the last expression are now simplified separately. We need the following facts about the predicate $\text{ok}$; provided $y \in \text{subs} \, x$ we have

$$\text{ok} \, ([a] + x) \, y = (a \in x \rightarrow \text{ok} \, x \, y, \ \text{False}),$$

(4)

and

$$\text{ok} \, ([a] + x) \, ([a] + y) = (a \in x \rightarrow \text{ok} \, (x - [a]) \, y \land a \notin y, \ \text{ok} \, x \, y).$$

(5)

Here use is made of the McCarthy conditional form $(p \rightarrow a, b)$. Proofs of (4) and (5) are straightforward given the definitions of set and nodup, and are omitted.

Now we argue

$$\begin{align*}
\land /\text{ok} \, ([a] \oplus x) & \prec \text{subs} \, x \\
= & \{\text{(4)}\} \\
\land / (a \in x \rightarrow \text{ok} \, x \prec \text{subs} \, x, \ {\{\}}) \\
= & \{\text{conditionals}\} \\
& (a \in x \rightarrow \land /\text{ok} \, x \prec \text{subs} \, x, \ \land /{\{\}}) \\
= & \{\text{(1) and introducing } \omega = \land /{\{\}}\} \\
& (a \in x \rightarrow \text{remdup} \, x, \ \omega).
\end{align*}$$

The step labelled ‘conditionals’ above refers to the law

$$f(p \rightarrow a, b) = (p \rightarrow \text{fa}, \text{fb}),$$

which is valid provided $p$ is a total predicate.

The second term is simplified as follows

$$\begin{align*}
\land /\text{ok} \, ([a] \oplus x) & \prec ([a] \oplus *) \text{subs} \, x \\
= & \{\text{law: } p \prec \text{fa} \cdot xs = \text{f} \cdot (p \cdot f) \prec xs\} \\
\land /([a] \oplus *) \cdot (\text{ok} \, ([a] \oplus x) \cdot ([a] \oplus *)) \prec \text{subs} \, x \\
= & \{\text{(5) and law: } (p \land q) \prec xs = p \prec q \prec xs\} \\
\land /([a] \oplus *) \cdot (a \in x \rightarrow \text{ok} \, (x - [a]) \prec (a \notin) \prec \text{subs} \, x, \ \text{ok} \, x \prec \text{subs} \, x) \\
= & \{\text{claim: see below}\} \\
\land /([a] \oplus *) \cdot (a \in x \rightarrow \text{ok} \, (x - [a]) \prec \text{subs} \, (x - [a]), \ \text{ok} \, x \prec \text{subs} \, x) \\
= & \{\text{(3)}\} \\
[a] \oplus & \land / (a \in x \rightarrow \text{ok} \, (x - [a]) \prec \text{subs} \, (x - [a]), \ \text{ok} \, x \prec \text{subs} \, x) \\
= & \{\text{conditionals}\} \\
(a \in x \rightarrow [a] \oplus & \land /\text{ok} \, (x - [a]) \prec \text{subs} \, (x - [a]), [a] \oplus \land /\text{ok} \, x \prec \text{subs} \, x) \\
= & \{\text{(1)}\} \\
(a \in x \rightarrow & [a] \oplus \text{remdup} \, (x - [a]), \ [a] \oplus \text{remdup} \, x).
\end{align*}$$

The claim referred to in this calculation is the identity

$$(a \notin) \prec \text{subs} \, x = \text{subs} \, (x - [a]),$$

The equation is an instance of a more general law

$$\text{all} \, p \prec \cdot \text{subs} = \text{subs} \cdot p \prec ,$$

(6)

in which

$$\text{all} \, p = \land / \cdot p \cdot *.$$

Thus, $\text{all} \, p \, x$ is the condition that every element of $x$ satisfies the predicate $p$. The claim is a special case of (6), since $(a \notin) = \text{all} \, (a \ni)$ and $x - [a] = (a \ni) \prec x.$
Putting the two pieces of the derivation together, we have derived the following program for remdup

\[
\text{remdup} \left[ a \right] = [ ] \\
\text{remdup} \left( [a] \uplus x \right) = \begin{cases} 
\text{remdup} x \cap [a] \uplus \text{remdup} (x - [a]), & \text{if } a \in x \\
[a] \uplus \text{remdup} x, & \text{otherwise.}
\end{cases}
\] (7)

Though it appears somewhat long, the above derivation can fairly be called routine. Most of the laws used are standard, and much of the derivation is mechanical. The only non-obvious part perhaps resides in the two equations (4) and (5) for ok. However, it is here that the problem-specific information really enters the picture, so it is not surprising that some work has to be done to see what is required.

It appears quite difficult to analyse the running time of (7). The reason is that evaluation of \( x \cap y \) does not require complete evaluation of both \( x \) and \( y \) unless \( x = y \). It can be shown, however, that algorithm (7) takes exponential time in the worst case. Nevertheless, our work has not been wasted, since (7) is used as the starting point for a second derivation, one that achieves the goal of a quadratic time algorithm.

4 Second derivation

In order to reach our goal, let us take a closer look at the computation of (7). Suppose that \( a, b \in x \) with \( a < b \). Unfolding the computation of \( \text{remdup} \left( [a, b] \uplus x \right) \), we obtain

\[
\text{remdup} \left( [a, b] \uplus x \right) = \text{remdup} x \\
\cap [b] \uplus \text{remdup} (x - [b]) \\
\cap [a] \uplus \text{remdup} (x - [a]) \\
\cap [a, b] \uplus \text{remdup} (x - [a, b]).
\]

Since \( a < b \), the second term is lexically larger than the third and can be omitted. Thus, we can write the above expression as follows

\[
\text{remdup} \left( [a, b] \uplus x \right) = [ ] \uplus \text{remdup} (x - []) \\
\cap [a] \uplus \text{remdup} (x - [a]) \\
\cap [a, b] \uplus \text{remdup} (x - [a, b]).
\]

The form of this expression suggests an appropriate generalization: for a strictly increasing sequence \( w \) such that \( \text{set w} \subseteq \text{set x} \), define

\[
f w x = \cap /g x * \text{inits w},
\]

where

\[
g x y = y \uplus \text{remdup} (x - y).
\] (8) (9)

This is a large generalization to take in at one go, but the way it is discovered is standard practice: symbolically unfold the computation to see what subcomputations are required and how they can be combined. That \( f \) is a generalization of \( \text{remdup} \) follows from
\[
\text{remdup } x \\
= \{ (9) \} \\
g \times [] \\
= \{ \text{definition of } \cap / \text{on singletons} \} \\
\cap / [g \times []] \\
= \{ \text{definition of the map operator } * \} \\
\cap / [g \times *[[]]] \\
= \{ \text{definition of } \text{inits } [] \text{ and (8)} \} \\
f [] x.
\]

Having found our generalization, we now need to develop an efficient algorithm for \( f \). Again, we head for a standard recursion of the form

\[
f w [ ] = \ldots \\
f w ([a] ++ x) = \ldots
\]

The first step, namely to derive \( f w [ ] = [] \), is straightforward and details are omitted. For the second step we have by instantiation in (8) that

\[ f w ([a] ++ x) = \cap / [g ([a] ++ x) * \text{inits } w]. \]

To simplify this expression we need to consider the value of \( g ([a] ++ x) y \). After some routine manipulations using (9) and (7) we obtain

\[
g ([a] ++ x) y = \begin{cases} 
g \times y & \text{if } a \in y \\
g \times y \cap g \times (y ++ [a]) & \text{if } a \notin y \land a \in x \\
g \times (y ++ [a]) & \text{if } a \notin y \land a \notin x. \end{cases}
\]

Since \( g ([a] ++ x) y \) is required only for \( y \in \text{inits } w \), and the above expression divides into cases depending on whether \( a \in y \), we are led to consider those initial segments of \( w \) which might contain \( a \) and those that do not. Suppose we split \( w \) into two parts \( w = u ++ v \), where

\[(u, v) = (\text{takewhile } (\leq a) w, \text{dropwhile } (\leq a) w).\]

Note that if \( a \in w \), then \( a \) will be the last element of \( u \), since \( w \) is assumed to be a sequence of values in strictly increasing order.

Next we use the fact that

\[\text{inits } (u ++ v) = \text{inits } u ++ (u ++) * \text{inits } v,\]

where \( \text{inits } u \) is the list of initial segments of \( u \) excluding \( u \) itself, to rewrite \( f w ([a] ++ x) \) as the minimum of two terms

\[
f w ([a] ++ x) = \cap / [g ([a] ++ x) * \text{inits } u] \\
\cap / [g ([a] ++ x) * (u ++) * \text{inits } v].
\]

The remainder of the derivation is devoted to simplifying (11). To start with, consider the first term on the right of (11). Since if \( a \) appears at all in \( u \), it appears only as the last element of \( u \), we have that no sequence in \( \text{inits } u \) contains \( a \). Hence, using (10) and simplifying, we have

\[
\cap / [g ([a] ++ x) * \text{inits } u] = \begin{cases} 
\tau \cap \tau_2 & \text{if } a \in x \\
\tau_2 & \text{if } a \notin x,
\end{cases}
\]
where

\[ t_1 = \cap / g x \cdot \text{init}^{-} u \]
\[ t_2 = \cap / g x \cdot (++) [a] \cdot \text{init}^{-} u. \]

A similar simplification yields

\[ \cap / g ([a]++ x) \cdot (u++) \cdot \text{init} v = \begin{cases} 
  t_3, & \text{if } a \in u \\
  t_4 \cap t_3, & \text{if } a \notin u \land a \in x \\
  t_4, & \text{if } a \notin u \land a \notin x,
\end{cases} \]

where

\[ t_3 = \cap / g x \cdot (u++) \cdot \text{init} v \]
\[ t_4 = \cap / g x \cdot (++) [a] \cdot (u++) \cdot \text{init} v. \]

Putting these results together, we have

\[ f_w ([a]++ x) = \begin{cases} 
  t_1 \cap t_2 \cap t_3, & \text{if } a \in w \land a \in x \\
  t_2 \cap t_3, & \text{if } a \in w \land a \notin x \\
  t_1 \cap t_2 \cap t_3 \cap t_4, & \text{if } a \notin w \land a \in x \\
  t_2 \cap t_4, & \text{if } a \notin w \land a \notin x.
\end{cases} \] (12)

To simplify the right-hand side of (12) we need the following properties of \( g \) (the proofs are left to the reader)

\[ y \subseteq z \Rightarrow g x y \subseteq g x z \] (13)
\[ g x (u++ v) = u++ g (x-u) v. \] (14)

Implications (13) refers to the partial order \( \subseteq \) defined by (2). For ease of reference, here are the relevant terms \( t_1, \ldots, t_4 \) again

\[ t_1 = \cap / g x \cdot \text{init}^{-} u \]
\[ t_2 = \cap / g x \cdot (++) [a] \cdot \text{init}^{-} u \]
\[ t_3 = \cap / g x \cdot (u++) \cdot \text{init} v \]
\[ t_4 = \cap / g x \cdot (++) [a] \cdot (u++) \cdot \text{init} v. \]

To begin with, suppose \( t_2 = g x (u'++ [a]) \), for some proper prefix \( u' \) of \( u \). Thus \( u = u'++ [b]++ u'' \), where \( b \leq a \) by definition of \( u \). Hence, \( u \subseteq u'++ [a] \), and so by (13) we have

\[ g x u \leq g x (u'++ [a]). \]

But, by definition of \( t_3 \) we have \( t_3 \leq g x u \), and so it follows that

\[ t_2 \cap t_3 = t_3. \]

Equation (12) can therefore be simplified to read

\[ f_w ([a]++ x) = \begin{cases} 
  t_1 \cap t_3, & \text{if } a \in w \land a \in x \\
  t_3, & \text{if } a \in w \land a \notin x \\
  t_1 \cap t_3 \cap t_4, & \text{if } a \notin w \land a \in x \\
  t_2 \cap t_4, & \text{if } a \notin w \land a \notin x.
\end{cases} \] (15)
Now we simplify each term. First
\[
\begin{align*}
t1 \cap t3 &= \{\text{definition of } t1 \text{ and } t3\} \\
&= \ni /g x \ast \text{init}s \ast u \ni \ni /g x \ast (u \ast \ast) \ast \text{init}s \ast v \\
&= \{\text{decomposition of init}s\} \\
&= \ni /g x \ast \text{init}s (u \ast \ast v) \\
&= \{(8) \text{ and } w = u \ast \ast v\} \\
f \nu x.
\end{align*}
\]
Second
\[
\begin{align*}
t3 &= \{\text{definition of } t3\} \\
&= \ni /g x \ast (u \ast \ast) \ast \text{init}s \ast v \\
&= \{\text{distributive law; } f \ast g \ast = (f \ast g) \ast \}, \text{ and (14)}\} \\
&= \ni / (u \ast \ast) \ast g (x \ast y) \ast \text{init}s \ast v \\
&= \{(3)\} \\
&= u \ast \ast \ni /g (x \ast u) \ast \text{init}s \ast v \\
&= \{(8)\} \\
&= u \ast \ast f \nu (x \ast u).
\end{align*}
\]
To simplify the third term \( t1 \cap t3 \cap t4 \) of (15), observe that the first element of \( v \), if it exists, is greater than \( a \). Hence
\[
u \ast [a] \subseteq u \ast \ast v',
\]
for any non-empty initial segment \( v' \) of \( v \). Using (13) it follows that
\[
t3 \cap t4 = g x u \ni g x (u \ast [a]).
\]
Thus
\[
\begin{align*}
t1 \cap t3 \cap t4 &= \{\text{above}\} \\
&= t1 \ni g x u \ni g x (u \ast [a]) \\
&= \{\text{definition of } t1 \text{ and init}s\} \\
&= \ni /g x \ast \text{init}s (u \ast [a]) \\
&= \{(8)\} \\
&= f(u \ast [a]) \nu x.
\end{align*}
\]
To simplify the final term \( t2 \cap t4 \), assume \( a \notin w \land a \notin x \), so last \( u < a \). It follows that \( u \ast [a] \subseteq u' \ast [a] \) for any proper prefix \( u' \) of \( u \), and so
\[
\begin{align*}
t2 \cap t4,
&= \{\text{above and (13)}\} \\
&= g x (u \ast [a]) \\
&= \{(9) \text{ and } a \notin x\} \\
&= u \ast [a] \ast \text{remdup} (x \ast u) \\
&= \{(8)\} \\
&= u \ast [a] \ast f [] (x \ast u).
\end{align*}
\]
5 The result

Putting the above pieces together, we have that \( \text{remdup} = f[] \), where

\[
f_w[] = []
\]

\[
f_w([a] ++ x) =
\begin{cases}
  f_w x, & \text{if } a \in x \land a \in w \\
  u ++ f(v(x - u)), & \text{if } a \notin x \land a \in w \\
  f(u ++ [a]) x, & \text{if } a \in x \land a \notin w \\
  u ++ [a] ++ f[](x - u), & \text{if } a \notin x \land a \notin w
\end{cases}
\]

where \((u, v) = \text{split } a w\),

and

\[
\text{split } a w = (\text{take} \{ \leq a \} w, \text{dropwhile} \{ \leq a \} w).
\]

This program takes \( O(n^2) \) steps for a sequence of length \( n \). If we ignore the time spent evaluating \( x - u \), then \( O(n) \) steps are performed at each iteration since \( \neq w \leq n \). This gives \( O(n^2) \) steps. However, the total time required for evaluating expressions of the form \( x - u \) is also \( O(n^2) \) steps, since the sum of the lengths of the \( us \) for which this operation is performed is, at most, \( n \). Hence, the total running time is \( O(n^3) \) steps.

Well, it was a lot of work to achieve the above result. Is there a simpler solution?

References
