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AN IMBEDDING THEOREM FOR ANISOTROPIC ORLICZ-SOBOLEV SPACES

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Let G be a convex function of m variables, let Ω be a domain in $\mathbb{R}^n$, and let $L_G(\Omega)$ denote the vector-valued Orlicz space determined by G. We give an imbedding theorem for the space $W_G^1(\Omega)$ of weakly differentiable functions u provided with the norm $\|(u, Du)\|_G$, where $m = n + 1$ and Du denotes the gradient of u. This theorem is a variant of an imbedding theorem by N.S. Trudinger for the completion of $C_0^1(\Omega)$ in the norm $\|Du\|_G$, where m=n.

1. PRELIMINARIES

We shall use the definitions and properties of vector-valued Orlicz spaces as given in [**2**], with an additional monotonicity requirement. References on Orlicz spaces can be found in [**2**].

DEFINITION: A function $G : \mathbb{R}^m \to [0, \infty]$ is said to be a *G-function* of m variables if

(i) $G(0) = 0$;
(ii) $\lim_{|x|\to\infty} G(x) = \infty$;
(iii) G is convex;
(iv) G is symmetric;
(v) $G^{-1}(\infty)$ is bounded away from zero;
(vi) G is lower semicontinuous.

G is called a Young function if $m = 1$.

We further assume that

(vii) G is monotone increasing in each variable separately.

If Ω is a domain in $\mathbb{R}^n$, $u_1, \ldots, u_m$ are measurable on Ω, and $u = (u_1, \ldots, u_m)$, then $L_G(\Omega)$ is defined by

$$L_G(\Omega) = \{u : \int_\Omega G(\alpha u)\, dx < \infty \text{ for some } \alpha > 0\}.$$

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DEFINITION: The *Luxemburg norm* $\|u\|_{G,\Omega} = \|u\|_G$ is defined by

(1) $$\|u\|_{G,\Omega} = \inf\{k > 0 : \int_\Omega G(u/k)\,dx \leqslant 1\}.$$

DEFINITION: The *conjugate function* G_+^* of G is defined by

$$G_+^*(x) = \sup_{y_i \geqslant 0}\{x \cdot y - G(y)\}.$$

For $u \in L_G(\Omega)$, $v \in L_{G_+^*}(\Omega)$, the following generalised Hölder inequality holds:

(2) $$\int_\Omega u \cdot v\,dx \leqslant 2\,\|u\|_G\,\|v\|_{G_+^*}\,.$$

DEFINITION: If G is a G-function of $n+1$ variables, then $W_G^1(\Omega)$ denotes the set of weakly differentiable functions u for which $(u, D_1u, \dots, D_nu) = (u, Du)$ belongs to $L_G(\Omega)$. A norm is defined on $W_G^1(\Omega)$ by

(3) $$\|u\|_{W_G^1(\Omega)} = \|u\|_{W_G^1} = \|(u, Du)\|_G\,.$$

DEFINITION: A domain $\Omega \subset \mathbb{R}^n$ is said to be *admissible* if there exists a constant γ, depending only on n, such that, for any u in the Sobolev space $W^1L_1(\Omega)$,

(4) $$\|u\|_{n/(n-1)} \leqslant \gamma\left(\|u\|_1 + \sum_{i=1}^{n}\|D_iu\|_1\right).$$

(See [**1**].) It is known that (4) is true if Ω satisfies the *cone condition*, that is, if there exists a fixed cone $k_\Omega \subset \mathbb{R}^n$ such that each point x of Ω is the vertex of a cone $k_\Omega(x) \subset \Omega$, congruent to k_Ω.

We shall use the following extension of the chain rule (see [**1**]):

LEMMA. *If u is a weakly differentiable function on a domain $\Omega \subset \mathbb{R}^n$ and if g is a uniformly Lipschitz continuous function on $\mathbb{R}$, then $g \circ u$ is weakly differentiable, and*

$$Dg(u) = g'(u)Du \qquad \text{almost everywhere in } \Omega,$$

that is,

(5) $$D_ig(u) = g'(u)D_iu, \quad i = 1, \dots, n$$

where, in (5), the convention is made that the right side is zero if D_iu is zero, even if $g'(u)$ is undefined or infinite.

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2. An imbedding theorem

Theorem. *Let Ω be a bounded admissible domain in $\mathbb{R}^n$. Let G be a G-function of $n+1$ variables, and suppose F is a continuous, non-negative function on $[0,\infty)$ such that*

$$G_+^*[0, F(s), \ldots, F(s)] \leqslant s. \tag{6}$$

Let $m(s) = s^{1/n}F(s)$, and suppose A is a Young function such that

$$\int_0^{|t|} \frac{ds}{m(s)} = kA^{-1}(|t|) \tag{7}$$

for some constant k. Then there exists a constant C, depending only on n, such that

$$\|u\|_A \leqslant C\,\|u\|_{W_G^1} \tag{8}$$

for any $u \in W_G^1(\Omega)$.

Proof: We suppose first that $u \in W_G^1(\Omega)$ is bounded. Let $\lambda = \|u\|_A$. Since Ω is bounded,

$$\int_\Omega A(u/\lambda)\,dx = 1. \tag{9}$$

From (7) and the definition of m,

$$\left(A^{1-1/n}\right)' = k(1-1/n)F(A). \tag{10}$$

Let $C = A^{1-1/n}$ and let $g = C(u/\lambda)$. By (5), $g \in W_{L_1}^1(\Omega)$, and since we assumed Ω is admissible,

$$\begin{aligned}
\|g\|_{n/(n-1)} &\leqslant \gamma\left[\int_\Omega \left(\sum_{i=1}^n |\,D_i g\,|\right) dx + \|g\|_1\right] \\
&= \frac{\gamma}{\lambda}\int_\Omega \left(\sum_{i=1}^n |\,C'(u/\lambda)D_i u\,|\right) dx + \gamma\,\|g\|_1 \\
&\leqslant \frac{2\gamma}{\lambda}\,\|(0, C'(u/\lambda), \ldots, C'(u/\lambda))\|_{G_+^*}\,\|(u, Du)\|_G + \gamma\,\|g\|_1
\end{aligned}$$

from Hölder's inequality (2). Then using the definition of C and (10), we get
(11)

$$\|g\|_{n/(n-1)} \leqslant \frac{2\gamma}{\lambda}k(1-1/n)\,\|(0, F[A(u/\lambda)], \ldots, F[A(u/\lambda)])\|_{G_+^*}\,\|(u, Du)\|_G + \gamma\,\|g\|_1$$

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Using (6) and (9), we have

$$\int_\Omega G_+^*\{(0, F[A(u/\lambda)], \dots, F[A(u/\lambda)])\}\,dx \leqslant \int_\Omega A(u/\lambda)\,dx = 1$$

and so

$$\|(0, F[A(u/\lambda)], \dots, F[A(u/\lambda)])\|_{G_+^*} \leqslant 1 \tag{12}$$

from the definition of the Luxemburg norm (1). From the definition of G and from (9),

$$\|g\|_{n/(n-1)} = \left[\int_\Omega A(u/\lambda)\,dx\right]^{(n-1)/n} = 1. \tag{13}$$

Since $\dfrac{g(t/\lambda)/(t/\lambda)}{A(t/\lambda)/(t/\lambda)} = \dfrac{1}{A^{1/n}(t/\lambda)} \to 0$ as $t \to \infty$, there exists t_0 such that

$$\frac{g(t/\lambda)}{t/\lambda} \leqslant \frac{1}{2\gamma}\frac{A(t/\lambda)}{t/\lambda}$$

for $t \geqslant t_0$. Further, by L'Hospital's rule, $g(t/\lambda) \to 0$ as $t \to 0$, so that $K = \sup\limits_{t\in(0,t_0)} [g(t/\lambda)/(t/\lambda)]$ is finite. Thus, for all $t > 0$,

$$g(t/\lambda) \leqslant \frac{1}{2\gamma}A(t/\lambda) + Kt/\lambda.$$

Replacing t in the last equation by u, integrating over Ω, and using (9) gives

$$\|g\|_1 \leqslant 1/(2\gamma) + (K/\lambda)\,\|u\|_1\,. \tag{14}$$

Thus from (11), (12), (13), and (14) we obtain

$$1 \leqslant \frac{2\gamma}{\lambda}\,\|(u, Du)\|_G + \frac{1}{2} + \frac{\gamma K}{\lambda}\,\|u\|_1\,,$$

that is,

$$\lambda \leqslant 4\gamma\,\|(u, Du)\|_G + 2\gamma K\,\|u\|_1\,. \tag{15}$$

But

$$\|u\|_1 \leqslant 2\,\|(1, 0, \dots, 0)\|_{G_+^*}\,\|(u, Du)\|_G\,;$$

hence (15) may be written as

$$\|u\|_A \leqslant C\,\|(u, Du)\|_G\,, \tag{16}$$

where C depends only on n.

If u is not bounded, we define

$$u_l = \begin{cases} u, & \text{for } |u| < l \\ l \operatorname{sign} u, & \text{for } |u| \geqslant l. \end{cases}$$

By the chain rule (5), u_l belongs to $W_G^1(\Omega)$, and by the monotone convergence theorem, $\|u_l\|_A \to \|u\|_A$ and $\|(u_l, Du_l)\|_G \to \|(u, Du)\|_G$, so that (16) is true for all $u \in W_G^1(\Omega)$. Referring to (3), we see that this establishes (8), so the theorem is proved. ∎

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