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INVARIANT KÄHLER STRUCTURES ON THE COTANGENT BUNDLES OF COMPACT SYMMETRIC SPACES

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Abstract. For rank-one symmetric spaces M of the compact type all Kähler structures F^λ, defined on their punctured tangent bundles T^0M and invariant with respect to the normalized geodesic flow on T^0M, are constructed. It is shown that this class $\{F^\lambda\}$ of Kähler structures is stable under the reduction procedure.

§1. Introduction

Let G/K be a Riemannian symmetric space, where G is a semisimple Lie group, with the standard G-invariant metric $\mathbf{g}$. This metric defines the geodesic flow with the Hamiltonian H on the tangent bundle $T(G/K)$ as a symplectic manifold with the symplectic 2-form Ω (that comes from the canonical symplectic structure on the cotangent bundle using the metric to identify these two bundles).

Geometric constructions which come from geometric quantization naturally lead to complex structures defined on the punctured tangent bundle $T^0(G/K) = T(G/K) - \{\text{zero section}\}$. Such structure J_S for the spheres was found by Souriau [So]. Later it was observed by Rawnsley [Ra1], that the norm function $\sqrt{H}$ is strictly plurisubharmonic with respect to the above complex structure J_S and thus defines the Kähler metric on T^0S^n with Ω as the Kähler form. He also observed that J_S is invariant with respect to the Hamiltonian flow $X_{\sqrt{H}}$ of the norm function $\sqrt{H}$ (the normalized geodesic flow) and used the Kähler structure J_S to quantize the geodesic flow on the spheres [Ra2].

Subsequently, Furutani and Tanaka [FT] defined a Kähler structure J_S with the analogous properties on the punctured tangent bundle of complex

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and quaternionic projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ and used them for quantization. In [IM] Ii and Morikawa describe these structures on the punctured tangent bundles (of Riemannian manifolds G/K) by means of horizontal and vertical lifts.

In [Sz1] Szőke explored the relationship of J_S and a so-called adapted complex structure J_A (associated with the Riemannian metric $\mathbf{g}$) on the respective tangent bundle $T(G/K)$. He showed that for all compact, rank-one symmetric spaces (also for the Cayley projective plane $\mathbb{C}aP^2$) the family of complex structures obtained by pushing forward the adapted complex structure with respect to an appropriate family of diffeomorphisms has a limit and this limit complex structure coincides with J_S.

The purpose of this paper is to describe all G-invariant Kähler structures on $T^0(G/K)$ (with Ω as the Kähler form) preserved by the normalized geodesic flow $X_{\sqrt{H}}$. We prove that such Kähler structures F exist only on the punctured tangent bundles of rank-one symmetric spaces of the compact type. There is a one-to-one correspondence between the space of such structures and the space of smooth functions with positive real part of the form $\lambda \circ \sqrt{H}$ (Theorem 12). In particular, the structure J_S coincides with F^λ, where $\lambda(r) = r$. Moreover, among these structures there exists a unique metric compatible structure (see Definition 4) F^λ, $\lambda = 1$ defined on $T^0(G/K)$ (Proposition 19). The Hamiltonian H is strictly plurisubharmonic with respect to this complex structure F^1 and $\sqrt{H}$ satisfies the Monge-Ampere equation on $T^0(G/K)$. This class $\{F^\lambda\}$ of Kähler structures is stable with respect to reduction procedure. In particular, the reduction procedure under the action of $U(1)$ for J_S on T^0S^{2n+1} gives the Kähler structure J_S on $T^0\mathbb{C}P^n$, under the action of $Sp(1)$ for J_S on T^0S^{4n+3} gives the Kähler structure J_S on $T^0\mathbb{H}P^n$. As an application of the methods developed in section 2, we obtain a description of the adapted complex structures on the tangent bundle $T(G/K)$ of any homogeneous space of a compact Lie group G (Proposition 21).

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§2. G-invariant Kähler structures on $\mathbf{T(G/K)}$

2.1. Positive-definite polarizations

Here we will review few facts about polarizations. For more detailed account of the material below, see for instance [Ga, GS]. Let (X, Ω) be a (real) symplectic manifold, F be a (smooth) complex subbundle of the complexified tangent bundle $T^{\mathbb{C}}X$. F is (defines) a *complex structure* on X if F is involutive (i.e. is closed under the Lie bracket: $[F, F] \subset F$), $F \cap \overline{F} = 0$ and $F + \overline{F} = T^{\mathbb{C}}X$. We say that a subbundle F is *integrable* if 1) $F \cap \overline{F}$ has constant rank; 2) the subbundles F and $F + \overline{F}$ are involutive. A subbundle F is *Lagrangian* if $\Omega(F, F) = 0$ and $\dim_{\mathbb{C}} F = \frac{1}{2} \dim_{\mathbb{R}} X$.

A *polarization* of X is a complex integrable Lagrangian subbundle F of $T^{\mathbb{C}}X$. A polarization F is *positive-definite* if $-i\Omega_x(Z, \overline{Z}) > 0$ for all nonzero $Z \in F(x)$ and $x \in X$.

Definition 1. [GS] A symplectic manifold is a (positive) Kähler manifold if it possesses a positive-definite polarization.

To reconcile this definition with the standard one we will use the next lemma (see [GS, Lemma 4.3]):

Lemma 1. *Let F be a positive-definite polarization on a (real) symplectic manifold X. Then for every $x \in X$ there exists a unique linear mapping $J_x : T_xX \to T_xX$ such that*

1) $J_x^2 = -Id_x$;
2) $F(x) = \{Y + iJ_x(Y), Y \in T_xX\}$;
3) $\Omega_x(J_x(Y_1), J_x(Y_2)) = \Omega_x(Y_1, Y_2)$ *for any* $Y_1, Y_2 \in T_xX$;
4) *the quadratic form* $B_x(Y_1, Y_2) = \Omega_x(J_xY_1, Y_2)$ *is symmetric and positive-definite.*

By Lemma 1, F is an (integrable) complex subbundle of $(0, 1)$ vectors of the complex structure J and the quadruple (X, J, B, Ω) is a Kähler manifold in the usual sense.

Definition 2. [GSt] Suppose that F is a complex structure on X and $D \subset X$ is some domain. A smooth real function f on D satisfies the homogeneous complex Monge-Ampere equation if $(\partial\overline{\partial} f)^{(\dim X)/2} = 0$.

2.2. G-invariant complex structures

Let M be a homogeneous space of a real reductive connected Lie group G, i.e. $M = G/K$. Suppose that K is a (closed) reductive subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of the groups G and K respectively. There exists a faithful representation of $\mathfrak{g}$ such that its associated bilinear form Φ is nondegenerate on $\mathfrak{g}$ (if $\mathfrak{g}$ is semi-simple we can take as Φ the Killing form associated with the adjoint representation of $\mathfrak{g}$). The form Φ is nondegenerate on $\mathfrak{k}$. Let $\langle,\rangle = c\Phi$, where $c \in \mathbb{R}$ is a nonzero constant. This form $\langle,\rangle$ defines the G-invariant pseudo-Riemannian metric $\mathbf{g}$ on G/K. The metric $\mathbf{g}$ identifies the cotangent bundle T^*M and the tangent bundle TM and thus we can also talk about the canonical 1-form θ on TM, that is the form defined by

$$\theta(Y) \stackrel{\text{def}}{=} \mathbf{g}(x, p_* Y), \qquad Y \in T_x(TM), \tag{1}$$

where $p : TM \to M$ denotes the natural projection. The form θ and the symplectic form $\Omega \stackrel{\text{def}}{=} d\theta$ are G-invariant with respect to the natural action of G on TM (extension of the action of G on M).

Denote by $\mathfrak{m}$ the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle,\rangle$, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the $\operatorname{Ad} K$-invariant direct sum decomposition of $\mathfrak{g}$. Consider the trivial vector bundle $G \times \mathfrak{m}$ with the two Lie group actions (which commute) on it: the left G-action, $l_h : (g, w) \mapsto (hg, w)$ and the right K-action $r_k : (g, w) \mapsto (gk, \operatorname{Ad} k^{-1}(w))$. Let $\pi : G \times \mathfrak{m} \to G \times_K \mathfrak{m}$ be the natural projection. It is well known that $G \times_K \mathfrak{m}$ and TM are isomorphic. Using the corresponding G-equivariant diffeomorphism $\phi : G \times_K \mathfrak{m} \to TM$, $[(g, w)] \mapsto \frac{d}{dt}\Big|_0 g \exp(tw) K$ and the projection π define the G-equivariant submersion $\Pi : G \times \mathfrak{m} \to TM$, $\Pi = \phi \circ \pi$. Denote by $\tilde{\theta}$ the 1-form $\Pi^*\theta$ and by $\tilde{\Omega}$ its differential $d\tilde{\theta}$. Let ξ^l be the left-invariant vector field on the Lie group G defined by a vector $\xi \in \mathfrak{g}$.

Lemma 2. *The 1-form $\tilde{\theta}$ on the manifold $G \times \mathfrak{m}$ has the form*

$$\tilde{\theta}_{(g,w)}(\xi^l(g), u) = \langle w, \xi \rangle, \tag{2}$$

$$\tilde{\Omega}_{(g,w)}((\xi_1^l(g), u_1), (\xi_2^l(g), u_2)) = \langle \xi_2, u_1 \rangle - \langle \xi_1, u_2 \rangle - \langle w, [\xi_1, \xi_2] \rangle, \tag{3}$$

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where $g \in G, w \in \mathfrak{m}$, $\xi, \xi_1, \xi_2 \in \mathfrak{g}$, $u, u_1, u_2 \in \mathfrak{m} = T_w\mathfrak{m}$. The kernel $\mathcal{K} \subset T(G \times \mathfrak{m})$ of the 2-form $\tilde{\Omega}$ is generated by the global (left) G-invariant vector fields ζ^L, $\zeta \in \mathfrak{k}$ on $G \times \mathfrak{m}$, $\zeta^L(g,w) = (\zeta^l(g), [w,\zeta])$.

Proof. The formula for the form $\tilde{\theta}$ is an evident consequence of (1). To prove the formula for $\tilde{\Omega}$ fix the basis $\{\eta_j\}$ in $\mathfrak{g}$ and the corresponding dual basis $\{\omega_j\}$ in $\mathfrak{g}^*$. Then $\tilde{\theta} = \sum_j f_j \omega_j^L$, where $f_j(g,w) = \langle w, \eta_j \rangle$ is a smooth function and ω_j^L is the pullback along the projection $G \times \mathfrak{m} \to G$ of the left G-invariant 1-form ω_j^l on the Lie group G. Now it is sufficient to see that $df_j(g,w)(\xi^l(g), u) = \langle u, \eta_j \rangle$ and to use the Maurer-Cartan formula: $d\omega_j^l(\xi^l, \eta^l) = -\omega_j([\xi,\eta])$. Since Ω is a symplectic form, the kernel of $\tilde{\Omega}$ is the kernel of Π_*. $\square$

Let F be a polarization on TM. Since F is an integrable subbundle of $T^{\mathbb{C}}(TM)$, it is defined by the differential ideal $\mathcal{I}(F) \subset \Lambda T^{\mathbb{C}*}(TM)$ (closed relatively to the exterior differentiation). The kernel $\mathcal{F}$ of the differential ideal $\Pi^*\mathcal{I}(F)$ is an integrable subbundle of $T^{\mathbb{C}}(G \times \mathfrak{m})$. We will denote $\mathcal{F}$ also by $\Pi_*^{-1}(F)$. This subbundle is uniquely defined by two conditions: 1) $\dim_{\mathbb{C}} \mathcal{F} = \dim_{\mathbb{R}} G$; 2) $\Pi_*(\mathcal{F}) = F$. It is evident that $\tilde{\Omega}(\mathcal{F},\mathcal{F}) = 0$ and the subbundle $\mathcal{F}$ contains $\mathcal{K}^{\mathbb{C}}$. Moreover, $\mathcal{F}$ is right K-invariant.

We can substantially simplify matters by working on the trivial vector bundle $G \times \mathfrak{m}$ with the subbundle $\mathcal{F}$ rather than on the tangent bundle $T(G/K)$ with the polarization F. To this end we need

Lemma 3. *Let $\mathcal{F}$ be an integrable complex subbundle of $T^{\mathbb{C}}(G \times \mathfrak{m})$ such that* 1) *$\mathcal{F}$ is right K-invariant;* 2) *$\mathcal{K}^{\mathbb{C}} \subset \mathcal{F}$;* 3) *$\dim_{\mathbb{C}} \mathcal{F} = \dim_{\mathbb{R}} G$;* 4) *$\tilde{\Omega}(\mathcal{F},\mathcal{F}) = 0$. Then $F = \Pi_*(\mathcal{F})$ is a polarization on TM.*

Conversely, any polarization F on TM defines an integrable subbundle $\mathcal{F} = \Pi_*^{-1}(F)$ with properties 1)–4)*.*

Proof. Since $\mathcal{F}$ is right K-invariant and the kernel $\mathcal{K}$ of Π_* is contained in $\mathcal{F}$, the image $F = \Pi_*(\mathcal{F})$ of $\mathcal{F}$ is a well defined subbundle of $T^{\mathbb{C}}(TM)$ of dimension $\dim G - \dim K$. $F \cap \overline{F}$ has constant rank because $\mathcal{K} \subset \mathcal{F}$ is a real subbundle. It then immediately follows from 4) that the subbundle F is Lagrangian. To prove the smoothness and involutiveness of F we notice that Π is a submersion, i.e. for any point $(g,w) \in G \times \mathfrak{m}$ there exist a neighborhood U of (g,w), coordinates $x_1, \dots, x_N$ on U and coordinates $x_1, \dots, x_{N'}$ on the open set $U' = \Pi(U)$ such that $x_j(g,w) = 0$, $j = 1, \dots, N$ and $\Pi|U$ in these coordinates has a form $\Pi : (x_1, x_2, \dots, x_N) \mapsto (x_1, x_2, \dots, x_{N'})$. Let

$Y(x_1,\dots,x_N) = \sum_{j=1}^{N} a_j(x_1,\dots,x_N)\partial/\partial x_j$ be any section of $\mathcal{F}|U$. The subbundle $\mathcal{K}$ is spanned on U by $\partial/\partial x_j, j = N'+1,\dots,N$ and $\mathcal{F}|U$ is preserved by these $\partial/\partial x_j$. Therefore, the smooth vector field

$$(4) \qquad Y_0(x_1,\dots,x_N) = \sum_{j=1}^{N} a_j(x_1,\dots,x_{N'},0,\dots,0)\partial/\partial x_j,$$

is also a section of $\mathcal{F}|U$. ($\mathcal{F}$ is preserved by $\partial/\partial x_j$ iff $\mathcal{F}$ is preserved by the corresponding local one-parameter group [Ga].) Thus, $\Pi_* Y_0(x_1,..,x_{N'}) = \sum_{j=1}^{N'} a_j(x_1,..,x_{N'},0,..,0)\partial/\partial x_j$ is a smooth section of $F|U'$. Involutiveness of F follows easily from (4). In the same manner we obtain that $\Pi_*(\mathcal{F}+\overline{\mathcal{F}})$ is involutive, i.e. F is a polarization. ∎

Remark 4. If the Lie subgroup K is connected, condition 1) of the lemma is a consequence of the integrability of $\mathcal{F}$ and 2). In this case any leaf of $\mathcal{K}$ is K-orbit and our lemma may be obtained as a simple modification of [Ga, Lemma in s. III.17] or the proof of [GS, Theorem 3.5].

PROPOSITION 5. *Let F be a polarization on TM, $\mathcal{F} = \Pi_*^{-1}(F)$. Then*
1) *F is G-invariant iff $\mathcal{F}$ is left G-invariant;*
2) *F is a complex structure on TM iff $\mathcal{F}\cap\overline{\mathcal{F}} = \mathcal{K}^{\mathbb{C}}$ ($\mathcal{F}+\overline{\mathcal{F}} = T^{\mathbb{C}}(G\times\mathfrak{m})$);*
3) *F is a positive-definite polarization on TM iff $-i\tilde{\Omega}(Z,\overline{Z}) \geq 0$ for all vector fields (sections) $Z \in \Gamma\mathcal{F}$ and $\tilde{\Omega}(Z,\overline{Z}) = 0 \Leftrightarrow Z \in \Gamma\mathcal{K}^{\mathbb{C}} \subset \Gamma\mathcal{F}$.*

Proof. Taking into account that the submersion Π is G-equivariant we conclude that the subbundles $\mathcal{F}$ and $\Pi_*(\mathcal{F})$ are (left) G-invariant simultaneously. If $\mathcal{F}\cap\overline{\mathcal{F}} = \mathcal{K}^{\mathbb{C}}$ then $\Pi_*(\mathcal{F})\cap\Pi_*(\overline{\mathcal{F}}) = 0$, so that F is a complex structure. The latter assertion of the lemma is evident. ∎

Thus we see that there is one-to-one correspondence between the set of all G-invariant polarizations F on TM and the set of all integrable (left) G-invariant subbundles $\mathcal{F} \subset T^{\mathbb{C}}(G\times\mathfrak{m})$ for which conditions 1)–4) of Lemma 3 hold.

Let F be a G-invariant polarization, $\mathcal{F} = \Pi_*^{-1}(F)$. Our interest in the next subsection centers on what will be shown to be an important G-invariant subbundle of $\mathcal{F}$. Define this subbundle $\mathcal{P} \subset \mathcal{F}$ as a complementary G-invariant subbundle to $\mathcal{K}^{\mathbb{C}}$ in $\mathcal{F}$ such that $\mathcal{P}(e,w) \subset \mathfrak{m}^{\mathbb{C}}\times\mathfrak{m}^{\mathbb{C}} \subset T_e^{\mathbb{C}}G\times T_w^{\mathbb{C}}\mathfrak{m}$. Since $\mathcal{K}$ is generated by the vector fields $\zeta^L, \zeta\in\mathfrak{k}$, $\zeta^L(g,w) = (\zeta^l(g),[w,\zeta])$ the subbundle $\mathcal{P}$ is unique. By definition, $\mathcal{F} = \mathcal{K}^{\mathbb{C}}+\mathcal{P}$. Moreover, these two subbundles $\mathcal{K},\mathcal{P}$ (and $\mathcal{F}$ by definition) are right K-invariant because the decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ is $\operatorname{Ad}K$-invariant.

2.3. G-invariant complex structures on the tangent bundle of a symmetric space

We continue with the previous notation but throughout the remainder of this section it is assumed in addition that $M = G/K$ is a symmetric space, in particular, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

The next lemma and proposition describe all positive-definite G-invariant polarizations on the tangent bundle TM to a symmetric space M in terms of smooth family of $\mathbb{C}$-linear operators on $\mathfrak{m}^{\mathbb{C}}$.

Lemma 6. *Let $\mathcal{F} = \Pi_*^{-1}(F)$, where F is a G-invariant positive-definite polarization on TM, $\mathcal{F} = \mathcal{K}^{\mathbb{C}} + \mathcal{P}$. Suppose that G/K is a symmetric space ($[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$). Then for every $w \in \mathfrak{m}$ there exists a unique nondegenerate $\mathbb{C}$-linear mapping $P_w : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ such that the subbundle $\mathcal{P}$ is generated by nowhere vanishing on $G \times \mathfrak{m}$ (left) G-invariant vector fields $\xi^L, \xi \in \mathfrak{m}$, where $\xi^L(g, w) = (\xi^l(g), iP_w(\xi))$. Moreover, $\mathrm{Ad}_{\mathfrak{m}} k \cdot P_w \cdot \mathrm{Ad}_{\mathfrak{m}} k^{-1} = P_{\mathrm{Ad}\, k(w)}$ for all $w \in \mathfrak{m}, k \in K$, where $\mathrm{Ad}_{\mathfrak{m}} k = \mathrm{Ad}\, k|\mathfrak{m}$.*

Proof. Define the real subspaces V_1, V_2 of $T_{(e,w)}(G \times \mathfrak{m})$ putting $V_1 = \{(\xi, 0), \xi \in \mathfrak{m} \subset T_e G\}$ and $V_2 = \{(0, u), u \in \mathfrak{m} = T_w \mathfrak{m}\}$. Note that $\mathcal{P}(e, w) \subset V_1^{\mathbb{C}} \oplus V_2^{\mathbb{C}}$ and the spaces $\mathcal{P}(e, w)$, $V_1^{\mathbb{C}}$, $V_2^{\mathbb{C}}$ have the same dimensions.

Now to prove the existence of P_w it suffices to prove that $\mathcal{P}(e, w) \cap V_k^{\mathbb{C}} = 0, k = 1, 2$. By (3) $\tilde{\Omega}(V_k, V_k) = 0, k = 1, 2$ ($[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} = \mathfrak{m}^{\perp}$). Therefore $\tilde{\Omega}(Z, \overline{Z}) = 0$ for all $Z \in \mathcal{P}(e, w) \cap V_k^{\mathbb{C}}$ and by Proposition 5 $Z \in (\mathcal{P} \cap \mathcal{K}^{\mathbb{C}})(e, w) = 0$.

By the definition of the right K-action the mapping r_{k*} takes each $(\xi^l(g), u)$ at (g, w) to $((\mathrm{Ad}\, k^{-1}(\xi))^l(gk), \mathrm{Ad}\, k^{-1}(u))$ at $(gk, \mathrm{Ad}\, k^{-1}(w))$, where $\xi \in \mathfrak{g}, w, u \in \mathfrak{m}, k \in K$. Then $\mathcal{P}$ is right K-invariant iff $\mathrm{Ad}_{\mathfrak{m}} k \cdot P_w \cdot \mathrm{Ad}_{\mathfrak{m}} k^{-1} = P_{\mathrm{Ad}\, k(w)}$. ∎

Let $W \subset \mathfrak{m}$ be an open $\mathrm{Ad}\, K$-invariant subset of $\mathfrak{m}$. Let $P : W \to GL(\mathfrak{m}^{\mathbb{C}})$, $w \mapsto P_w$ be a smooth map such that $\mathrm{Ad}_{\mathfrak{m}} k \cdot P_w \cdot \mathrm{Ad}_{\mathfrak{m}} k^{-1} = P_{\mathrm{Ad}\, k(w)}$ for all $w \in W, k \in K$. Denote by $\mathcal{K}_W$ the restriction $\mathcal{K}|(G \times W)$.

Definition 3. We will say that complex subbundles $\mathcal{F}$ of $T^{\mathbb{C}}(G \times W)$ and F of $T^{\mathbb{C}}(\Pi(G \times W))$ are defined by the map P if 1) $\mathcal{F} = \mathcal{K}_W^{\mathbb{C}} + \mathcal{P}$, where $\mathcal{P}$ is generated by nowhere vanishing on $G \times W$ (left) G-invariant vector fields $\xi^L, \xi \in \mathfrak{m}$, $\xi^L(g, w) = (\xi^l(g), iP_w(\xi))$; 2) $F = \Pi_*(\mathcal{F})$. Such (left G-invariant) subbundles $\mathcal{F}$ and F will be denoted by $\mathcal{F}(P)$ and $F(P)$ respectively.

We wish to study the complex structures $F(P)$, which are not necessarily positive-definite polarizations. Fix bases $\{X_\beta\}$ of $\mathfrak{k}$ and $\{W_b\}$ in $\mathfrak{m}$. Let $\{w_b\}$ be the coordinates in $\mathfrak{m}$ with respect to the basis $\{W_b\}$. For any vector-function $\tau : W \to \mathfrak{m}^{\mathbb{C}}, \tau = \sum_b \tau_b W_b$ by $\overrightarrow{\tau}$ we denote the vector field $\overrightarrow{\tau}(w) \overset{\mathrm{def}}{=} \sum_b \tau_b(w)\frac{\partial}{\partial w_b}$. We will say that a vector field $Y \in \mathit{\Gamma} T(G \times \mathfrak{m})$ is horizontal (resp. vertical) if $pr_{v*}(Y) = 0$ (resp. if $pr_{h*}(Y) = 0$), where $pr_v : G \times \mathfrak{m} \to \mathfrak{m}$ (resp. $pr_h : G \times \mathfrak{m} \to G$) is the usual projection. For any $\xi \in \mathfrak{m}$ denote by $P(\xi)$ the vector-function $P(\xi) : w \mapsto P_w(\xi)$.

Proposition 7. *Suppose that $M = G/K$ is a symmetric space. Let $\mathcal{F} = \mathcal{F}(P)$ be a complex subbundle of $T^{\mathbb{C}}(G \times W)$ defined by P. Then the subbundle $F = \Pi_*(\mathcal{F})$ is*

1) *involutive iff $[\xi^L, \eta^L] \subset \mathit{\Gamma}\mathcal{K}_W^{\mathbb{C}}$, i.e. the Lie bracket identities $\left[\overrightarrow{P(\xi)}, \overrightarrow{P(\eta)}\right] = -\overrightarrow{[w, [\xi, \eta]]}$ hold on W for all $\xi, \eta \in \mathfrak{m}$;*
2) *Lagrangian iff for each $w \in W$ P_w is symmetric with respect to the bilinear form $\langle , \rangle$ on $\mathfrak{m}$;*
3) *a complex structure iff 1) holds and the real part $\operatorname{Re} P_w$ of the linear mapping P_w is nondegenerate for all $w \in W$, i.e. $\mathcal{F} \cap \overline{\mathcal{F}} = \mathcal{K}_W^{\mathbb{C}}$.*

The subbundle $F(P)$ is a positive-definite polarization iff conditions 1), 2) hold and $\langle (\operatorname{Re} P_w)(\xi), \xi \rangle > 0$ for all $w \in W$ and all nonzero $\xi \in \mathfrak{m}$.

Proof. We will try to explain what the various items in this proposition mean. By Definition 3 the subbundle $\mathcal{P}$ is right K-invariant. Therefore the subbundle $\mathcal{K}_W$ (generated by this action of the Lie group K) preserves the subbundle $\mathcal{F}$: $[\zeta^L, \mathcal{P}] \subset \mathcal{P}$, $\zeta \in \mathfrak{k}$.

Let $\xi, \eta \in \mathfrak{m}$. Since the vector fields ξ^L, η^L are left G-invariant, their horizontal and vertical components are independent of $w \in W$ and $g \in G$ respectively: $\xi^L(g, w) = \sum_b \xi_b W_b^l(g) + i \sum_b \Big(P_w(\xi)\Big)_b \frac{\partial}{\partial w_b}$. Therefore the horizontal component of $[\xi^L, \eta^L]$ is the vector field $[\xi, \eta]^l$. On the other hand, any G-invariant section of $\mathcal{F}$ is defined by its horizontal component, so from the relations $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ we obtain that $[\xi^L, \eta^L]$ is a section of $\mathcal{F}(P)$ iff condition 1) of Proposition 7 holds.

Since $\mathcal{K}$ is the kernel of $\tilde{\Omega}$, the subbundle $F(P)$ is Lagrangian iff

$$\tilde{\Omega}_{(e,w)}\Big((\xi, iP_w(\xi)), (\eta, iP_w(\eta))\Big) = \langle \eta, iP_w(\xi) \rangle - \langle \xi, iP_w(\eta) \rangle - \langle w, [\xi, \eta] \rangle = 0.$$

But $\mathfrak{m} \perp [\mathfrak{m}, \mathfrak{m}]$, hence $\langle \xi, iP_w(\eta) \rangle = \langle \eta, iP_w(\xi) \rangle$, i.e. we derive 2). Using analogous arguments we obtain that

$$\begin{aligned}
&-i\tilde{\Omega}_{(e,w)}\Big((\xi+i\eta, iP_w(\xi+i\eta)), (\xi-i\eta, -i\overline{P}_w(\xi-i\eta))\Big)\\
&= i\langle\xi+i\eta, -i\overline{P}_w(\xi-i\eta)\rangle - i\langle\xi-i\eta, iP_w((\xi+i\eta)\rangle\\
&= \Big(\langle(\overline{P}_w+P_w)(\xi),\xi\rangle + \langle(\overline{P}_w+P_w)(\eta),\eta\rangle\Big)\\
&+ i\Big(\langle(\overline{P}_w-P_w)(\xi),\eta\rangle - \langle(\overline{P}_w-P_w)(\eta),\xi\rangle\Big).
\end{aligned}$$

By condition 2) of the proposition, the linear operator $\overline{P}_w - P_w$ is symmetric and, therefore, $F(P)$ is positive-definite iff $\langle(\operatorname{Re} P_w)(\xi),\xi\rangle > 0$ for all $\xi \neq 0$ from $\mathfrak{m}$ (see also Proposition 5). It is easy to check that condition 3) means the following: $\mathcal{F}+\overline{\mathcal{F}} = T^{\mathbb{C}}(G\times W)$. Thus $F+\overline{F} = T^{\mathbb{C}}(\Pi(G\times W))$. Now the proposition follows from Lemma 3 and Proposition 5. □

Remark 8. Since $\mathcal{P}$ is right K-invariant $[\zeta^L,\mathcal{P}]\subset\mathcal{P}$, $\zeta\in\mathfrak{k}$. If the Lie subgroup K is connected then $\mathcal{P}$ is right K-invariant iff $[\zeta^L,\mathcal{P}]\subset\mathcal{P}$ [Ga]. Therefore the condition $\operatorname{Ad}_{\mathfrak{m}} k\cdot P_w\cdot \operatorname{Ad}_{\mathfrak{m}} k^{-1} = P_{\operatorname{Ad} k(w)}$, $w\in W, k\in K$ with connected K in the definition of P can be replaced by the Lie bracket identities: $\overrightarrow{P([\zeta,\xi])} = \Big[\overrightarrow{[w,\zeta]}, \overrightarrow{P(\xi)}\Big]$, where $\zeta\in\mathfrak{k},\xi\in\mathfrak{m}$.

2.4. Invariant Kähler structures on punctured tangent bundles of symmetric spaces

In this subsection we suppose in addition that M is a semisimple Riemannian symmetric space, i.e. the connected Lie group G is semisimple and the form $\langle,\rangle$ on $\mathfrak{m}$ is positive definite (for appropriate choice of the constant $c : \langle,\rangle = c\Phi$).

Let X_H be the Hamiltonian vector field on $TM = T(G/K)$ of the Hamiltonian function $H : TM \to \mathbb{R}$, $H(gK,Y) = \mathbf{g}(Y,Y), Y\in T_{gK}(G/K)$ associated with the given metric $\mathbf{g}$. This vector field X_H ($\Omega(X_H,\cdot) = -dH$) defines the geodesic flow on TM. Here we will describe all Kähler structures F on $TM-M$ invariant under the Hamiltonian flow of the function $H^a, a\in\mathbb{R}$, i.e. such that $[X_{H^a},F]\subset F$. In this case the vector field X_{H^a} generates the local one-parameter group of (F-) biholomorphic mappings.

To simplify substantially the computation we will work on the trivial vector bundle $G\times\mathfrak{m}$ as in the previous subsection. To this end we consider on $G\times\mathfrak{m}$ the function $\tilde{H}$ and the vector field $\tilde{X}_H$ putting

$$\tilde{H}(g,w) = \langle w,w\rangle \qquad \text{and} \qquad \tilde{X}_H(g,w) = 2(w^l(g),0). \tag{5}$$

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It is immediate that $\tilde{H} = \Pi^* H$. Since the form $\langle , \rangle$ is $\operatorname{Ad} G$-invariant, by (3)

$$\tilde{\Omega}(\tilde{X}_H(g,w),(\eta^l(g),u)) = -\langle 2w,u\rangle - \langle w,[2w,\eta]\rangle = -2\langle w,u\rangle = -d\tilde{H}(\eta^l(g),u),$$

where $\eta \in \mathfrak{g}, u \in \mathfrak{m} = T_w\mathfrak{m}$. Thus $\Pi^*\Omega(\tilde{X}_H,\cdot) = -d\tilde{H} = -\Pi^* dH$. But Ω is nondegenerate and therefore $\Pi_*\tilde{X}_H$ is a well-defined vector field and $\Pi_*\tilde{X}_H = X_H$ by definition.

Lemma 9. *Let F be a polarization on TM, $\mathcal{F} = \Pi_*^{-1}(F)$. Let $\tilde{Z}, Z$ be Π-related vector fields on $G \times \mathfrak{m}$ and TM respectively, i.e. $\Pi_*(\tilde{Z}) = Z$. Then F is Z-invariant iff subbundle $\mathcal{F}$ is $\tilde{Z}$-invariant.*

Proof. The subbundle $\mathcal{F}$ locally is generated by smooth vector fields Y such that the images Π_*Y are vector fields which generate (locally) F (see the proof of Lemma 3). Now to prove the lemma we have to use the well known assertion about Π-related vector fields [He, Prop.3.3]: for any vector fields Y_1, Y_2 on $G \times \mathfrak{m}$ such that Π_*Y_1, Π_*Y_2 are (well defined) vector fields we have: $\Pi_*[Y_1,Y_2] = [\Pi_*Y_1, \Pi_*Y_2]$. ∎

Lemma 10. *Let $\mathcal{F} = \mathcal{F}(P) \subset T^{\mathbb{C}}(G \times W)$ be a G-invariant subbundle defined by the map P. Then the subbundle $F = \Pi_*(\mathcal{F})$ is invariant with respect to the Hamiltonian vector field of the function $H^a, a \in \mathbb{R}-\{0\}$ iff*

$$(6)\quad P_w^2(\xi) = -\operatorname{ad}_w^2(\xi) - 2(a-1)\frac{\langle w, P_w(\xi)\rangle}{\langle w,w\rangle}P_w(w) \quad \forall \xi \in \mathfrak{m}, \forall w \in W.$$

Moreover, if $[X_{H^a}, F] \subset F$ then 1) *$a = 1/2$ and $0 \notin W$;* 2) *G/K is a rank-one symmetric space;* 3) *for all $w \in W$: $P_w(w) = \lambda(w)w$, where $\lambda(w) \in \mathbb{C}$.*

Proof. By Lemma 9 it is sufficient to show that $[\tilde{X}_{H^a}, \mathcal{F}] \subset \mathcal{F}$, where $\tilde{X}_{H^a} = a\tilde{H}^{a-1}\tilde{X}_H$. Since the vector field $\tilde{X}_{H^a}$ is right K-invariant, $[\tilde{X}_{H^a}, \mathcal{K}] \subset \mathcal{K}$. Next, we have to calculate the commutators $Y = [\tilde{X}_{H^a}, \xi^L]$, $\xi \in \mathfrak{m}$. Using the notation of the proof of Proposition 7 and putting $\eta_b(w) = (P_w(\xi))_b$ we obtain:

$$\begin{aligned} Y &= \Big[2a\tilde{H}^{a-1}(w)w_bW_b^l(g), \xi_bW_b^l(g) + i\eta_b(w)\frac{\partial}{\partial w_b}\Big] \\ &= 2a\tilde{H}^{a-1}(w)\Big([w,\xi]_\beta X_\beta^l(g) - i\eta_b(w)W_b^l(g)\Big) \\ &\quad - 4ai(a-1)\tilde{H}^{a-2}(w)\langle w,\eta(w)\rangle w_bW_b^l(g), \end{aligned}$$

i.e. the commutator Y is a horizontal vector field. But any G-invariant section of $\mathcal{F}$ is defined by its horizontal part, and hence $(X_{\mathfrak{k}} + X_{\mathfrak{m}}, 0) \in \mathcal{F}(e, w) \subset \mathfrak{g}^{\mathbb{C}} \times \mathfrak{m}^{\mathbb{C}}$, where $X_{\mathfrak{k}} \in \mathfrak{k}^{\mathbb{C}}$, $X_{\mathfrak{m}} \in \mathfrak{m}^{\mathbb{C}}$, iff $[w, X_{\mathfrak{k}}] + iP_w(X_{\mathfrak{m}}) = 0$ (see Definition 3). Since in our case $Y(e, w) = 2a\langle w, w\rangle^{a-1}(X_{\mathfrak{k}} + X_{\mathfrak{m}}, 0)$ with $X_{\mathfrak{k}} = [w, \xi]$ and $X_{\mathfrak{m}} = -iP_w(\xi) - 2i(a-1)\langle w, w\rangle^{-1}\langle w, P_w(\xi)\rangle w$ we obtain equation (6). On the other hand, the operator $P_w : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}$ is nondegenerate. Therefore, the vector $P_w(w)$ and the number $\Delta = -2(a-1)\langle w, P_w(w)\rangle / \langle w, w\rangle$ are nonzero for all $w \in W - \{0\}$. Since $\mathrm{ad}_w(w) = 0$, the vector $P_w(w)$ is an eigenvector of P_w with the eigenvalue Δ. Moreover, from (6) for $\xi = P_w(w)$ it follows that $\Delta^2 P_w(w) = -\mathrm{ad}_w^2(P_w(w)) + \Delta^2 P_w(w)$, i.e. $\mathrm{ad}_w^2(P_w(w)) = 0$. But P_w is nondegenerate, therefore the kernels of $\mathrm{ad}_w^2 |\mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$ and $\mathrm{ad}_w |\mathfrak{m} : \mathfrak{m} \to \mathfrak{k}$ coincide with the one-dimensional space $\langle w\rangle$. Hence, 1) G/K is a rank-one symmetric space; 2) $P_w(w) = \lambda(w)w$, where $\lambda : W \to \mathbb{C}$ is a smooth function. It then follows immediately from (6) for $\xi = w \neq 0$ that $a = 1/2$. $0 \notin W$ because $\dim \mathfrak{m} \geq 2$. □

Remark 11. If the symmetric space G/K has rank one then any $\mathrm{Ad}\, K$-invariant function $\lambda(w)$ on W is a function of $\langle w, w\rangle$ [He], i.e. λ as a function on $G \times W$ is a function of $\tilde{H}$. Moreover, any G-invariant function f on $T(G/K)$ is defined uniquely by some $\mathrm{Ad}\, K$-invariant function on $\mathfrak{m}$, i.e. $f = f(H)$.

We wish to describe now all G and $X_{\sqrt{H}}$-invariant positive-definite polarizations on $T(G/K)$. Denote by $\mathfrak{m}^0$ the set of all nonzero elements of $\mathfrak{m}$ and by $T^0M \overset{\text{def}}{=} TM - \{\text{zero section}\}$ the punctured tangent bundle of M. Put $|w| = \sqrt{\langle w, w\rangle}$ for $w \in \mathfrak{m}$.

THEOREM 12. *Let $M = G/K$ be a rank-one semisimple Riemannian symmetric space. Assume that F is a G-invariant positive-definite polarization defined on the G-invariant open subset $\Pi(G \times W)$, $0 \notin W$ of TM. Let $P_w : \mathfrak{m}^{\mathbb{C}} \to \mathfrak{m}^{\mathbb{C}}, w \in W$ be the corresponding family of linear mappings. If F is invariant with respect to the Hamiltonian vector field of the function $\sqrt{H}$ then G/K has the compact type and*

$$P_w(\xi) = \sqrt{-\mathrm{ad}_w^2}\,(\xi) + \frac{\langle w, \xi\rangle}{\langle w, w\rangle}\lambda(w)w, \tag{7}$$

where $\lambda : W \to \mathbb{C}$, $\lambda(w) = \lambda(|w|)$ is a function with a positive real part.

Conversely, the complex subbundle $F = F(P)$, where P is determined by (7) *and G/K has the compact type, is a positive-definite polarization on $\Pi(G \times W)$.*

Proof. We continue with the notation of the proof of Lemma 10. Since the form $\langle , \rangle$ is $\operatorname{Ad} G$-invariant and positive-definite on $\mathfrak{m}$, the direct sum decomposition $\mathfrak{m} = \langle w \rangle \oplus \langle w \rangle^\perp$ is ad_w^2-invariant. This fact and (6) taken together with the latter assertion of Proposition 7 implies that for all $w \in W$ 1) the symmetric operator $(-\operatorname{ad}_w^2)$ has in $\langle w \rangle^\perp$ only positive eigenvalues, i.e. G/K is a symmetric space of the compact type [He]; 2) the real part of $\lambda(w)$ is positive. Hence $P_w = \sqrt{-\operatorname{ad}_w^2}$ on $\langle w \rangle^\perp$, where $\sqrt{-\operatorname{ad}_w^2}|\langle w \rangle^\perp$ is a unique positive definite square root of $(-\operatorname{ad}_w^2)|\langle w \rangle^\perp$.

It is easy to check that P_w satisfy the condition $\operatorname{Ad}_{\mathfrak{m}} k \cdot P_w \cdot \operatorname{Ad}_{\mathfrak{m}} k^{-1} = P_{\operatorname{Ad} k(w)}$ for all $w \in W, k \in K$ iff $\lambda(w)$ is a $\operatorname{Ad} K$-invariant function because $\operatorname{Ad} k \cdot \operatorname{ad}_w = \operatorname{ad}_{\operatorname{Ad} k(w)} \cdot \operatorname{Ad} k$.

Conversely, suppose that the complex subbundle $F = F(P)$ is defined by the mapping $P : \mathfrak{m}^0 \to GL(\mathfrak{m}^{\mathbb{C}})$ (7) and G/K is a rank-one symmetric space of the compact type. Taking into the account the proof of the first part of the theorem, it is sufficient to show only that the subbundle $\mathcal{F}$ is involutive. We prove this using the result [So, Ra1, FT]: for any rank-one symmetric space G/K of the compact type with a classical Lie group G, there exists the G-invariant Kähler structure (on the punctured tangent bundle $T^0(G/K)$) J_S which is invariant under Hamiltonian flow of the function $\sqrt{H}$. For the Cayley plane F_4/Spin_9 such Kähler structure J_S exists by virtue of [Sz1, Theorem 3.2] (see also Proposition 22 from the next section).

Let F^S be a corresponding complex subbundle of $(0,1)$ vectors of the complex structure J_S, $\mathcal{F}^S = \Pi_*^{-1}(F^S)$. By Lemma 6 and already proved first part of this theorem, $\mathcal{F}^S = \mathcal{F}^S(P^S)$, where P^S is defined by equation (7) with some $\operatorname{Ad} K$-invariant function $\lambda^S : \mathfrak{m}^0 \to \mathbb{C}$ with a positive real part. Let $\mathcal{K}_0 = \mathcal{K}|(G \times \mathfrak{m}^0)$.

Denote by $\mathcal{P}^S$ the subbundle of $\mathcal{F}^S$ defined as $\mathcal{P}$ for $\mathcal{F}$. Let $\mathcal{P}^H$ be the subbundle of $\mathcal{P}^S$ of all vectors from $\mathcal{P}^S$ tangent to level surfaces of $\tilde{H}$, in particular, $d\tilde{H}(\mathcal{P}^H) = 0$. It is evident that $\mathcal{P}^H$ has codimension one in $\mathcal{P}^S$ ($\tilde{H} = \Pi^* H : d\tilde{H}(\mathcal{K}) = 0$) and the subbundle $\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$ (of codimension one in $\mathcal{F}^S$) is involutive. Moreover, $\mathcal{P}^H$ is generated by the local G-invariant vector fields $(\xi^l(w), iP_w^S(\xi(w)))$, where $\xi(w) \perp w$ (the subspace $\langle w \rangle^\perp \subset \mathfrak{m}$ is invariant with respect to $P_w^S|\langle w \rangle^\perp = \sqrt{-\operatorname{ad}_w^2}|\langle w \rangle^\perp$). It then follows

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immediately from the equality $P^S_w|\langle w\rangle^\perp = P_w|\langle w\rangle^\perp$ that $\mathcal{P}^H \subset \mathcal{P}$, $(\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H) \subset \mathcal{F}$.

The subbundle $\mathcal{F}^S$ is generated by $\mathcal{K}_0^{\mathbb{C}}+\mathcal{P}^H$ and the global nowhere vanishing on $G\times\mathfrak{m}^0$ left G-invariant vector field $Y^S = Y^S(\lambda^S) \in \Gamma\mathcal{P}^S, Y^S(g,w) = \tilde{H}^{-1/2}(w)(w^l(g), i\lambda^S(w)w)$ (recall that $P^S_w(w) = \lambda^S(w)w$). $\mathcal{F}^S$ is involutive iff $[Y^S, \mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H] \subset \mathcal{F}^S$. But

$$\begin{aligned} Y^S(g,w) &= \tilde{H}^{-1/2}(w)(w^l(g),0) + \tilde{H}^{-1/2}(w)(0, i\lambda^S(w)w) \\ &= \frac{1}{2}\tilde{H}^{-1/2}\tilde{X}_H + \tilde{H}^{-1/2}(w)(0, i\lambda^S(w)w). \end{aligned}$$

By Lemma 9 the subbundle $\mathcal{F}^S$ and, consequently, $\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$ are preserved by the vector field $\frac{1}{2}\tilde{H}^{-1/2}\tilde{X}_H$ ($d\tilde{H}(\tilde{X}_H) = 0$ and $\Pi_*(\frac{1}{2}\tilde{H}^{-1/2}\tilde{X}_H) = X_{\sqrt{H}}$). Therefore $\mathcal{F}^S$ is involutive iff $[Z, \mathcal{K}_0^{\mathbb{C}}+\mathcal{P}^H] \subset \mathcal{F}^S$, where $Y^S = \frac{1}{2}|w|^{-1}\tilde{X}_H + i|w|^{-1}\lambda^S(w)Z$, $Z(g,w) = (0,w)$ (by Remark 11 λ^S is constant along sections of $\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$). But for any section X of $\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$

$$d\tilde{H}([Z,X]) = Zd\tilde{H}(X) - Xd\tilde{H}(Z) = -X(2\tilde{H}) = 0$$

and therefore $[Z, \mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H] \subset \mathcal{F}^S \cap \ker d\tilde{H} = \mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$. Thus $\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$ is preserved by the vector field Z independent of λ^S. Hence $\mathcal{K}_0^{\mathbb{C}} + \mathcal{P}^H$ is invariant with respect to the vector field $Y(\lambda)$ from $\mathcal{F}$, i.e. $\mathcal{F}$ is involutive.

Remark 13. The formula similar to (7) with $\lambda(w) = \alpha|w|$, $\alpha \in \mathbb{R}^+$ for the structure J_S was found by Szőke in [Sz1] using the limit arguments. The Kähler structure (7) with $\lambda(w) = \alpha|w|$, $\alpha \in \mathbb{R}^+$ for a classical rank-one symmetric space of the compact type is described in [IM] using a Riemannian geometric method. The polarization $\Pi_*(\mathcal{P}^H) + \langle X_H\rangle$ on the punctured tangent bundle T^0S^n to the sphere was used by Ii [Ii] for the geometric quantization of the geodesic flow. In [PM, Chapter 2] this polarization was obtained using a one-parameter family of polarizations on $T\mathbb{R}^{n+1}$ (invariant with respect to Hamiltonian flows of $n+1$-dimensional harmonic oscillators).

Denote by F^λ the positive-definite polarization on $\Pi(G\times W) \subset T^0M$, where $M = G/K$ is a rank-one symmetric space of the compact type, determined by some function $\lambda = \lambda(|w|)$ (7). By the theorem above, any positive-definite polarization F on some G-invariant open subset D of T^0M has the form F^λ with the function $\lambda : W \to \mathbb{C}, \lambda = \lambda(|w|)$, where $G\times W = \Pi^{-1}(D)$.

As was observed by Rawnsley [Ra1] and Furutani-Tanaka [FT] for the Kähler structure J_S on T^0M, where $M = G/K$ is a classical rank-one symmetric space of the compact type, the strictly plurisubharmonic function $2\sqrt{H}$ satisfies the condition $\Omega = -i\partial\overline{\partial}2\sqrt{H}$. Moreover, the one-form $\operatorname{Im}\overline{\partial}2\sqrt{H}$ is the canonical one-form θ. The question arises: what Kähler structures F^λ admit such a function?

Proposition 14. *The Kähler structure F^λ on G-invariant open subset D of $T^0(G/K)$ admits a function $Q = q \circ \sqrt{H}$ such that $-i\partial\overline{\partial}Q = \Omega$, where*

$$q(r) = \int r\frac{\lambda(r) + \overline{\lambda}(r)}{|\lambda|^2(r)}dr. \tag{8}$$

This function is a unique G-invariant function with this property (up to a constant of integration) if $W(D)$ is connected. Moreover, if λ is a real function then $\operatorname{Im}\overline{\partial}Q = \theta$.

Proof. By definition $\overline{\partial}Q|F^\lambda = dQ|F^\lambda$ and $\overline{\partial}Q|\overline{F^\lambda} = 0$. Denote by $\tilde{\Delta}$ the one-form $\Pi^*(\overline{\partial}Q)$ on $G \times W$. Then for any $\xi \in \mathfrak{m}$, $\zeta \in \mathfrak{k}$:

$$\tilde{\Delta}_{(g,w)}(\xi^l(g), -i\overline{P}_w(\xi)) = 0, \qquad \tilde{\Delta}_{(g,w)}(\zeta^l(g), [w,\zeta]) = 0$$
$$\text{and}\quad \tilde{\Delta}_{(g,w)}(\xi^l(g), iP_w(\xi)) = |w|^{-1}q'\langle w, iP_w(\xi)\rangle = i|w|^{-1}q'\lambda\langle w,\xi\rangle$$

because the operator P_w is symmetric with respect to $\langle,\rangle$ (here q' and λ denote the derivative $q'(|w|)$ and the number $\lambda(|w|)$). Now using definition (7) of P_w, the invariance of the space $\langle w\rangle^\perp$ with respect to P_w we obtain that $\tilde{\Delta}_{(e,w)}(\xi_0, 0) = 0$, $\tilde{\Delta}_{(e,w)}(0, \xi_0) = 0$ for all $\xi_0 \in \langle w\rangle^\perp \subset \mathfrak{m}$; $\tilde{\Delta}_{(e,w)}(\zeta, 0) = 0, \zeta \in \mathfrak{k}$ because $[w,\zeta]\perp w$ and

$$\tilde{\Delta}_{(e,w)}(0,w) = |w|^{-1}q'\frac{\lambda}{\lambda+\overline{\lambda}}\langle w,w\rangle, \qquad \tilde{\Delta}_{(e,w)}(w, i\lambda w) = i|w|^{-1}q'\lambda\langle w,w\rangle.$$

Therefore for any $\eta \in \mathfrak{g}, u \in \mathfrak{m}$ we have

$$\begin{aligned}\tilde{\Delta}_{(e,w)}(\eta,u) &= \frac{q'}{|w|(\lambda+\overline{\lambda})}\Big(i|\lambda|^2\langle w,\eta\rangle + \lambda\langle w,u\rangle\Big)\\ &= \frac{q'}{|w|(\lambda+\overline{\lambda})}\Big(i|\lambda|^2\tilde{\theta} + \frac{1}{2}\lambda d\tilde{H}\Big)_{(e,w)}(\eta,u)\end{aligned}$$

(see definitions (2) and (5)). Since the 2-forms $\tilde{\Omega} = d\tilde{\theta}$ and $d\tilde{H}\wedge\tilde{\theta}$ are linearly independent on $G \times W$ ($\dim\mathfrak{m} \geq 2$), the function $Q, Q = q\circ\sqrt{H}$ satisfies the condition $\Omega = -id\overline{\partial}Q$ iff $q'|\lambda|^2/(|w|(\lambda+\overline{\lambda})) \equiv 1$. In this case if λ is a real function then $\operatorname{Im}\overline{\partial}Q = \theta$. ∎

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We denote by $\sigma : T(G/K) \to T(G/K)$ the involution which maps any tangent vector Y at gK onto $-Y$ at gK. It is evident that $\sigma_*(F^\lambda) = \overline{F^\lambda}$ iff $P_w^\lambda = \overline{P_{-w}^\lambda}$, i.e. if λ is a real function.

THEOREM 15. *Let $M = G/K$ be a rank-one symmetric space of the compact type and let $q : \mathbb{R}^+ \to \mathbb{R}$ be a smooth strictly increasing function. For the class $[q + C]_{C\in\mathbb{R}}$ of functions there exists a unique G-invariant positive-definite polarization $F = F(q)$ on the punctured tangent bundle T^0M such that*

1) *F is invariant with respect to the Hamiltonian vector field $X_{\sqrt{H}}$ of $\sqrt{H}$;*
2) *the one-form $\operatorname{Im}\overline{\partial}Q$, with $Q = (q \circ \sqrt{H})$, is the canonical one-form θ;*
3) *σ is an antiholomorphic involution.*

Moreover, for this polarization $F(q)$: a) *$F = F^\lambda$, where $\lambda(w) = 2|w|/q'(|w|)$, $w \in \mathfrak{m}^0$;* b) *$\sqrt{Q}$ satisfies the homogeneous complex Monge-Ampere equation on T^0M iff $q(r) = c_0r^2$, $c_0 > 0$, i.e. $Q = c_0H$.*

Proof. This theorem summarizes results which have already been proved with the exception of the assertion b). To prove b) we note that $\operatorname{Im}\overline{\partial}Q$ is the canonical one-form θ on T^0M and, consequently, by [GSt, (5.2) and (5.5)] $\sqrt{Q}$ satisfies the homogeneous complex Monge-Ampere equation iff Q is a homogeneous polynomial of degree 2 in the impulse-coordinates on $TM \simeq T^*M$. ∎

Remark 16. Since the entire construction and the proof is compatible with taking quotients with respect to a discrete group of G, the results of subsections 2.3 and 2.4 are valid for such quotients as well.

Remark 17. The Kähler structure J_S on T^0M with the complex subbundle of $(0,1)$ vectors F^S 1) is invariant with respect to $X_{\sqrt{H}}$; for this structure 2) $\Omega = -i\partial\overline{\partial}2\sqrt{H}$; 3) $\sigma_*(F^S) = \overline{F^S}$ [Ra1, FT]. Therefore, $F^S = F^\lambda$, where $\lambda(w) = |w|$ (for some constant c which defines $\langle,\rangle$). For example, if $G/K = SO(n+1)/SO(n)$ then $\operatorname{ad}_w^2(\xi) = -\Big(\langle\langle w,w\rangle\rangle\xi - \langle\langle w,\xi\rangle\rangle w\Big)$ and $\sqrt{-\operatorname{ad}_w^2(\xi)} = \|w\|\xi - \frac{\langle\langle w,\xi\rangle\rangle}{\|w\|}w$, where $w,\xi \in \mathfrak{m}^0$, $\langle\langle w,\xi\rangle\rangle = -\frac{1}{2}\operatorname{Tr} w\xi$ (the normalized trace form associated with the faithful standard representation). By [IM] J_S is defined by the operator $P_w(\xi) = \alpha|w|\xi$ for some $\alpha \in \mathbb{R}$. Now from (7) we find that $P_w(\xi) = \|w\|\xi$ and $\lambda(w) = \lambda^S(w) = \|w\|$. In the same manner we can consider the symmetric space $G/K = SU(n+1)/S(U(1) \times U(n))$ and using the commutator formula $\operatorname{ad}_w^2(\xi) =$

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$-\Big(\langle\!\langle w,w\rangle\!\rangle\xi - \langle\!\langle w,\xi\rangle\!\rangle w + 3\langle\!\langle Jw,\xi\rangle\!\rangle Jw\Big)$ and $\langle\!\langle Jw,Jw\rangle\!\rangle = \langle\!\langle w,w\rangle\!\rangle$ obtain that $\sqrt{-\operatorname{ad}_w^2}(\xi) = \|w\|\xi - \frac{\langle\!\langle w,\xi\rangle\!\rangle}{\|w\|}w + \frac{\langle\!\langle Jw,\xi\rangle\!\rangle}{\|w\|}Jw$, where $\langle\!\langle w,\xi\rangle\!\rangle = -\frac{1}{2}\operatorname{Tr} w\xi$ ($w,\xi \in \mathfrak{m}^0 \subset su(n{+}1)$) and $J : \mathfrak{m} \to \mathfrak{m}$ determines the complex structure on G/K. By [IM] J_S is defined by the operator $P_w(\xi) = \alpha\left(|w|\xi + \frac{\langle Jw,\xi\rangle}{|w|}Jw\right)$ for some $\alpha \in \mathbb{R}$. Thus, by (7) $\lambda(w) = \lambda^S(w) = \|w\|$. Similarly (see [IM]), for the homogeneous space $G/K = Sp(n+1)/(Sp(1)\times Sp(n))$ (the quaternion Kähler manifold) the complex structure J_S is defined by $\lambda(w) = \lambda^S(w) = \|w\|$, where $\langle\!\langle w,\xi\rangle\!\rangle = -\frac{1}{2}\operatorname{Tr} w\xi$ ($w,\xi \in sp(n+1) \subset \operatorname{End}(\mathbb{H}^{n+1})$).

§3. G-invariant metric compatible complex structures on $\mathbf{T(G/K)}$

3.1. The main lemma

We continue with the notation of section 2 but throughout this section, unless otherwise indicated, it is assumed that G is a real reductive connected Lie group and K its (closed) reductive subgroup. Suppose that the form $\langle,\rangle$ on the Lie algebra $\mathfrak{g}$ defines the G-invariant Riemannian metric $\mathbf{g}$ on $M = G/K$. Since $\langle,\rangle = c\Phi$ (c is a nonzero constant), an arbitrary geodesic $\gamma : \mathbb{R} \to G/K$ through $gK \in G/K$ can be written as $g\exp(t\xi)K$ for some $\xi \in \mathfrak{m}$. For the geodesic γ we can define a map $\hat\gamma : \mathbb{C} \to T(G/K)$, $(x{+}iy) \mapsto y\dot\gamma(x)$.

Definition 4. [DSz] Let F_A be a complex structure on some domain $D \subset T(G/K)$. We will say that the complex structure F_A on D is *metric compatible* if for every geodesic γ in G/K the map $\hat\gamma$ is holomorphic on $\hat\gamma^{-1}(D)$. If the domain D containes the zero section G/K of $T(G/K)$ such complex structure is called *adapted*.

For a Riemannian manifold G/K on some G-invariant domain $D \subset T(G/K)$ which contains the neighborhood of the zero-section G/K there exists an adapted structure F_A and this structure on D is unique (metric compatible structures are not uniquelly defined by the metric as we will show below). F_A is G-invariant. Moreover, F_A has the following additional properties [GSt, Sz2, Sz3, DSz]:

1) $\sigma : D^\sigma \to D^\sigma$, where $D^\sigma \overset{\text{def}}{=} D\cap\sigma(D)$, is an antiholomorphic involution;
2) the one-form $\operatorname{Im}\overline{\partial}H$ is the canonical one-form θ;
3) F_A is a Kähler structure with respect to the canonical symplectic form $\Omega = d\theta$;
4) the function $\sqrt{H}$ satisfies the homogeneous complex Monge-Ampere equation on $D{-}G/K$;

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5) if the Lie group G is compact then the structure F_A is defined on the whole space $T(G/K)$ (i.e. $D = T(G/K)$) and the complexification $G^{\mathbb{C}}$ of G acts on $T(G/K)$ by biholomorphic transformations.

Note. For 2) it is important that $T^*(G/K)$ and $T(G/K)$ are identified using the metric $\mathbf{g}$.

Lemma 18. *Let F be a complex structure on some G-invariant domain $D \subset T(G/K)$ and $\mathcal{F} = \Pi_*^{-1}(F)$ (the integrable subbundle of $T^{\mathbb{C}}(G \times W)$). The complex structure F is metric compatible on D iff the (left) G-invariant and right K-invariant vector field $A^L, A^L(g,w) = (w^l(g), iw)$ is a section of $\mathcal{F}$.*

Proof. Let $\xi \in \mathfrak{m}$. Since $g \exp(x+ty)\xi K = g \exp x\xi \exp ty\xi K$, the mapping $\hat{\gamma} : x + iy \mapsto (d/dt)_0 g \exp(x+ty)\xi K$ may be considered as the restriction of a composition of the two maps $\tilde{\gamma} : \mathbb{C} \to G \times \mathfrak{m}$, $x + iy \mapsto (g \exp x\xi, y\xi)$ and $\Pi : G \times \mathfrak{m} \to T(G/K)$. By definition the mapping $\hat{\gamma}$ is holomorphic if the vector $\hat{\gamma}_*(\partial/\partial x + i\partial/\partial y)$ belongs to F or, equivalently, iff $\tilde{\gamma}_*(\partial/\partial x + i\partial/\partial y)$ belongs to $\mathcal{F}$. Now the assertion of the lemma follows from the simple equalities:

$$\tilde{\gamma}_*(\partial/\partial x)(x,y) = (\xi^l(g \exp x\xi), 0) \in T_{g \exp x\xi}G \times T_{y\xi}\mathfrak{m}$$

and

$$\tilde{\gamma}_*(\partial/\partial y)(x,y) = (0, \xi) \in T_{g \exp x\xi}G \times T_{y\xi}\mathfrak{m}.$$

Proposition 19. *Let $M = G/K$ be a rank-one symmetric space of the compact type. Then there exists a unique G-invariant metric compatible structure F on the punctured tangent bundle T^0M such that F is invariant with respect to the Hamiltonian vector field $X_{\sqrt{H}}$ of $\sqrt{H}$. This structure coincides with the structure F^λ, where $\lambda \equiv 1$ on $\mathfrak{m}^0$.*

Moreover, for this structure 1) $\Omega = -i\partial\overline{\partial}H$; 2) $\operatorname{Im}\overline{\partial}H$ *is the standard canonical one-form* θ; 3) $\sqrt{H}$ *satisfies the homogeneous complex Monge-Ampere equation on* T^0M.

Proof. The proof follows immediately from Lemma 18 and Theorem 15 because $P_w(w) = w$ iff $q(r) = r^2 + C$. ∎

Note. The metric compatible complex structure F^1 on T^0M cannot be extended to TM to make it adapted. In fact F^1 is different from the complex structure that is adapted to the G-invariant Riemannian metric $\mathbf{g}$ on $M = G/K$. This can be read off for example by looking at the corresponding J tensors. The J tensor of F^1 can be calculated from formula (7) in Theorem 12 and for the J tensor of the adapted (to the Riemannian metric $\mathbf{g}$) complex structure see [Sz1].

3.2. Adapted structures on $T(G/K)$

The all facts in this subsection are no doubt known [Sz2, IM]. But our approch, which we will use in the next subsections, is new. In this subsection G is a compact connected Lie group.

Let $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ be the complexifications of the (algebraic) Lie groups G and $K \subset G$ respectively. In particular, K is a maximal compact subgroup in the (algebraic) Lie group $K^{\mathbb{C}}$ and the intersection of K with each connected component of $K^{\mathbb{C}}$ is not empty [On, Ch.5]. The group $G^{\mathbb{C}}$ considered as a real Lie group we denote $G^{\mathbb{R}}$. The canonical complex structure F_c on $G^{\mathbb{R}}$ is defined by left $G^{\mathbb{R}}$-invariant $(0,1)$ vector fields $\xi^l + i(I\xi)^l$, where $\xi \in \mathfrak{g}$ and I is a complex structure on $\mathfrak{g}^{\mathbb{C}}$. Fix a positive-definite form $\langle , \rangle = c\Phi$ on $\mathfrak{g}$.

Since G and K are maximal compact Lie subgroups of $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ respectively, by the Mostow's result [Mo1, Th. 4] topologically $K^{\mathbb{C}} = K\exp(i\mathfrak{k})$ and $G^{\mathbb{C}} = G\exp(i\mathfrak{m})\exp(i\mathfrak{k})$ ($[i\mathfrak{k}, [i\mathfrak{k}, i\mathfrak{k}]] \subset i\mathfrak{k}$), i.e. the mappings

$$
\begin{aligned}
(9) \qquad & G \times \mathfrak{m} \times \mathfrak{k} \to G^{\mathbb{C}}, (g, w, \zeta) \mapsto g\exp(iw)\exp(i\zeta), \\
& K \times \mathfrak{k} \to K^{\mathbb{C}}, (k, \zeta) \mapsto k \cdot \exp(i\zeta)
\end{aligned}
$$

are diffeomorphisms. Then the mapping

$$
(10) \qquad G \times_K \mathfrak{m} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \qquad [(g, w)] \mapsto g\exp(iw)K^{\mathbb{C}}
$$

is a G-equivariant diffeomorphism [Mo2, Lemma 4.1]. This mapping supplies manifold $T(G/K)$ with the complex structure F_c^K.

LEMMA 20. *Let G be a compact Lie group, F_c be the two-sided invariant complex structure on $TG \simeq G \times \mathfrak{g}$ induced by the complex structure on $G^{\mathbb{C}}$. Then F_c is generated by the left G-invariant vector fields ξ_c^L, $\xi \in \mathfrak{g}$,*

$$
(11) \qquad \xi_c^L(g, w) = \left(\xi^l(g), \frac{z(\sin z + i\cos z)}{(\sin z + i\cos z) - i}(\xi) \right), \text{ where } z = \mathrm{ad}_w .
$$

Proof. We have to calculate the image of the $G^{\mathbb{R}}$-invariant vector field $\xi^l + i(I\xi)^l$, $\xi \in \mathfrak{g}$ under the diffeomorphism $\phi : G^{\mathbb{C}} \to G \times \mathfrak{g}$, $g\exp(iw) \mapsto (g, w)$. By (9) $\exp(iw)\exp(t\xi) = g(t)\exp(iv(t))$, where $g_0 = e$, $v_0 = w$. Then from

$$
\exp t\xi = \Big(\exp(-iw)g(t)\exp iw\Big)\Big(\exp(-iw)\exp iv(t)\Big)
$$

we obtain two real equations $\xi = e^{-i\,\mathrm{ad}_w}(g_0') + \dfrac{1-e^{-i\,\mathrm{ad}_w}}{\mathrm{ad}_w}(v_0')$ for the tangent vectors $g_0', v_0' \in \mathfrak{g}$.

Replacing $\xi \mapsto i\xi$ we obtain an analogous equation for $i\xi$. It remains to find the solutions of these simple equations and rewrite them in the form (11). □

Since the vector field $A^L, A^L(g,w) = (w^l(g), iw)$ is a section of F_c, F_c is an adapted structure. To prove that this structure is a positive-definite polarization we have to consider the Lie group G with the two-side invariant metric $\mathbf{g}$ as a symmetric space. Then we obtain on $\mathfrak{m}_G = \{(w,-w) \in \mathfrak{g}\times\mathfrak{g}\}$ the family of positive-definite operators (with respect to the form $\langle,\rangle$)

$$P_w(\xi,-\xi) = \left(\frac{\mathrm{ad}_w\cos(\mathrm{ad}_w)}{\sin(\mathrm{ad}_w)}(\xi), -\frac{\mathrm{ad}_w\cos(\mathrm{ad}_w)}{\sin(\mathrm{ad}_w)}(\xi)\right),$$

(The submersion $\Pi : G\times G\times\mathfrak{m}_G \to G\times\mathfrak{g}$ has the form $(g_1,g_2,(w,-w)) \mapsto (g_1g_2^{-1}, 2\,\mathrm{Ad}_{g_2} w))$. Now by Lemma 18 and Proposition 7 we get

Corollary 20.1. [Sz2] *F_c is a positive-definite polarization and the adapted structure on TG.*

Because of the evident relation $\exp(iw)\exp w = \exp w \exp(iw)$, $w \in \mathfrak{m}$ on the complex Lie group $G^{\mathbb{C}}$ we can conclude that the vector field $A^L, A^L(g,w) = (w^l(g), iw)$ is a section of $\Pi_*^{-1}(F_c^K)$, i.e. by Lemma 18 F_c^K is an adapted structure.

Corollary 20.2. [Sz2] *F_c^K is an adapted structure defined on the whole $T(G/K)$.*

We can rewrite the vector fields (11) in a more convenient form. It is clear that the subbundle F_c on $G\times\mathfrak{g}$ is generated by left G-invariant vector fields

$$\widehat{\xi}_c(g,w) = \left(\left(\frac{(\sin z + i\cos z) - i}{z(\sin z + i\cos z)}(\xi)\right)^l(g), \xi\right), \tag{12}$$

where $\xi, w \in \mathfrak{g}$, $z = \mathrm{ad}_w$.

Proposition 21. *Let G be a compact connected Lie group. Let $\mathcal{F}^K$ be the subbundle on $G\times\mathfrak{m}\subset G\times\mathfrak{g}$ generated by $\mathcal{K}^{\mathbb{C}}$ and vector fields (12), where $\xi\in\mathfrak{m}$. Then the subbundle $F^K = \Pi_*(\mathcal{F}^K)$ determines a Kähler structure on $T(G/K)$. This structure is adapted and $F^K = F_c^K$.*

We will prove the proposition in the next section using the reduction [Ag]. Here note only that the case $K = \{e\}$ is already proved.

If $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ formula (12) can be simplified:

COROLLARY 21.1. *If* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ *then* $\mathcal{F}^K = \mathcal{K}^{\mathbb{C}} + \mathcal{P}^K$, *where* $\mathcal{P}^K$ *is generated by the left* G*-invariant vector fields* $\xi^L, \xi \in \mathfrak{m}$ *on* $G \times \mathfrak{m}$:

$$(13)\quad \xi^L(g, w) = (\xi^l(g), iP_w^K(\xi)),\ \textit{where}\ P_w^K(\xi) = \frac{z \cos z}{\sin z}(\xi),\ z = \mathrm{ad}_w\,.$$

Note that the formula similar to (13) was obtained in [DSz] in terms of the Jacobi operator and the curvature tensor, and in [IM] by means of horizontal and vertical lifts of vector fields for the rank–one classical symmetric spaces of the compact type.

3.3. Invariant structures on $T^0(G/K)$

Let $M = G/K$ be a rank-one symmetric space of the compact type. Following [Sz1] we consider the family Ψ_ε^α of (left) G-equivariant and (right) K-equivariant diffeomorphisms

$$(14)\quad \Psi_\varepsilon^\alpha : G \times \mathfrak{m}^0 \to G \times \mathfrak{m}^0, \quad \Psi_\varepsilon^\alpha(g, w) = \Big(g, \varepsilon \frac{\exp|\alpha w|}{|\alpha w|} w\Big),\ \alpha \in \mathbb{R}^+.$$

We will prove that the following limit subbundle $\mathcal{F}''_\alpha = \lim_{\varepsilon \to 0} \Psi^\alpha_{\varepsilon *} \mathcal{F}^K$ on $G \times \mathfrak{m}^0$ exists and $\mathcal{F}''_\alpha = \mathcal{F}(P^\alpha)$ (7) with P^α defined by the function $\lambda(w) = \alpha|w|$. It is easy to check that $(\Psi^\alpha_{\varepsilon *}\xi^L)(g, w) = (\xi^l(g), iP_w^{\alpha,\varepsilon}(\xi))$ for all $\xi \in \mathfrak{m}$, $|w| > \varepsilon/\alpha$, where $P_w^{\alpha,\varepsilon} : \mathfrak{m} \to \mathfrak{m}$ are the linear operators given by

$$P_w^{\alpha,\varepsilon}(\xi) = \frac{1}{\delta_\varepsilon(|w|)} \cdot P^K_{\delta_\varepsilon(|w|)w}(\xi) + \Big(1 - \frac{1}{|\alpha w|\delta_\varepsilon(|w|)}\Big) \frac{\langle w, \xi\rangle}{|w|}\alpha w$$

with $\delta_\varepsilon(|w|) = \ln(|\varepsilon^{-1}\alpha w|)/|\alpha w| > 0$. Now we obtained that

$$\lim_{\varepsilon \to 0}(\Psi^\alpha_{\varepsilon *}\xi^L)(g, w) = (\xi^l(g), i\Big(\sqrt{-\,\mathrm{ad}_w^2}\,(\xi) + \frac{\langle w, \xi\rangle}{|w|}\alpha w\Big))$$

because $P_w^K(\xi) = \dfrac{\mathrm{ad}_w \cos(\mathrm{ad}_w)}{\sin(\mathrm{ad}_w)}(\xi)$. Since all subbundles $\Psi^\alpha_{\varepsilon *}\mathcal{F}^K$ are involutive, by Proposition 7: $\Big[\overrightarrow{P^{\alpha,\varepsilon}(\xi)}, \overrightarrow{P^{\alpha,\varepsilon}(\eta)}\Big](w) = -[w, [\xi, \eta]]$. To prove that the subbundle $\mathcal{F}(P^\alpha)$ is also involutive remark that $\mathrm{ad}_w \cot(\mathrm{ad}_w) = \widehat{w} \coth \widehat{w}$, where $\widehat{w} = \sqrt{-\,\mathrm{ad}_w^2}$. Hence for $w_t^\varepsilon = w + t \cdot a^\varepsilon w$, $a^\varepsilon \in \mathbb{R}$ we have

$$\frac{d}{dt}\Big|_{t=0} \coth(\delta_\varepsilon(|w_t^\varepsilon|)\widehat{w_t^\varepsilon}) = -a^\varepsilon \cdot \sinh^{-2}(\delta_\varepsilon(|w|)\widehat{w}) \cdot O(\delta_\varepsilon(|w|)) \quad \text{as} \quad \varepsilon \to 0.$$

On the other hand, if $w_t^\varepsilon = \operatorname{Ad} k_t^\varepsilon(w)$, where $k_t^\varepsilon = \exp t\zeta^\varepsilon$, $\zeta^\varepsilon \in \mathfrak{k}$, then

$$\frac{d}{dt}\Big|_{t=0} \operatorname{Ad} k_t^\varepsilon \coth(\delta_\varepsilon(|w|)\widehat{w}) \operatorname{Ad}(k_t^\varepsilon)^{-1} = [\operatorname{ad}_{\zeta^\varepsilon}, \coth(\delta_\varepsilon(|w|)\widehat{w})].$$

So that $\lim\limits_{\varepsilon\to 0}\left(\frac{d}{dt}\Big|_{t=0} P^{\alpha,\varepsilon}_{w_t^\varepsilon}\right) = \frac{d}{dt}\Big|_{t=0}\left(\lim\limits_{\varepsilon\to 0} P^{\alpha,\varepsilon}_{w_t^\varepsilon}\right)$ for any smooth curve $w_t^\varepsilon = w + tP_w^{\alpha,\varepsilon}(\xi)$ $\left(\delta'_\varepsilon(r) = O(\delta_\varepsilon(r)) \text{ as } \varepsilon \to 0,\ \langle w\rangle \oplus \operatorname{ad}_w(\mathfrak{k}) = \mathfrak{m}\right)$. Therefore $\left[\overrightarrow{P^\alpha(\xi)}, \overrightarrow{P^\alpha(\eta)}\right](w) = -[w,[\xi,\eta]]$ on $\mathfrak{m}^0$.

PROPOSITION 22. *The limit subbundle* $\mathcal{F}''_\alpha = \lim_{\varepsilon\to 0}\Psi^\alpha_{\varepsilon *}\mathcal{F}^K$ *on* $G\times\mathfrak{m}$ *coincides with* $\mathcal{F}(P^\alpha)$, *where* $P^\alpha_w(\xi) = \sqrt{-\operatorname{ad}_w^2}(\xi) + \dfrac{\langle w,\xi\rangle}{|w|}\alpha w$. *This subbundle* $\mathcal{F}(P^\alpha)$ *is involutive.*

Remark 23. In [Sz1] it is proved that the limit structure $\Pi_*(\mathcal{F}''_1)$ 1) is an $X_{\sqrt{H}}$-invariant complex structure 2) for classical rank-one symmetric spaces coincides with the Kähler structure J_S [So, Ra1, FT] (for an appropriate metric $\langle,\rangle$).

§4. The reduction

4.1. The reduction and polarizations

In this subsection we shall give an exposition of the results by Gotay and Guillemin-Sternberg [Go, GS] modified and developed to our needs. Suppose that X is the cotangent bundle T^*N of a manifold N. Let $p : T^*N \to N$ be the canonical projection. Denote by G and S real reductive connected Lie groups which act on N and suppose that these actions commute. The actions of G, S on N naturally extend to the actions of G, S on T^*N. These actions on T^*N are symplectic since they preserve the canonical 1-form θ and thus also the symplectic 2-form $\Omega = d\theta$. For each vector ξ belonging to the Lie algebra $\mathfrak{s}$ of S the 1-parameter subgroup $\exp t\xi$ induces the Hamiltonian vector field $\hat{\xi}$ on T^*N with the Hamiltonian function $f_\xi = \theta(\hat{\xi})$: $df_\xi = -\hat{\xi}\rfloor d\theta$. Hence the action of S on $X = T^*N$ is Hamiltonian (or Poisson) [Go, GS] and therefore defines the moment map $\mathbf{J} : X \to \mathfrak{s}^*$ from X to the dual space of the Lie algebra $\mathfrak{s}$ by $\mathbf{J}(x)(\xi) = f_\xi(x)$. The map $\mathbf{J}$ is S-equivariant, i.e. intertwines the action of S on X and the co-adjoint action of S on $\mathfrak{s}^*$.

Suppose that the action of S on N is free and proper. Then every $\mu \in \mathfrak{s}^*$ is a regular value of $\mathbf{J}$ [Go, Prop.2.2], in particular, $\mathbf{J}^{-1}(\mu)$ is a submanifold.

For the reasons cited in the Introduction, we restrict our attention to the submanifold $X_0 = \mathbf{J}^{-1}(0)$. X_0 is G-invariant. Indeed, by the definition of the 1-form θ: $x \in X_0$ iff $x(p_*(\hat{\xi}_x)) = 0, \forall \xi \in \mathfrak{s}$. But the actions of G and S on N commute and therefore the vector fields $\hat{\xi}$, $p_*(\hat{\xi})$ are G-invariant. Next, by equivariance, X_0 is stable under the action of S so that the orbit space $X_0' = X_0/S$ is a well defined smooth manifold and the projection mapping $\boldsymbol{\pi}$ is a principal S-fibration. Since the fibers are the leaves of the null-foliation, there exists a unique symplectic form Ω_0' on X_0' such that $\boldsymbol{\pi}^*\Omega_0' = \mathbf{j}^*\Omega$, $\mathbf{j}$ being the inclusion mapping of X_0 into X. Since the actions of G and S on $X = T^*N$ commute, there exists a unique action of G on X_0' such that the projection $\boldsymbol{\pi}$ is G-equivariant.

Let $\Omega' = d\theta'$ be the canonical symplectic structure on T^*N', where $N' = N/S$.

Proposition 24. [Go] *The reduced phase space (X_0', Ω_0') is symplectomorphic to (T^*N', Ω'). Moreover, under this identification of X_0' with T^*N' we have the following identity for the canonical 1-forms: $\boldsymbol{\pi}^*\theta' = \mathbf{j}^*\theta$.*

The construction of this symplectomorphism in [Go] is based on the fact that the pullback bundle

$$p_S^*(T^*N') = \mathbf{J}^{-1}(0), \tag{15}$$

where $p_S : N \to N/S$ is the canonical submersion. Quotienting by S in (15) then gives $T^*N' \simeq X_0'$. Since the actions of S and G on N commute, the G-action on $X_0' \simeq T^*N'$ is the extension of natural action of G on the quotient space N/S. From this and (15) we obtain

Proposition 25. *Let H be a function on $X = T^*N$ invariant under the actions of G and S. Assume that H is a homogeneous polynomial of degree 2 in the impulse coordinates. Then the reduced function H' on $X_0' = T^*N'$, i.e. such that $\boldsymbol{\pi}^*H' = \mathbf{j}^*H$, is G-invariant and a homogeneous polynomial of degree 2 in the impulse coordinates.*

Let F be a positive-definite S-invariant polarization on X and S be a compact Lie group.

Theorem 26. [GS] *There is canonically associated with F a positive-definite polarization F' on the reduced space X_0'.*

This polarization F' is described as follows [GS]: For each point $x \in X_0$ let $F_0(x) = (T^{\mathbb{C}}_x X_0) \cap F(x)$. Then F_0 is an S-invariant subbundle of $T^{\mathbb{C}} X_0$ of the dimension $(\dim X'_0)/2$ and $F' = \boldsymbol{\pi}_*(F_0)$ (the intersection of $F(x)$ with the complexified kernel of $\boldsymbol{\pi}_*(x)$, generated by $\hat{\xi}(x)$, vanishes). It is evident that if F is G-invariant then F' also is G-invariant, because the actions of G and S on N commute.

4.2. The reduced polarizations on $\mathbb{C}P^n$ and $\mathbb{H}P^n$

In this subsection we will prove that the Kähler structures F^λ on the punctured tangent bundles to the symmetric spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$ may be obtained using the reduction from the analogous structures on the punctured tangent bundles to the spheres. Below we will consider the bundles, mappings on these three types of manifolds and will use the notation introduced earlier for $T(G/K)$ but with indexes $\mathbb{R}, \mathbb{C}, \mathbb{H}$ for the sphere, complex and quaternionic projective spaces respectively.

Let G'' be the Lie group $SO(2n+2)$ and K'' be its subgroup isomorphic to $SO(2n+1)$. On the $2n+1$-dimensional sphere $N = G''/K''$ we have a transitive action of the subgroup $G \subset G''$ isomorphic to $SU(n+1)$. The intersection $G \cap K''$ is the subgroup $K \simeq SU(n)$. Therefore $N = G/K = SU(n+1)/SU(n)$. The complex projective space $\mathbb{C}P^n$ is the homogeneous space $N' = G/K' = SU(n+1)/S(U(1) \times U(n))$, i.e $N' = N/S$, where S is a Lie subgroup of G isomorphic to $U(1)$ (we consider the right action of S). Consider also the natural left G-action on N. It is clear that these actions of G and S commute. Let $\Phi_{\mathbb{R}}$ and $\Phi_{\mathbb{C}}$ be the normalized trace forms $-\frac{1}{2}\operatorname{Tr}$ of the real semisimple Lie algebras $\mathfrak{g}'' = so(2n+2)$ and $\mathfrak{g} = su(n+1)$ respectively (associated with the faithful standard representations). Using the forms $\Phi_{\mathbb{R}}$ and $\Phi_{\mathbb{C}}$ we identify the cotangent bundles $T^*N = T^*(G''/K'')$ and $T^*N' = T^*(G/K')$ with the corresponding tangent bundles $X = TN$ and $X'_0 = TN'$ as in subsection 2.2. Let $\theta_{\mathbb{R}}$ and $\theta'_{\mathbb{C}}$ be the canonical 1-forms on TN and TN' (depending of these identifications). Denote by $H_{\mathbb{R}}$ and $H'_{\mathbb{C}}$ the corresponding to $\Phi_{\mathbb{R}}$ and $\Phi_{\mathbb{C}}$ Hamiltonians of the geodesic flows on the manifolds $TN = T(G''/K'')$ and $TN' = T(G/K')$ respectively. But identifying $T^*N = T^*(G/K)$ with $TN = T(G/K)$ using the form $\Phi_{\mathbb{C}}$ we obtain another canonical 1-form $\theta_{\mathbb{C}}$ and the Hamiltonian $H_{\mathbb{C}}$ on TN. It is clear that $C_{\mathbb{R}\mathbb{C}}\theta_{\mathbb{C}} = \theta_{\mathbb{R}}$ and $C_{\mathbb{R}\mathbb{C}}H_{\mathbb{C}} = H_{\mathbb{R}}$ for some constant $C_{\mathbb{R}\mathbb{C}}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \oplus \mathfrak{m}'$ be the orthogonal direct sum decomposition of $\mathfrak{g}$ with respect to the form $\Phi_{\mathbb{C}}$ of $\mathfrak{g}$, where $\mathfrak{k}$ is the Lie algebra of the Lie subgroup $K \subset G$. Let $\mathfrak{m} = \mathfrak{s} \oplus \mathfrak{m}'$. Denote by $\mathfrak{k}'$ the Lie algebra of

$K' \subset G$: $\mathfrak{k}' = \mathfrak{k} \oplus \mathfrak{s}$. Since $[\mathfrak{k}', \mathfrak{m}] \subset \mathfrak{m}$, we can consider the trivial vector bundle $G \times \mathfrak{m}$ with the right K'-action $r_{k'} : (g, w) \mapsto (gk', \operatorname{Ad} k'^{-1}(w))$. Let $\mathbf{J} : T(G/K) \to \mathfrak{s}^*$ be the moment map associated with the (right) S-action on $N = G/K$. Using the (left) G-equivariant submersion $\Pi : G \times \mathfrak{m} \to T(G/K)$, $(g, w) \mapsto \frac{d}{dt}\Big|_0 g\exp(tw)K$ and Lemma 2 we obtain that $(\mathbf{J} \circ \Pi)(g, w)(\eta) = \tilde{\theta}_{\mathbb{C}(g,w)}(\eta^l(g), [w, \eta]) = \Phi_{\mathbb{C}}(w, \eta)$ for any $\eta \in \mathfrak{s}$. Therefore $\Pi(G \times \mathfrak{m}') = \mathbf{J}^{-1}(0) = X_0$ and, consequently, $X_0' = TN' \simeq G \times_{K'} \mathfrak{m}'$. Moreover, by (2) $\boldsymbol{\pi}^*\theta_{\mathbb{C}}' = \mathbf{j}^*\theta_{\mathbb{C}}$. Because the mapping $T(G/K) \to T(G''/K'')$, $\frac{d}{dt}\Big|_0 g\exp(t\xi)K \mapsto \frac{d}{dt}\Big|_0 g\exp(t\xi)K''$, $g \in G$, $\xi \in \mathfrak{m}$, is a diffeomorphism (G acts transitively on G''/K'') and $H_{\mathbb{R}} = C_{\mathbb{RC}} H_{\mathbb{C}}$, the reduced Hamiltonian $H_{\mathbb{R}}'$ on X_0' coincides with $C_{\mathbb{RC}} H_{\mathbb{C}}'$. To find the constant $C_{\mathbb{RC}}$ consider the standard embedding $su(n+1) \to so(2n+2)$, $\xi = (A + iB) \mapsto \xi'' = \left(\begin{array}{c|c} A & -B \\ \hline B & A \end{array}\right)$ of $\mathfrak{g}$ in $\mathfrak{g}''$. Let $\mathfrak{k}'' = so(2n+1)$ be the Lie algebra of $K'' \subset G''$. Since for the trace-forms (on $\mathfrak{g}$ and $\mathfrak{g}''$) $\operatorname{Tr}\xi^2 = \operatorname{Tr}(\xi'')^2_{\mathfrak{m}''}$, where $\xi \in \mathfrak{m} \subset \mathfrak{g}$ and $()_{\mathfrak{m}''}$ is the projection into $\mathfrak{m}''$ along $\mathfrak{k}''$ in $\mathfrak{g}''$ determined by the orthogonal direct sum decomposition $\mathfrak{g}'' = \mathfrak{k}'' \oplus \mathfrak{m}''$ [He], we obtain that $C_{\mathbb{RC}} = 1$. Note that $\operatorname{Tr}\xi^2 = \frac{1}{2}\operatorname{Tr}(\xi'')^2$.

Let $F_{\mathbb{R}} = F_{\mathbb{R}}(q)$, where $q : \mathbb{R}^+ \to \mathbb{R}$ is a smooth strictly increasing function, be the Kähler structure on $T^0N \subset X$ such that the one-form $\operatorname{Im}\overline{\partial}(q \circ \sqrt{H_{\mathbb{R}}})$ is the canonical one-form $\theta_{\mathbb{R}}$. By Theorem 15 the structure $F_{\mathbb{R}}$ is S-invariant, by Theorem 26 there exists the reduced Kähler structure $F_{\mathbb{R}}'$ on $T^0N' \subset X_0'$ such that $F_{\mathbb{R}}' = \boldsymbol{\pi}_*(F_{\mathbb{R}0})$, i.e. $\boldsymbol{\pi}^*(\operatorname{Im}\overline{\partial}(q \circ \sqrt{H_{\mathbb{R}}'})) = \mathbf{j}^*(\operatorname{Im}\overline{\partial}(q \circ \sqrt{H_{\mathbb{R}}}))$ (to prove this it is sufficient to use the definition of $\overline{\partial} : \overline{\partial}H|F = dH|F$, $\overline{\partial}H|\overline{F} = 0$). But $\boldsymbol{\pi}^*\theta_{\mathbb{C}}' = \mathbf{j}^*\theta_{\mathbb{C}}$ so that $\operatorname{Im}\overline{\partial}(q \circ \sqrt{H_{\mathbb{R}}'}) = C_{\mathbb{RC}}\theta_{\mathbb{C}}'$ is the canonical one-form on T^0N' up to the constant $C_{\mathbb{RC}} = 1$. Since the considered Kähler structure $F_{\mathbb{R}}$ and the function $H_{\mathbb{R}}$ are G and $X_{\sqrt{H_{\mathbb{R}}}}$-invariant, $F_{\mathbb{R}}'$, $H_{\mathbb{R}}'$ are G and $X_{\sqrt{H_{\mathbb{R}}'}}$-invariant. Therefore by Theorem 15, $F_{\mathbb{R}}'$ is the structure $F_{\mathbb{C}}' = F_{\mathbb{C}}'(q)$ on T^0N' such that $-i\partial\overline{\partial}(q \circ \sqrt{H_{\mathbb{C}}'}) = \Omega$.

THEOREM 27. *Let $N = S^{2n+1}$ and $F_{\mathbb{R}} = F_{\mathbb{R}}(q)$ be the Kähler structure on $(T^0N, d\theta_{\mathbb{R}})$. The reduced manifold $(\mathbf{J}^{-1}(0) \cap T^0N)/S$, where $S \simeq U(1)$, is isomorphic to $(T^0N', d\theta_{\mathbb{C}}')$, $N' = \mathbb{C}P^n$ with the reduced Kähler structure $F_{\mathbb{R}}' = F_{\mathbb{C}}'(q)$.*

For the Kähler structure $J_S \in \{F_{\mathbb{R}}(q)\}$ on T^0S^{2n+1} the reduced Kähler structure coincides with the structure J_S on $T^0\mathbb{C}P^n$.

Proof. The first part of the theorem is proved. Taking into account Remark 17 to prove the latter assertion it is sufficient to see that the Kähler structure J_S on T^0N is defined by the function $\lambda^S = \sqrt{H_{\mathbb{R}}} = \|\cdot\|$ and $q^S(r) = 2r$, (the latter depends essentially of the forms $\Phi_{\mathbb{R}}$, $\Phi_{\mathbb{C}}$ which we used to identify T^*N and TN, T^*N' and TN'). □

Now let G'' be the Lie group $SO(4n+4)$ and K'' be its subgroup isomorphic to $SO(4n+3)$. On the $4n+3$-dimensional sphere $N = G''/K''$ we have the transitive action of the subgroup $G \subset G''$ isomorphic to $Sp(n+1)$. The intersection $G \cap K''$ is the subgroup $K \simeq Sp(n)$. Therefore $N = Sp(n+1)/Sp(n)$. The quaternionic projective space $\mathbb{H}P^n$ is the homogeneous space $N' = Sp(n+1)/(Sp(1) \times Sp(n))$, i.e $N' = N/S$, where S is a Lie subgroup of G isomorphic to $Sp(1)$ (we consider the right action of S). Let $\Phi_{\mathbb{R}}$ and $\Phi_{\mathbb{H}}$ be the normalized trace forms $-\frac{1}{2}\operatorname{Tr}$ of the real semisimple Lie algebras $\mathfrak{g}'' = so(4n+4)$ and $\mathfrak{g} = sp(n+1)$ respectively (associated with the faithful standard representations by real and quaternionic matrices respectively). Using the forms $\Phi_{\mathbb{R}}$ and $\Phi_{\mathbb{H}}$ we identify the cotangent bundles $T^*N = T^*(G''/K'')$ and $T^*N' = T^*(G/K')$ with the corresponding tangent bundles $X = TN$ and $X_0' = TN'$. Let $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{H}}'$ be the canonical 1-forms on TN and TN' (depending on these identifications). Denote by $H_{\mathbb{R}}$ and $H_{\mathbb{H}}'$ the corresponding to $\Phi_{\mathbb{R}}$ and $\Phi_{\mathbb{H}}$ Hamiltonians of the geodesic flows on TN and TN' respectively. It is evident now that we are in the similar situation as above when we considered the pair $(S^{2n+1}, \mathbb{C}P^n)$. Replacing, where it is necessary, $\mathbb{C} \mapsto \mathbb{H}$, and finding $C_{\mathbb{RH}} = 1$, we obtain

THEOREM 28. *Let $N = S^{4n+3}$ and $F_{\mathbb{R}} = F_{\mathbb{R}}(q)$ be the Kähler structure on $(T^0N, d\theta_{\mathbb{R}})$. The reduced manifold $(\mathbf{J}^{-1}(0) \cap T^0N)/S$, where $S \simeq Sp(1)$, is isomorphic to $(T^0N', d\theta_{\mathbb{H}}')$, where $N' = \mathbb{H}P^n$, with the reduced Kähler structure $F_{\mathbb{R}}' = F_{\mathbb{H}}'(q)$.*

For the Kähler structure $J_S \in \{F_{\mathbb{R}}(q)\}$ on T^0S^{4n+3} the reduced Kähler structure coincides with the structure J_S on $T^0\mathbb{H}P^n$.

4.3. The reduction and adapted structures

Let G be a real reductive connected Lie group, K its (closed) reductive subgroup. We can identify TG with $X = G \times \mathfrak{g}$ using the left action of G on TG. Consider the right K-action on $G \times \mathfrak{g}$. Since the canonical 1-form θ on $G \times \mathfrak{g}$ has form (2), $\mathbf{J}(g, w)(\zeta) = \langle w, \zeta \rangle$, $\zeta \in \mathfrak{k}$ is the moment map for this K-action. It is evident that $X_0 = G \times \mathfrak{m}$ and, consequently, $X_0' = \mathbf{J}^{-1}(0)/K$ is the space $G \times_K \mathfrak{m}$ isomorphic to $T(G/K)$.

Let us return to the proof and notation of Proposition 21. The subbundle $\mathcal{F}^K$ is generated by $\mathcal{K}^{\mathbb{C}}$ and the subbundle $F_c \cap T^{\mathbb{C}}(G \times \mathfrak{m})$. By

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Theorem 26 and Lemma 20 $F^K = \Pi_*(\mathcal{F}^K)$ is a positive-definite polarization, by Lemma 18 F^K is an adapted structure defined on whole $T(G/K)$. Since such adapted structure is unique [LSz], $F^K = F_c^K$.

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