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The Fefferman–Stein Type Inequalities for Strong and Directional Maximal Operators in the Plane

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Abstract. The Fefferman–Stein type inequalities for strong maximal operators and directional maximal operators are verified with an additional composition of the Hardy–Littlewood maximal operator in the plane.

1 Introduction

The purpose of this paper is to develop a theory of weights for strong maximal operators and directional maximal operators in the plane. We first fix some notation. By *weights* we will always mean non-negative and locally integrable functions on $\mathbb{R}^n$. Given a measurable set E and a weight w, $w(E) = \int_E w(x)\,dx$, $|E|$ denotes the Lebesgue measure of E and 1_E denotes the characteristic function of E. Let $0 < p \le \infty$ and w be a weight. We define the weighted Lebesgue space $L^p(\mathbb{R}^n, w)$ to be a Banach space equipped with the norm (or quasi norm)

$$\|f\|_{L^p(\mathbb{R}^n,w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \Big)^{1/p}.$$

For a locally integrable function f on $\mathbb{R}^n$, we define the Hardy–Littlewood maximal operator $\mathfrak{M}_{\mathcal{Q}}$ by

$$\mathfrak{M}_{\mathcal{Q}} f(x) = \sup_{Q\in\mathcal{Q}} 1_Q(x) \fint_Q |f(y)|\,dy,$$

where $\mathcal{Q}$ is the set of all cubes in $\mathbb{R}^n$ (with sides not necessarily parallel to the axes) and the barred integral $\fint_Q f(y)\,dy$ stands for the usual integral average of f over Q. For a locally integrable function f on $\mathbb{R}^n$, we define the strong maximal operator $\mathfrak{M}_{\mathcal{R}}$ by

$$\mathfrak{M}_{\mathcal{R}} f(x) = \sup_{R\in\mathcal{R}} 1_R(x) \fint_R |f(y)|\,dy,$$

where $\mathcal{R}$ is the set of all rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

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Let $\mathfrak{T}: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $p > 1$, be a sublinear operator. It is a fundamental problem in weight theory to determine some maximal operator $\mathfrak{M}_{\mathfrak{T}}$ capturing certain geometric characteristics of $\mathfrak{T}$, such that

$$\int_{\mathbb{R}^n} |\mathfrak{T} f(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}_{\mathfrak{T}} w(x)\,dx \tag{1.1}$$

holds for an arbitrary weight w. It is well known that

$$\int_{\mathbb{R}^n} \mathfrak{M}_{\mathcal{Q}} f(x)^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}_{\mathcal{Q}} w(x)\,dx$$

holds for an arbitrary weight w and $p > 1$, and further that

$$\sup_{t>0} t\, w\big(\{x \in \mathbb{R}^n : \mathfrak{M}_{\mathcal{Q}} f(x) > t\}\big) \le C \int_{\mathbb{R}^n} |f(x)| \mathfrak{M}_{\mathcal{Q}} w(x)\,dx. \tag{1.2}$$

These are called the Fefferman–Stein inequalities and are toy models of (1.1) (see [3]).

Problem 1.1 ([4, p. 472]) *Does the analogue of the Fefferman–Stein inequality hold for the strong maximal operator,* i.e.,

$$\int_{\mathbb{R}^n} \mathfrak{M}_{\mathcal{R}} f(x)^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p \mathfrak{M}_{\mathcal{R}} w(x)\,dx, \quad p > 1, \tag{1.3}$$

for arbitrary $w(x) \ge 0$*?*

Concerning Problem 1.1, it is known that (1.3) holds for all $p > 1$ if $w \in A_\infty^*$; see [8] (also [11, 12]).

We say that w belongs to the class A_p^* whenever

$$[w]_{A_p^*} = \sup_{R \in \mathcal{R}} \fint_R w(x)\,dx \Big(\fint_R w(x)^{-1/(p-1)}\,dx \Big)^{p-1} < \infty, \quad 1 < p < \infty,$$

$$[w]_{A_1^*} = \sup_{R \in \mathcal{R}} \frac{\fint_R w(x)\,dx}{\operatorname{ess\,inf}_{x \in R} w(x)} < \infty.$$

It follows by Hölder's inequality that the A_p^* classes are increasing; that is, for $1 \le p \le q < \infty$, we have $A_p^* \subset A_q^*$. Thus, one defines $A_\infty^* = \bigcup_{p>1} A_p^*$.

The endpoint behavior of $\mathfrak{M}_{\mathcal{R}}$ close to L^1 is given by Mitsis [10] (for $n = 2$) and Luque and Parissis [9] (for $n > 2$); that is,

$$w\big(\{x \in \mathbb{R}^n : \mathfrak{M}_{\mathcal{R}} f(x) > t\}\big) \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{t} \Big(1 + \Big(\log^+ \frac{|f(x)|}{t} \Big)^{n-1} \Big) \mathfrak{M}_{\mathcal{R}} w(x)\,dx, \quad t > 0,$$

holds for any $w \in A_\infty^*$, where $\log^+ t = \max(0, \log t)$.

In this paper we will establish the following concerning Problem 1.1.

Theorem 1.2 *Let w be any weight on $\mathbb{R}^2$ and set $W = \mathfrak{M}_{\mathcal{R}} \mathfrak{M}_{\mathcal{Q}} w$. Then*

$$w\big(\{x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{R}} f(x) > t\}\big) \le C \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \Big(1 + \log^+ \frac{|f(x)|}{t} \Big) W(x)\,dx, \quad t > 0,$$

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holds, where the constant $C > 0$ does not depend on w and f.

By interpolation, we have the following corollary.

Corollary 1.3 *Let w be any weight on $\mathbb{R}^2$ and set $W = \mathfrak{M}_{\mathcal{R}}\mathfrak{M}_{\mathcal{Q}}w$. Then, for $p > 1$, there exists a constant $C_p > 0$ such that*

$$\|\mathfrak{M}_{\mathcal{R}} f\|_{L^p(\mathbb{R}^2,w)} \le C_p \|f\|_{L^p(\mathbb{R}^2,W)}$$

holds for all $f \in L^p(\mathbb{R}^2, W)$.

Let Σ be a set of unit vectors in $\mathbb{R}^2$, *i.e.,* a subset of the unit circle S^1. Associated with Σ, we define $\mathcal{B}_\Sigma$ to be the set of all rectangles in $\mathbb{R}^2$ whose longest side is parallel to some vector in Σ. For a locally integrable function f on $\mathbb{R}^2$, we also define the directional maximal operator $\mathfrak{M}_\Sigma$ associated with Σ as

$$\mathfrak{M}_\Sigma f(x) = \sup_{R \in \mathcal{B}_\Sigma} 1_R(x) \fint_R |f(y)|\, dy.$$

Many authors have studied this operator, (see [1, 2, 6, 7, 13, 14]), and Katz showed that $\mathfrak{M}_\Sigma$ is bounded on $L^2(\mathbb{R}^2)$ with the operator norm $O(\log N)$ for any set Σ with cardinality N.

For fixed sufficiently large integer N, let

$$\Sigma_N = \left\{ \left(\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N} \right) : j = 0, 1, \dots, N-1 \right\}$$

be the set of N uniformly spread directions on the circle S^1. In this paper we shall prove the following, which is a weighted version of the classical result due to Strömberg [14].

Theorem 1.4 *Let $N > 10$ and w be any weight on $\mathbb{R}^2$. Set $W = \mathfrak{M}_{\Sigma_N}\mathfrak{M}_{\mathcal{Q}}w$. Then*

$$\sup_{t>0} t\, w\Big(\{ x \in \mathbb{R}^2 : \mathfrak{M}_{\Sigma_N} f(x) > t \} \Big)^{1/2} \le C(\log N)^{1/2} \|f\|_{L^2(\mathbb{R}^2,W)} \tag{1.4}$$

holds for all $f \in L^2(\mathbb{R}^2, W)$, where the constant $C > 0$ does not depend on w and f.

By interpolation, we have the following corollary.

Corollary 1.5 *Let $N > 10$ and w be any weight on $\mathbb{R}^2$. Set $W = \mathfrak{M}_{\Sigma_N}\mathfrak{M}_{\mathcal{Q}}w$. Then, for $2 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\|\mathfrak{M}_{\Sigma_N} f\|_{L^p(\mathbb{R}^2,w)} \le C_p (\log N)^{1/p} \|f\|_{L^p(\mathbb{R}^2,W)}$$

holds for all $f \in L^p(\mathbb{R}^2, W)$.

The letter C will be used for the positive finite constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences.

2 Proof of Theorem 1.2

Our proof relies upon the refinement of the arguments in [10]. With a standard argument, we can assume that the basis $\mathcal{R}$ is the set of all dyadic rectangles R (cartesian products of dyadic intervals) having long side pointing in the x_1-direction. We denote by P_i, $i = 1, 2$, the projection on the x_i-axis. Fix $t > 0$ and assume given the finite collection of dyadic rectangles $\{R_i\}_{i=1}^M \subset \mathcal{R}$ such that

$$\fint_{R_i} |f(y)|\,dy > t, \quad i = 1, 2, \dots, M. \tag{2.1}$$

It suffices to estimate $w(\bigcup_{i=1}^M R_i)$ (see the next section for details).

First, relabeling if necessary, we can assume that the R_i are ordered so that their long sidelengths $|P_1(R_i)|$ decrease. We now give a selection procedure to find a subcollection $\{\widetilde{R}_i\}_{i=1}^N \subset \{R_i\}_{i=1}^M$.

Take $\widetilde{R}_1 = R_1$ and let $\widetilde{R}_2$ be the first rectangle R_j such that $|R_j \cap \widetilde{R}_1| < \frac{1}{3}|R_j|$. Suppose that we have now chosen the rectangles $\widetilde{R}_1, \widetilde{R}_2, \dots, \widetilde{R}_{i-1}$. We select $\widetilde{R}_i$ to be the first rectangle R_j occurring after $\widetilde{R}_{i-1}$ so that $\big|\bigcup_{k=1}^{i-1} R_j \cap \widetilde{R}_k\big| < \frac{1}{3}|R_j|$. Thus, we see that

$$\Big|\bigcup_{j=1}^{i-1} \widetilde{R}_i \cap \widetilde{R}_j\Big| < \frac{1}{3}|\widetilde{R}_i|, \quad i = 2, 3, \dots, N. \tag{2.2}$$

We claim that

$$\bigcup_{i=1}^M R_i \subset \Big\{ x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{Q}}[1_{\bigcup_{i=1}^N \widetilde{R}_i}](x) \geq \frac{1}{3} \Big\}. \tag{2.3}$$

Indeed, choose any point x inside a rectangle R_j that is not one of the selected rectangles $\widetilde{R}_i$. Then there exists a unique $K \leq N$ such that

$$\Big|\bigcup_{i=1}^K R_j \cap \widetilde{R}_i\Big| \geq \frac{1}{3}|R_j|.$$

Since, $|P_1(R_j)| \leq |P_1(\widetilde{R}_i)|$ for $i = 1, 2, \dots, K$, we have

$$P_1(R_j) \cap P_1(\widetilde{R}_i) = P_1(R_j) \text{ when } R_j \cap \widetilde{R}_i \neq \varnothing,$$

where we have used the dyadic structure;

(2.4) If both I and J are the dyadic interval, then $I \cap J \in \{I, J, \varnothing\}$.

Thus,

$$\bigcup_{i=1}^K R_j \cap \widetilde{R}_i = \bigcup_{i=1}^K P_1(R_j) \times \big(P_2(R_j) \cap P_2(\widetilde{R}_i)\big) = P_1(R_j) \times \bigcup_{i=1}^K P_2(R_j) \cap P_2(\widetilde{R}_i).$$

Hence,

$$\Big|\bigcup_{i=1}^K P_2(R_j) \cap P_2(\widetilde{R}_i)\Big| \geq \frac{1}{3}|P_2(R_j)|.$$

Thanks to the fact that $|P_2(R_j)| \leq |P_1(R_j)| \leq |P_1(\widetilde{R}_i)|$, this implies that

$$\Big|\bigcup_{i=1}^K Q \cap \widetilde{R}_i\Big| \geq \frac{1}{3}|Q|,$$

where Q is a unique dyadic cube containing x and having the side length $|P_2(R_j)|$. This proves (2.3).

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It follows from (2.3) and (1.2) that

$$w\Big(\bigcup_{i=1}^{M} R_i\Big) \le w\Big(\Big\{x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{Q}}[1_{\bigcup_{i=1}^{N}\widetilde{R}_i}](x) \ge \frac{1}{3}\Big\}\Big)$$
$$\le CU\Big(\bigcup_{i=1}^{N} \widetilde{R}_i\Big) \le C\sum_{i=1}^{N} U(\widetilde{R}_i),$$

where $U = \mathfrak{M}_{\mathcal{Q}} w$. We shall evaluate the quantity

$$\text{(i)} = \sum_{i=1}^{N} U(\widetilde{R}_i).$$

Let $\mu_U(x)$ be the weighted multiplicity function associated with the family $\{\widetilde{R}_i\}$; that is,

$$\mu_U(x) = \sum_{i=1}^{N} \frac{U(\widetilde{R}_i)}{|\widetilde{R}_i|} 1_{\widetilde{R}_i}(x).$$

By (2.1), choosing δ_0 small enough determined later,

$$\text{(i)} \le \sum_{i=1}^{N} \frac{U(\widetilde{R}_i)}{|\widetilde{R}_i|} \int_{\widetilde{R}_i} \frac{|f(y)|}{t}\, dy$$
$$= \delta_0 \int_{\mathbb{R}^2} \mu_U(x) W(x)^{-1} \cdot \frac{|f(x)|}{\delta_0 t} \cdot W(x)\, dx.$$

Using the elementary inequality

$$ab \le (e^a - 1) + b(1 + \log^+ b), \quad a, b \ge 0,$$

we get

$$\text{(i)} \le \delta_0 \int_{\mathbb{R}^2} \big(\exp(\mu_U(x) W(x)^{-1}) - 1\big) W(x)\, dx$$
$$+ \delta_0 \int_{\mathbb{R}^2} \frac{|f(x)|}{\delta_0 t} \Big(1 + \log^+ \frac{|f(x)|}{\delta_0 t}\Big) W(x)\, dx$$
$$\le \delta_0 \int_{\mathbb{R}^2} \big(\exp(\mu_U(x) W(x)^{-1}) - 1\big) W(x)\, dx$$
$$+ (1 - \log \delta_0) \int_{\mathbb{R}^2} \frac{|f(x)|}{t} \Big(1 + \log^+ \frac{|f(x)|}{t}\Big) W(x)\, dx.$$

We have to evaluate the quantity

$$\text{(ii)} = \int_{\mathbb{R}^2} \big(\exp(\mu_U(x) W(x)^{-1}) - 1\big) W(x)\, dx.$$

We expand the exponential in a Taylor series. Then

$$\text{(ii)} = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^2} (\mu_U(x) W(x)^{-1})^k W(x)\, dx$$
$$= \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^2} \mu_U(x)^k W(x)^{1-k}\, dx.$$

Fix $k \ge 2$. We use an elementary inequality

$$\Big(\sum_{i=1}^{\infty} a_i\Big)^k \le k \sum_{i=1}^{\infty} a_i \Big(\sum_{j=1}^{i} a_j\Big)^{k-1},$$

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where $\{a_i\}_{i=1}^{\infty}$ is a sequence of summable nonnegative reals. Then

$$\begin{aligned}&\int_{\mathbb{R}^2} \mu_U(x)^k W(x)^{1-k}\,dx\\ &\le k\sum_{i=1}^{N}\frac{U(\widetilde{R}_i)}{|\widetilde{R}_i|}\int_{\widetilde{R}_i}\Big(\sum_{j=1}^{i}\frac{U(\widetilde{R}_j)}{|\widetilde{R}_j|}1_{\widetilde{R}_j}(x)\Big)^{k-1}W(x)^{1-k}\,dx\\ &\le k\sum_{i=1}^{N}\frac{U(\widetilde{R}_i)}{|\widetilde{R}_i|}\int_{\widetilde{R}_i}\Big(\sum_{j=1}^{i}1_{\widetilde{R}_j}(x)\Big)^{k-1}\,dx,\end{aligned}$$

where we have used

$$\sum_{j=1}^{i}\frac{U(\widetilde{R}_j)}{|\widetilde{R}_j|}1_{\widetilde{R}_j}(x)\le\Big(\sum_{j=1}^{i}1_{\widetilde{R}_j}(x)\Big)W(x).$$

We claim that, for $n = 1, 2, \dots, N$,

$$|X_{i,n}| \le 3^{1-n}|\widetilde{R}_i|, \tag{2.5}$$

where

$$X_{i,n} = \Big\{x\in\widetilde{R}_i : \sum_{j=1}^{i}1_{\widetilde{R}_j}(x)\ge n\Big\}.$$

Indeed, first we notice that, for any k and j with $N \ge k > j \ge 1$, if $\widetilde{R}_k \cap \widetilde{R}_j \ne \varnothing$, then

$$\widetilde{R}_k\cap\widetilde{R}_j = P_1(\widetilde{R}_k)\times P_2(\widetilde{R}_j),$$

because we have $P_1(\widetilde{R}_k)\subset P_1(\widetilde{R}_j)$ and, by (2.2), $|P_2(\widetilde{R}_k)\cap P_2(\widetilde{R}_j)| < \frac{1}{3}|P_2(\widetilde{R}_k)|$. With this in mind, we can observe the following.

There exists a set of dyadic intervals $\{I_{j\,k}\}$ with $j = 1, 2, \dots, n$ and $k = 1, 2, \dots, K_j$ that satisfies the following:

- the dyadic intervals $I_{j\,k}$ are pairwise disjoint for varying k;
- for each $I_{j\,k}$, $j > 1$, there exists a unique $I_{(j-1)\,l} \supsetneq I_{j\,k}$;
- for each $I_{j\,k}$ there exists a unique number $i_{j\,k} \le i$ such that $I_{j\,k} = P_2(\widetilde{R}_{i_{j\,k}})$;
- $P_2(X_{i,1}) = I_{11}$, $P_2(X_{i,2}) = \bigcup_{k=1}^{K_2} I_{2\,k}$, $\dots$, $P_2(X_{i,j}) = \bigcup_{k=1}^{K_j} I_{j\,k}$, $\dots$, $P_2(X_{i,n}) = \bigcup_{k=1}^{K_n} I_{n\,k}$;
- if $I_{j\,k}\subset I_{(j-1)\,l}$, then $i_{j\,k} < i_{(j-1)\,l}$ and $\widetilde{R}_{i_{(j-1)\,l}}\cap\widetilde{R}_{i_{j\,k}}\ne\varnothing$.

It follows from the last relation and (2.2) that

$$3\sum_{k=1}^{K_j}|I_{j\,k}| < \sum_{k=1}^{K_{j-1}}|I_{(j-1)\,k}|,\qquad j = 2, 3, \dots, n.$$

This gives us that

$$3^{n-1}\sum_{k=1}^{K_n}|I_{n\,k}| < |I_{11}|,$$

which yields (2.5).

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It follows from (2.5) that

$$\frac{U(\widetilde{R}_i)}{|\widetilde{R}_i|}\int_{\widetilde{R}_i}\Big(\sum_{j=1}^{i}1_{\widetilde{R}_j}(x)\Big)^{k-1}dx \le \frac{U(\widetilde{R}_i)}{|\widetilde{R}_i|}\sum_{n=1}^{N}n^{k-1}|X_{i,n-1}|$$
$$\le U(\widetilde{R}_i)\sum_{n=1}^{N}n^{k-1}3^{2-n}.$$

Altogether, the quantity (ii) can be majorized by

$$\text{(i)}\times\Big[1+\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\sum_{n=1}^{N}n^{k-1}3^{2-n}\Big].$$

There holds

$$\sum_{k=2}^{\infty}\frac{1}{(k-1)!}\sum_{n=1}^{N}n^{k-1}3^{2-n}\le 9\sum_{n=1}^{\infty}\Big(\frac{e}{3}\Big)^{n} =: C_0.$$

If we choose δ_0 so that $\delta_0(1+C_0)=\frac{1}{2}$, we obtain

$$\text{(i)}\le C\int_{\mathbb{R}^2}\frac{|f(x)|}{t}\Big(1+\log^{+}\frac{|f(x)|}{t}\Big)W(x)\,dx.$$

This completes the proof. ∎

Remark Since our proof relies only upon the dyadic structure (2.4), it can be applied the basis $\mathcal{R}$ of the form the set of all rectangles in $\mathbb{R}^n$ whose sides are parallel to the coordinate axes and that are congruent to the rectangle $(0,a)^{n-1}\times(0,b)$ with varying $a,b>0$.

3 Proof of Theorem 1.4

We follow the argument in [5, Chapter 10, Theorem 10.3.5]. To avoid problems with antipodal points, it is convenient to split Σ_N as the union of eight sets, in each of which the angle between any two vectors does not exceed $\pi/4$. It suffices therefore to obtain (1.4) for each such subset of Σ_N. Let us fix one such subset of Σ_N, which we call Σ_N^1.

To prove (1.4), we fix $t>0$ and start with a compact subset K of the set $\{x\in\mathbb{R}^2 : \mathfrak{M}_{\Sigma_N^1}f(x)>t\}$. Then for every $x\in K$, there exists an open rectangle R_x that contains x and whose longest side is parallel to a vector in Σ_N^1. By compactness of K, there exists a finite subfamily $\{R_\alpha\}_{\alpha\in\mathcal{A}}$ of the family $\{R_x\}_{x\in K}$ such that

$$\fint_{R_\alpha}|f(y)|\,dy>t \tag{3.1}$$

for all $\alpha\in\mathcal{A}$ and such that the union of the R_α's covers K.

In the sequel we denote by θ_α the angle between the x_1-axis and the vector pointing in the longer direction of R_α for any $\alpha\in\mathcal{A}$. We also denote by l_α the shorter side of R_α and by L_α the longer side of R_α for any $\alpha\in\mathcal{A}$.

We shall select the subfamily $\{R_\beta\}_{\beta\in\mathcal{B}}$ as follows. Without loss of generality, we can assume that $\mathcal{A}=\{1,2,\dots,\ell\}$ with $L_j\ge L_{j+1}$ for all $j=1,2,\dots,\ell-1$. Let $\beta_1=1$ and choose β_2 to be the first number in $\{\beta_1+1,\beta_1+2,\dots,\ell\}$ such that

$$|R_{\beta_1}\cap R_{\beta_2}|\le\frac{1}{2}|R_{\beta_2}|.$$

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We next choose β_3 to be the first number in $\{\beta_2 + 1, \beta_2 + 2, \dots, \ell\}$ such that

$$|R_{\beta_1} \cap R_{\beta_3}| + |R_{\beta_2} \cap R_{\beta_3}| \le \frac{1}{2}|R_{\beta_3}|.$$

Suppose we have chosen the numbers $\beta_1, \beta_2, \dots, \beta_{j-1}$. Then we choose β_j to be the first number in $\{\beta_{j-1} + 1, \beta_{j-1} + 2, \dots, \ell\}$ such that

$$\sum_{k=1}^{j-1} |R_{\beta_k} \cap R_{\beta_j}| \le \frac{1}{2}|R_{\beta_j}|. \tag{3.2}$$

Since the set $\mathcal{A}$ is finite, this selection stops after m steps.

Define $\mathcal{B} = \{\beta_1, \beta_2, \dots, \beta_m\}$ and let

$$Y(x) = \sum_{\beta \in \mathcal{B}} 1_{(R_\beta)^*}(x),$$

where $(R_\beta)^*$ is the rectangle R_β expanded 5 times in both directions.

We claim that

$$w(K) \le w\Big(\bigcup_{\alpha \in \mathcal{A}} R_\alpha\Big) \le C(\log N) \int_{\mathbb{R}^2} Y(x) U(x)\, dx, \tag{3.3}$$

where $U(x) = \mathfrak{M}_{\mathcal{Q}} w(x)$. To verify this claim, we need the following lemma.

We set $\omega_k = \frac{2\pi 2^k}{N}$ for $k \in \mathbb{Z}^+$ and $\omega_0 = 0$. We let $M = [\frac{\log(N/8)}{\log 2}]$.

Lemma 3.1 ([5, Lemma 10.3.6]) *Let R_α be a rectangle in the family $\{R_\alpha\}_{\alpha \in \mathcal{A}}$ and let $0 \le k < M$. Suppose that $\beta \in \mathcal{B}$ is such that $\omega_k \le |\theta_\alpha - \theta_\beta| < \omega_{k+1}$ and such that $L_\beta \ge L_\alpha$. Let $s_\alpha = 8\max(l_\alpha, \omega_k L_\alpha)$. For an arbitrary $x \in R_\alpha$, let Q be a square centered at x with sides of length s_α parallel to the sides of R_α. Then we have*

$$\frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} \le 32 \frac{|(R_\beta)^* \cap Q|}{|Q|}.$$

We shall prove (3.3). By (1.2) it suffices to show that

$$\bigcup_{\alpha \in \mathcal{A}} R_\alpha \subset \Big\{x \in \mathbb{R}^2 : \mathfrak{M}_{\mathcal{Q}} Y(x) > \frac{C}{\log N}\Big\}. \tag{3.4}$$

Since we may assume that $C/(\log N) < 1$, the set $\bigcup_{\beta \in \mathcal{B}} R_\beta$ is contained in the set of the right hand side of (3.4). So, we fix $\alpha \in \mathcal{A} \setminus \mathcal{B}$. Then the rectangle R_α was not selected in the selection procedure.

By the construction and (3.2), we see that there exists j such that

$$\sum_{k=1}^{j} |R_{\beta_k} \cap R_\alpha| > \frac{1}{2}|R_\alpha|$$

and such that $L_{\beta_k} \ge L_\alpha$ for all $k = 1, 2, \dots, j$.

Let $\mathcal{B}_j = \{\beta_1, \beta_2, \dots, \beta_j\}$. It follows from Lemma 3.1 that

$$\frac{1}{2} < \sum_{\beta\in\mathcal{B}_j} \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|} = \sum_{k=0}^{M} \sum_{\substack{\beta\in\mathcal{B}_j:\\ \omega_k \le |\theta_\alpha - \theta_\beta| < \omega_{k+1}}} \frac{|R_\beta \cap R_\alpha|}{|R_\alpha|}$$

$$\le 32 \sum_{k=0}^{M} \sum_{\substack{\beta\in\mathcal{B}_j:\\ \omega_k \le |\theta_\alpha - \theta_\beta| < \omega_{k+1}}} \frac{|(R_\beta)^* \cap Q_k|}{|Q_k|},$$

where Q_k is a square determined by Lemma 3.1 with an arbitrary $x \in R_\alpha$. Since we have $M \le C(\log N)$ and

$$\sum_{\substack{\beta\in\mathcal{B}_j:\\ \omega_k \le |\theta_\alpha - \theta_\beta| < \omega_{k+1}}} \frac{|(R_\beta)^* \cap Q_k|}{|Q_k|} \le C\mathfrak{M}_{\mathcal{Q}} Y(x) \text{ for all } x \in R_\alpha,$$

we obtain

$$\mathfrak{M}_{\mathcal{Q}} Y(x) > \frac{C}{\log N} \text{ for all } x \in R_\alpha,$$

which implies (3.4) and, hence, (3.3).

We now evaluate

$$\text{(i)} = \int_{\mathbb{R}^2} Y(x)U(x)\,dx = \sum_{\beta\in\mathcal{B}} U((R_\beta)^*).$$

By (3.1) and Hölder's inequality we have

$$\text{(i)} \le \frac{1}{t} \sum_{\beta\in\mathcal{B}} U\big((R_\beta)^*\big) \fint_{R_\beta} |f(y)|\,dy$$

$$= \frac{1}{t} \int_{\mathbb{R}^2} \Big(\sum_{\beta\in\mathcal{B}} \frac{U((R_\beta)^*)}{|R_\beta|} 1_{R_\beta}(y) \Big) |f(y)|\,dy$$

$$\le \frac{1}{t} \bigg(\int_{\mathbb{R}^2} \Big(\sum_{\beta\in\mathcal{B}} \frac{U((R_\beta)^*)}{|R_\beta|} 1_{R_\beta}(y) \Big)^2 W(y)^{-1}\,dy \bigg)^{1/2} \|f\|_{L^2(\mathbb{R}^2, W)}.$$

Further, we have

$$\text{(ii)} = \int_{\mathbb{R}^2} \Big(\sum_{\beta\in\mathcal{B}} \frac{U((R_\beta)^*)}{|R_\beta|} 1_{R_\beta}(y) \Big)^2 W(y)^{-1}\,dy$$

$$= \sum_{j=1}^{m} \Big(\frac{U((R_{\beta_j})^*)}{|R_{\beta_j}|} \Big)^2 \int_{R_{\beta_j}} W(y)^{-1}\,dy$$

$$+ 2 \sum_{j=1}^{m} \frac{U((R_{\beta_j})^*)}{|R_{\beta_j}|} \sum_{k=1}^{j-1} \frac{U((R_{\beta_k})^*)}{|R_{\beta_k}|} \int_{R_{\beta_k}\cap R_{\beta_j}} W(y)^{-1}\,dy.$$

We notice that, for any $y \in R_{\beta_k} \cap R_{\beta_j}$

$$W(y) \ge \frac{U((R_{\beta_k})^*)}{|(R_{\beta_k})^*|} = \frac{U((R_{\beta_k})^*)}{25|R_{\beta_k}|}.$$

This yields

$$
\begin{aligned}
\text{(ii)} &\le 25 \sum_{j=1}^{m} U\big((R_{\beta_j})^*\big) + 50 \sum_{j=1}^{m} \frac{U((R_{\beta_j})^*)}{|R_{\beta_j}|} \sum_{k=1}^{j-1} |R_{\beta_k} \cap R_{\beta_j}| \\
&\le 50 \sum_{\beta \in \mathcal{B}} U\big((R_{\beta_j})^*\big),
\end{aligned}
$$

where we have used (3.2). Altogether, we obtain (i) $\le \frac{C}{t^2} \|f\|^2_{L^2(\mathbb{R}^2, W)}$, which yields, by (3.3),

$$
w(K) \le \frac{C(\log N)}{t^2} \|f\|^2_{L^2(\mathbb{R}^2, W)}.
$$

Since K was an arbitrary compact subset of $\{x \in \mathbb{R}^2 : \mathfrak{M}_{\Sigma^1_N} f(x) > t\}$, the same estimate is valid for the latter set, and we finish the proof. ∎

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