TENSOR PRODUCTS OF CLEAN RINGS

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Abstract. A ring is called clean if every element is the sum of an idempotent and a unit. It is an open question whether the tensor products of two clean algebras over a field is clean. In this note we study the tensor product of clean algebras over a field and we provide some examples to show that the tensor product of two clean algebras over a field need not be clean.

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1. Introduction. Throughout this paper, \( R \) is commutative ring and we use \( \text{Min}(R) \) to denote the set of minimal prime ideals of \( R \). We say \( R \) is quasi-local (resp. semi-local) if the set of maximal ideals of \( R \) has only one element (resp. finitely many elements). An element in \( R \) is called clean if it is the sum of a unit and an idempotent. Following Nicholson, cf. \([4]\), we call the ring \( R \) clean if every element in \( R \) is clean. Examples of clean rings include all zero-dimensional rings (i.e. every prime ideal is maximal) and local rings. Clean rings have been studied by several authors, for example \([4]\), \([2]\), and \([1]\). It is an open question whether the tensor product of two clean algebras over a field is clean, cf. \([2\text{, Question 3}]\). The main purpose of this note is to prove Theorem 1, while Theorem 2 and Proposition 3 are used in the proof of Theorem 1. As an application of Theorem 1 we use it to give an example of two clean algebras \( A \) and \( B \) over a field \( F \) where the tensor product \( A \otimes_F B \) is not clean, see Example 4. In this paper all algebras are unital.

THEOREM 1. Let \( F \) be an algebraically closed field. Let \( A \) and \( B \) be algebras over \( F \). If \( A \) and \( B \) have a finite number of minimal prime ideals (e.g. \( A \) and \( B \) Noetherian) then the following statements are equivalent:

(i) \( A \otimes_F B \) is clean.
(ii) The following hold
    (a) \( A \) and \( B \) are clean.
    (b) \( A \) or \( B \) is algebraic over \( F \).

To prove the above Theorem we first recall the following result from \([1]\) and prove Proposition 3.

THEOREM 2. ([1, Theorem 5]) Let \( R \) have a finite number of minimal prime ideals (e.g., \( R \) is Noetherian). Then the following conditions are equivalent.

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(i) $R$ is a finite direct product of quasi-local rings.
(ii) $R$ is a clean ring.
(iii) $R/p$ is quasi-local for each prime ideal $p$ of $R$.

**Proposition 3.** Let $A$ and $B$ be algebras over a field $F$. Let $\text{Min}(A \otimes_F B)$ be a finite set and assume that $A \otimes_F B$ is clean. Then the following hold.

(i) $A$ or $B$ is algebraic over $F$.
(ii) $A$ and $B$ are clean.
(iii) For any $m \in \text{Max}(A)$ and $n \in \text{Max}(B)$ the ring $A/m \otimes_F B/n$ is semi-local.

**Proof.** (i) By Theorem 2 we know that $A \otimes_F B$ is semi-local and hence by [3, Theorem 6] $A$ or $B$ is algebraic over $F$.

(ii) Assume that $A$ is algebraic over $F$. Then $\dim(A) = \dim(F) = 0$ and so $A$ is clean, cf. [1, Corollary 11]. We know that $\varphi : B \rightarrow (A \otimes_F B)$ is integral. Assume that $p_2 \in \text{Spec}(B)$. Since $\varphi$ is faithfully flat there exists $q \in \text{Spec}(A \otimes_F B)$ such that $q \cap B = p_2$. Since $\tilde{\varphi} : B/p_2 \rightarrow (A \otimes_F B)/q$ is integral and $(A \otimes_F B)/q$ is quasi-local, $B/p_2$ is quasi-local. On the other hand, since $\varphi$ is faithfully flat and $\text{Min}(A \otimes_F B)$ is finite, $\text{Min}(B)$ is finite too. Therefore, by Theorem 2, $B$ is clean.

(iii) By Theorem 2, $A \otimes_F B$ is semi-local and so $A/m \otimes_F B/n \cong (A \otimes_F B)/(m \otimes_F B + A \otimes_F n)$ is semi-local. □

**Proof of Theorem 1.** (i) $\implies$ (ii) First we show that $A \otimes_F B$ has a finite number of minimal prime ideals. Assume $q \in \text{Min}(A \otimes_F B)$ and set $q \cap A = p_1$ and $q \cap B = p_2$. Since $A \rightarrow A \otimes_F B$ is a faithfully flat homomorphism we have that $p_1 \in \text{Min}(A)$ and for the same reason $p_2 \in \text{Min}(B)$. In addition, $q \in \text{Min}(p_1 \otimes_F B + A \otimes_F p_2)$. Since $F$ is algebraically closed $A \otimes_F B/(p_1 \otimes_F B + A \otimes_F p_2) \cong A/p_1 \otimes_F B/p_2$ is an integral domain. Therefore $q = p_1 \otimes_F B + A \otimes_F p_2$. Now the assertion follows from Proposition 3.

(ii) $\implies$ (i). Assume that $q \in \text{Spec}(A \otimes_F B)$ and set $q \cap A = p_1$ and $q \cap B = p_2$. Then $p_1 \otimes_F B + A \otimes_F p_2 \subseteq q$. Since $A$ and $B$ are clean and $\text{Min}(A)$ and $\text{Min}(B)$ are finite we have that $A/p_1$ and $B/p_2$ are quasi-local. Let $m/p_1$ (resp. $n/p_2$) be the unique maximal ideal of $A/p_1$ (resp. $B/p_2$). Since one of $A$ or $B$ is algebraic over $F$ we have that one of $A/p_1$ or $B/p_2$ is algebraic over $F$. Since one of $A/m$ or $B/n$ is algebraic over $F$ we have $\dim(A/m \otimes_F B/n) = 0$. On the other hand, $F$ is algebraically closed so $A/m \otimes_F B/n$ is an integral domain. Therefore $A/m \otimes_F B/n$ is a field. Now by [5] the ring $A/p_1 \otimes_F B/p_2$ is quasi-local and hence $A \otimes_F B/(p_1 \otimes_F B + A \otimes_F p_2)$ is quasi-local. Now the assertion follows from Theorem 2. □

**Example 4.** Assume that $F = \mathbb{C}$ and $A = B = \mathbb{C}[x]$. Then by [1, Proposition 12] $A$ and $B$ are clean. We claim that $A \otimes_F B$ is not clean. Otherwise, since $\mathbb{C}$ is an algebraically closed field and $A(= B)$ is Noetherian, by Theorem 1, we have that $A$ or $B$ is algebraic over $\mathbb{C}$ and hence $A(= B)$ is equal to $\mathbb{C}$. That is a contradiction.

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