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Arithmetic properties of certain functions in several variables III

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We obtain a general transcendence theorem for the solutions of a certain type of functional equation. A particular and striking consequence of the general result is that, for any irrational number ω , the function

$$\sum_{h=1}^{\infty} [h\omega]z^h$$

takes transcendental values at all algebraic points α with $0 < |\alpha| < 1$.

Introduction

In this paper, we continue our study of the transcendency of functions in one or more complex variables which satisfy one of a certain general class of functional equations. The ideas for this work go back almost 50 years to 3 papers of Mahler [9], [10], and [11] in which he analyses solutions of functional equations of the form

$$f(Tz) = R\big(z;\ f(z)\big) ,$$

where T is a certain transformation of the n complex variables $z = \left(z_1, \ldots, z_n\right)$, and $R(z; w)$ is a rational function. In our earlier papers [7] and [8], we extended Mahler's results by widening the class of allowable transformations T . It is our object here to generalise the theory in another direction and, specifically, to answer a problem posed by Mahler, namely Problem 2 of [12]. In view of the technical nature of our

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general result, it seems appropriate to introduce the work of this paper by discussing a number of examples.

In [9], Mahler showed that the Fredholm series

$$f(z) = \sum_{h=0}^{\infty} z^{2^h} ,$$

which satisfies the functional equation

$$f\left(z^2\right) = f(z) - z ,$$

takes transcendental values at algebraic points α with $0 < |\alpha| < 1$, and in [10], he further showed that if $\alpha_1, \ldots, \alpha_m$ are multiplicatively independent algebraic numbers each satisfying $0 < |\alpha_j| < 1$, then the numbers $f(\alpha_1), \ldots, f(\alpha_m)$ are linearly independent over the field of algebraic numbers. In fact, as shown in [8], under these conditions the numbers $f(\alpha_1), \ldots, f(\alpha_m)$ are actually algebraically independent. A recent note [13, 14] of Mahler shows inter alia that the series

$$\sum_{h=0}^{\infty} (h!)^{-1} z^{2^h}$$

takes transcendental values at algebraic points α with $0 < |\alpha| < 1$. We can now generalise this result to show that series of the shape

$$\sum_{h=0}^{\infty} a_h z^{2^h} ,$$

where the $a_h = b_h/c_h$ are rational numbers satisfying

$$\log|b_h|, \log|c_h| = o\left(2^h\right) \quad (h \to \infty) ,$$

take transcendental values at algebraic points α with $0 < |\alpha| < 1$, providing, of course, that infinitely many of the a_h are non-zero. In place of the single functional equation satisfied by the Fredholm series $f(z)$, we now have a chain of functional equations for the functions

$$f_k(z) = \sum_{h=k}^{\infty} a_h z^{2^{h-k}} \quad (k \geq 0) ,$$

namely

$$f_k\left(z^2\right) = f_{k-1}(z) - a_{k-1}z \quad (k \geq 1) .$$

It is with just such systems of functional equations that we shall be concerned in this paper.

The ideas extend to functions in several complex variables. For example, denote by $\{f_h\}$ the sequence of Fibonacci numbers, defined by

$$f_0 = 0 , \quad f_1 = 1 , \quad f_{h+2} = f_{h+1} + f_h \quad (h \geq 0) .$$

Mahler [9] showed that the series

$$f\left(z_1, z_2\right) = \sum_{h=0}^{\infty} z_1^{f_h} z_2^{f_{h+1}} ,$$

which satisfies the functional equation

$$f\left(z_2, z_1z_2\right) = f\left(z_1, z_2\right) - z_2 ,$$

takes transcendental values at points (α, β) with α, β algebraic, $\alpha\beta \neq 0$, and

$$\log|\alpha| + \tfrac{1}{2}\left(1+5^{\frac{1}{2}}\right)\log|\beta| < 0 .$$

In particular, the series

$$g(z) = \sum_{h=0}^{\infty} z^{f_h}$$

takes transcendental values at algebraic points α with $0 < |\alpha| < 1$. In [8], we showed further that if $\alpha_1, \ldots, \alpha_m$ are algebraic numbers each satisfying $0 < \left|\alpha_j\right| < 1$ and the numbers $\left|\alpha_1\right|, \ldots, \left|\alpha_m\right|$ are multiplicatively independent, then the numbers $g\left(\alpha_1\right), \ldots, g\left(\alpha_m\right)$ are algebraically independent. We can now show that, under reasonable growth conditions on the algebraic coefficients a_h, numbers of the shape

$$\sum_{h=0}^{\infty} a_h\alpha^{f_h}$$

are transcendental for algebraic α with $0 < |\alpha| < 1$.

Mahler shows in [13, 14] that his techniques suffice to prove the transcendence of a class of numbers including the number

$$\sum_{h=0}^{\infty} (h!)^{-1} f_{2^h}^{-1} ,$$

an example given by Mignotte [15]. (Here $\{f_h\}$ is again the Fibonacci sequence.) In [8], we remark upon the amusing result that we can even show that the numbers

$$\sum_{h=0}^{\infty} f_{f_h}^{-k} \quad (k = 1, 2, \ldots)$$

are algebraically independent. We can now show, generalising Mahler's result [13, 14], that if $\{u_h\}$ is an integer sequence satisfying one of a wide class of linear recurrence relations and the algebraic coefficients a_h satisfy a reasonable growth condition, then sums of the shape

$$\sum_{h=0}^{\infty} a_h f_{u_h}^{-k} \quad (k = 1, 2, \ldots)$$

are transcendental. In particular, one can take for the sequence $\{u_h\}$ the sequence $\{p_h\}$ or $\{q_h\}$ of numerators or denominators respectively of the convergents of an irrational real number. In a similar spirit, subject to certain rather technical conditions on the functions involved, our results yield the transcendence of infinite products such as

$$\prod_{h=0}^{\infty} \left(1+(h!)^{-1} z^{u_h}\right)$$

and of continued fractions such as

$$1 + \frac{z^{u_0}}{1+} \, \frac{z^{u_1}}{1+} \, \frac{z^{u_2}}{1+\ldots}$$

at algebraic points α with $0 < |\alpha| < 1$, u_h being a linear recurrence of the type alluded to above.

Mahler's results do not seem at all well-known and one of the few examples referred to in the literature is the transcendence of the sums

$$f_\omega(\alpha) = \sum_{h=1}^{\infty} [h\omega]\alpha^h ,$$

for ω a real quadratic irrational and α algebraic with $0 < |\alpha| < 1$. (As usual, $[x]$ denotes the integer part of x.) We shall extend this result of Mahler's by proving the transcendence of the sums $f_\omega(\alpha)$ for arbitrary real irrational ω. This example is note-worthy in that it displays uncountably many transcendental numbers in one-to-one correspondence with the real irrational numbers. The result depends on the following construction, leading to a chain of functional equations. We assume, as we may, that $0 < \omega < 1$ and write

$$F_\omega(z_1, z_2) = \sum_{h_1=1}^{\infty} \sum_{1 \le h_2 < h_1\omega} z_1^{h_1} z_2^{h_2}$$

and

$$\omega_k = \frac{1}{a_k +} \frac{1}{a_{k+1} + \dots} \qquad (k \ge 1) ,$$

so that $\omega_1 = \omega$ and $\omega_{k+1} = \omega_k^{-1} - a_k$. Then, by elementary manipulation of the sums, we obtain

$$\begin{aligned} F_{\omega_k}(z_1, z_2) &= -F_{1/\omega_k}(z_2, z_1) + \frac{z_1 z_2}{(1-z_1)(1-z_2)} \\ &= -F_{\omega_{k+1}}\left(z_1^{a_k} z_2, z_1\right) + \frac{z_1^{a_k+1} z_2}{\left(1-z_1^{a_k} z_2\right)(1-z_1)} . \end{aligned}$$

If ω is a quadratic irrational, the ω_k are periodic so the chain of functional equations yields a single functional equation and consequently this case falls within the ambit of Mahler's work [9]. For arbitrary irrational ω, we need the general result of the present paper, which permits us to infer the transcendence of $F_\omega(\alpha, \beta)$ for algebraic α, β in the domain of convergence of the series, providing the partial quotients a_k are bounded. In the contrary case, with the a_k unbounded, more direct methods already establish the required result.

The paper is divided into 3 parts, as follows. Chapter 1 contains a number of definitions and preliminary observations which specify the type of functional equations we are able to treat. The main transcendence theorem itself is stated and proved in Chapter 2. Finally, Chapter 3 contains applications of the general theorem to more concrete situations of the type described above and concluding with an analysis of Mahler's series $f_\omega(z)$.

1. Preliminary definitions

1. COHERENT SEQUENCES OF MATRICES

Let $T = \left(t_{ij}\right)$ be an $n \times n$ matrix with non-negative integer entries. As usual, we define the spectral radius of T , denoted by $r(T)$, to be the maximum of the absolute values of the eigenvalues of T . We further define a transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ as follows: if $z = \left(z_1, \ldots, z_n\right)$ is a point of $\mathbb{C}^n$, then $w = Tz$ is the point with coordinates

$$w_i = \prod_{j=1}^{n} z_j^{t_{ij}} \quad (1 \le i \le n) .$$

We adopt the usual vector notation for multi-indices, that is, if $\mu = \left(\mu_1, \ldots, \mu_n\right)$ and $\nu = \left(\nu_1, \ldots, \nu_n\right)$, then we write

$$|\mu| = |\mu_1| + \ldots + |\mu_n| ,$$

$$\langle \mu, \nu \rangle = \mu_1\nu_1 + \ldots + \mu_n\nu_n ,$$

and

$$z^\mu = z_1^{\mu_1} \ldots z_n^{\mu_n} \quad \left(z \text{ in } \mathbb{C}^n \right).$$

Thus, for example, we have

$$(Tz)^\mu = z^{\mu T} \quad \left(z \text{ in } \mathbb{C}^n \right).$$

Finally, we denote by $\mathbb{C}^{*n}$ the set of points $z = \left(z_1, \ldots, z_n\right)$ of $\mathbb{C}^n$ with $z_1 \ldots z_n \neq 0$.

Let $\mathsf{T} = \{T_1, T_2, \ldots\}$ be a sequence of $n \times n$ non-negative integer matrices and define its associated sequence of matrices $\mathsf{S} = \{S_1, S_2, \ldots\}$ by

$$S_k = T_k T_{k-1} \ldots T_1 \quad (k \geq 1) . \tag{1}$$

It is convenient to write

$$r_k = r(S_k) \quad (k \geq 1) . \tag{2}$$

We denote by $\mathcal{U}(\mathsf{T})$ the set of all points z in $\mathbb{C}^{*n}$ with the following property: there is a positive n-tuple η, depending only on the sequence T and the point z, such that

$$\log\left|\left(S_k z\right)^{\mu}\right| \sim -r_k\langle \mu, \eta\rangle \quad (k \to \infty) \tag{3}$$

for each integer n-tuple μ. We call the sequence T a coherent sequence of matrices if it satisfies the following 2 conditions:

(i) the sequence r_k is strictly increasing and $r_k \to \infty$ as $k \to \infty$, and

(ii) the set $\mathcal{U}(\mathsf{T})$ is a non-empty neighbourhood of the origin in $\mathbb{C}^{*n}$.

There is a significant case in which the above definition can be formulated quite explicitly. Consider the sequence $\mathsf{T} = \{T, T, \ldots\}$ consisting of the repetitions of a single $n \times n$ non-negative integer matrix T. By a theorem of Frobenius (see, for example, [5], page 80), $r(T)$ is itself an eigenvalue of T. If $r(T)$ is greater than 1 and also greater than the absolute values of all the other eigenvalues of T, and T has a positive eigenvector belonging to the eigenvalue $r(T)$, then $\mathsf{T} = \{T, T, \ldots\}$ is a coherent sequence of matrices. Indeed, in this case, $S_k = T^k$, so the condition (i) above is immediate and condition (ii) can be verified by elementary linear algebra (see [7], Lemma 4). In this case, we can describe the vector η in (3) as the projection of the vector $\left(-\log|z_1|, \ldots, -\log|z_n|\right)$ on the eigenspace of T belonging to the eigenvalue $r(T)$.

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2. THE FUNCTIONAL EQUATIONS

Let $\mathsf{T} = \{T_1, T_2, \ldots\}$ be a coherent sequence of matrices and let $\mathsf{F} = \{f_0(z), f_1(z), \ldots\}$ be a sequence of functions of the n complex variables $z = (z_1, \ldots, z_n)$, each $f_k(z)$ being regular in some fixed neighbourhood of the origin. We say the sequence of functions F satisfies a recursive system of functional equations if

$$(4) \qquad f_k(T_k z) = \sum_{j=0}^{g_k} a_{kj}(z) f_{k-1}(z)^j \Big/ \sum_{j=0}^{g_k} b_{kj}(z) f_{k-1}(z)^j \quad (k \geq 1) ,$$

where the $a_{kj}(z)$ and $b_{kj}(z)$ $(0 \leq j \leq g_k)$ are polynomials with degrees at most d_k $(d_k \geq 1)$. We denote by $\Delta_k(z)$ the resultant of the 2 forms

$$\sum_{j=0}^{g_k} a_{kj}(z) u^j v^{g_k - j} \quad \text{and} \quad \sum_{j=0}^{g_k} b_{kj}(z) u^j v^{g_k - j} ,$$

so that $\Delta_k(z)$ is also a polynomial in z. By hypothesis, the functions of the sequence F have power series expansions at the origin, so we can write

$$(5) \qquad f_k(z) = \sum_{\mu} \tau_{\mu}^{(k)} z^{\mu} \quad (k \geq 0) .$$

Let $\mathsf{S} = \{S_1, S_2, \ldots\}$ be the associated sequence of matrices of the sequence T, defined by (1), and set $r_k = r(S_k)$, as before. The functional equations (4) yield, by induction, the further equations

$$(6) \qquad f_k(S_k z) = \sum_{j=0}^{h_k} A_{kj}(z) f_0(z)^j \Big/ \sum_{j=0}^{h_k} B_{kj}(z) f_0(z)^j \quad (k \geq 1) ,$$

where

$$(7) \qquad h_k = g_1 g_2 \cdots g_k \quad (k \geq 1)$$

and the $A_{kj}(z)$ and $B_{kj}(z)$ $(0 \leq j \leq h_k)$ are polynomials with degrees at most $c_1 e_k$, where c_1 is a positive constant depending only on the sequence T and

(8) $$e_k = h_k\left\{\frac{d_1}{h_1} + \frac{d_2r_1}{h_2} + \ldots + \frac{d_kr_{k-1}}{h_k}\right\} \quad (k \ge 1) .$$

The resultant of the two forms

$$\sum_{j=0}^{h_k} A_{kj}(z)u^jv^{h_k-j} \quad \text{and} \quad \sum_{j=0}^{h_k} B_{kj}(z)u^jv^{h_k-j}$$

is easily seen to be $\Delta_1(z)\Delta_2(S_1z) \ldots \Delta_k(S_{k-1}z)$.

Now introduce the new variables $t = (t_\mu)$, indexed by n-tuples $\mu = (\mu_1, \ldots, \mu_n)$ of non-negative integers, and set

$$F(z;\ t) = \sum_\mu t_\mu z^\mu .$$

Thus, by (5), we have

(9) $$F\left(z;\ \tau^{(k)}\right) = f_k(z) \quad (k \ge 0) .$$

Consider a function $E(z;\ t)$ of the shape

$$E(z;\ t) = \sum_{j=0}^{\rho} P_j(z;\ t)F(z;\ t)^j = \sum_\mu p_\mu(t)z^\mu ,$$

where the $P_j(z;\ t)$ are polynomials in z and in finitely many of the variables t_μ and the series on the right is the power series expansion of $E(z;\ t)$ at the origin. We say the sequence of functions F is strongly transcendental if, for each function $E(z;\ t)$ formed in the above manner, there is a constant m with the following property: whenever k is a sufficiently large positive integer and the polynomials $P_j\left(z;\ \tau^{(k)}\right)$ are not all identically zero, then there is an index μ with $|\mu| \le m$ such that $p_\mu\left(\tau^{(k)}\right) \ne 0$.

Finally, let α be a point of C^{*n} . We say that the point α is admissible (more explicitly, admissible with respect to the sequence of matrices T and the system of functional equations (4)) if it has the following 3 properties:

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(i) α is in the neighbourhood $U(T)$,

(ii) $\Delta_k(S_{k-1}\alpha) \neq 0$ for each positive integer k , and

(iii) if η is the vector determined by (3) for $z = \alpha$, then the coordinates of η are linearly independent over $\mathbb{Q}$.

We remark that conditions (i) and (ii) of the last definition are "regularity" conditions. Thus (i) ensures that $S_k\alpha \to 0$ as $k \to \infty$, so that $f_k(S_k\alpha)$ is defined for all sufficiently large k , and (ii) ensures that the rational functions in the functional equations (6) are well-defined at α , whenever $f_0(\alpha)$ exists. Condition (iii) amounts to a condition of "independence" for the coordinates of α . In particular, by applying a result of Baker on the logarithms of algebraic numbers, it can be shown that (iii) holds if $\alpha_1, \ldots, \alpha_n$ are algebraic numbers and $|\alpha_1|, \ldots, |\alpha_n|$ are multiplicatively independent (see the proof of Theorem 2 in [7]).

2. The transcendence theorem

3. STATEMENT OF THE MAIN THEOREM

Let K be an algebraic number field of finite degree, d say, over $\mathbb{Q}$. For each β in K , we can find a non-zero rational integer den β , a denominator for β , such that $(\text{den}\ \beta)\beta$ is an algebraic integer. We measure the size of β by

$$\|\beta\| = \max_{\sigma} \{|\sigma\beta|, |\text{den}\ \beta|\} ,$$

where σ runs through the d distinct embeddings of K into $\mathbb{C}$. For a polynomial $p(z) = \sum p_\mu z^\mu$ with coefficients in K , we define

$$\|p\| = \max_{\mu} \|p_\mu\| .$$

We can now formulate the main transcendence theorem.

THEOREM 1. *Let* $\mathrm{T} = \{T_1, T_2, \ldots\}$ *be a coherent sequence of matrices. Let* $\mathrm{F} = \{f_0(z), f_1(z), \ldots\}$ *be a strongly transcendental sequence of functions, each one being regular in some neighbourhood of the*

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origin and satisfying the recursive system of functional equations (4), *and suppose that all the coefficients* $\tau_{\mu}^{(k)}$ *of the power series expansions* (5) *of the* $f_k(z)$ *and all the coefficients of the polynomials* $a_{kj}(z)$ *and* $b_{kj}(z)$ *in the functional equations* (4) *lie in some fixed algebraic number field. Let* r_k, h_k , *and* e_k *be given by* (2), (7), *and* (8). *Assume that there is a positive constant* c_2 , *independent of* k , *such that*

$$h_k + e_k + \max_{0 \le j \le g_{k+1}} \{\log\|a_{k+1,j}\|, \log\|b_{k+1,j}\|\} \le c_2 r_k , \tag{10}$$

and also that, for each $\varepsilon > 0$, *there is a positive number* $C_1(\varepsilon)$ *such that*

$$\log\left\|\tau_{\mu}^{(k)}\right\| \le \varepsilon r_k(1+|\mu|) \tag{11}$$

for all non-negative integer n*-tuples* μ , *whenever* $k \ge C_1(\varepsilon)$. *Finally, let* α *be an admissible algebraic point. If* $f_0(\alpha)$ *exists, then* $f_0(\alpha)$ *is transcendental.*

The theorem generalises the main theorem of [7] (see [7], Theorem 1 and Lemma 11). Indeed, if the sequences $\mathsf{T} = \{T, T, \ldots\}$ and $\mathsf{F} = \{f(z), f(z), \ldots\}$ each consist of repetitions of a single entity, then the system of functional equations (4) becomes a single functional equation for $f(z)$ and the hypothesis of strong transcendence for the sequence F becomes the ordinary transcendence of the function $f(z)$, and this is just the situation we treat in [7]. The proof of the theorem depends, as in our earlier work, on the construction of a suitable auxiliary function which we shall show has properties incompatible with the assumption that $f_0(\alpha)$ is algebraic. This programme is carried out in the next 4 sections.

4. THE AUXILIARY FUNCTION

Throughout the remainder of this chapter, we assume that the sequence of matrices T , the sequence of functions F , and the point α satisfy all the requirements of Theorem 1 and we assume, in addition, that $f_0(\alpha)$ is algebraic. Let K be an algebraic number field of finite degree, d

say, over $\mathbb{Q}$ which contains all the coefficients $\tau_\mu^{(k)}$ of the power series expansions (5), all the coefficients of the polynomials appearing in the system of functional equations (4), the coordinates of the point α, and the number $f_0(\alpha)$. In the following work $c_1, c_2, \ldots$ denote positive constants depending only on the quantities introduced above and $C_1(\varepsilon), C_2(\varepsilon), \ldots$ denote positive constants which may in addition depend on the parameter ε. In particular, these constants do not depend on the parameters k and ρ which will appear shortly.

For any $m \geq 0$, we denote by $P(m)$ the ring of polynomials in the finitely many variables t_μ with $|\mu| \leq m$ and with coefficients in K.

LEMMA 1. *There is an infinite sequence N of positive integers such that the following 2 assertions for a polynomial $p(t)$ in $P(m)$ are equivalent:*

(i) $p\left(\tau^{(k)}\right) = 0$ *for infinitely many* k *in* N; *and*

(ii) $p\left(\tau^{(k)}\right) = 0$ *for all* k *in* N.

Proof. Let N_1 be the sequence of all positive integers and I_1 be the ideal of $P(m)$ comprising those polynomials $p(t)$ such that $p\left(\tau^{(k)}\right) = 0$ for all k in N_1. If there is no polynomial $p(t)$ in $P(m)\backslash I_1$ such that $p\left(\tau^{(k)}\right) = 0$ for infinitely many k in N_1, the construction stops. If there is such a polynomial, $p_1(t)$ say, then we let N_2 be the sequence of indices k in N_1 such that $p_1\left(\tau^{(k)}\right) = 0$, and we let I_2 be the ideal of $P(m)$ comprising those polynomials $p(t)$ such that $p\left(\tau^{(k)}\right) = 0$ for all k in N_2. Continuing the construction in the obvious way, we obtain a strictly ascending chain $I_1 \subset I_2 \subset \ldots$ of ideals of $P(m)$. Since $P(m)$ is noetherian, the construction necessarily terminates when we reach some ideal I_l and the corresponding sequence N_l clearly has the property required in the lemma.

Let $N(m)$ be some infinite sequence of positive integers with the

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property described in Lemma 1 and let $I(m)$ be the ideal of polynomials of $P(m)$ satisfying the equivalent conditions *(i)* and *(ii)* of the lemma. In general, there are many choices for $N(m)$, but having chosen one, we keep it fixed for the rest of the discussion. Clearly, $I(m)$ is a prime ideal of $P(m)$ and $I(m)$ is not the whole of $P(m)$, since it does not contain the constant polynomials. Thus $P(m)/I(m)$ is a non-trivial integral domain.

The next lemma gives the construction of the auxiliary function $E_\rho(z;\, t)$.

LEMMA 2. *For each positive integer* $\rho \ge c_3$, *there are* $\rho + 1$ *polynomials* $P_0(z;\, t), \ldots, P_\rho(z;\, t)$ *which, considered as polynomials in* z, *have degrees at most* ρ *in each variable and coefficients in* $P\left(\rho^{1+1/n}\right)$, *such that the function*

$$E_\rho(z;\, t) = \sum_{j=0}^{\rho} P_j(z;\, t)F(z;\, t)^j = \sum_\mu p_\mu(t)z^\mu \tag{12}$$

has the following 2 properties:

(i) all the coefficients $p_\mu(t)$ *with* $|\mu| \le \tfrac{1}{2}\rho^{1+1/n}$ *are in* $I\left(\rho^{1+1/n}\right)$; *and*

(ii) the function $E_\rho\left(z;\, \tau^{(k)}\right)$ *is not identically zero for all sufficiently large* k *in* $N\left(\rho^{1+1/n}\right)$.

Proof. Set $m = \rho^{1+1/n}$. We treat $P_0(z;\, t), \ldots, P_\rho(z;\, t)$ as polynomials in z whose coefficients are in $P(m)/I(m)$. With this interpretation, the $\rho + 1$ polynomials $P_j(z;\, t)$ together possess $(\rho+1)^{n+1}$ coefficients. Moreover, the $p_\mu(t)$ defined by (12) are polynomials in t and, for $|\mu| \le m$, we have $p_\mu(t)$ in $P(m)/I(m)$. So the requirement *(i)* of the lemma gives at most $\left(\tfrac{1}{2}\rho^{1+1/n}+1\right)^n$ homogeneous linear equations in the coefficients of the polynomials $P_j(z;\, t)$. If ρ is sufficiently large, the number of equations is less than the total number of coefficients, so the system has a non-trivial solution in the domain $P(m)/I(m)$. This achieves *(i)*. The construction also ensures that the polynomials $P_j\left(z;\ \tau^{(k)}\right)$ are not all identically zero for each sufficiently large k in $N(m)$. This remark, together with the hypothesis of strong transcendence of the sequence F , gives *(ii)*.

5. AN UPPER BOUND FOR $\left|E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)\right|$

LEMMA 3. *Let* S_k, r_k , *and* $\tau^{(k)}$ *be the quantities defined in* (1), (2), *and* (5), *and let* $E_\rho(z;\ t)$ *be the function constructed in Lemma* 2. *Then*

$$\log\left|E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)\right| \le -c_4 r_k \rho^{1+1/n}$$

whenever $\rho \ge c_3$ *and* k *is in* $N\left(\rho^{1+1/n}\right)$ *and sufficiently large compared to* ρ .

Proof. As in the proof of Lemma 2, we write

$$E_\rho(z;\ t) = \sum_\mu p_\mu(t) z^\mu ,$$

where the coefficients $p_\mu(t)$ are polynomials in t whose degrees can be bounded in terms of ρ alone. Let μ be an n-tuple of non-negative integers. By (3) and (11), we have

$$\log\left|\left(S_k\alpha\right)^\mu\right| \le -c_5 r_k|\mu| \quad\text{and}\quad \log\left|\tau_\mu^{(k)}\right| \le \varepsilon r_k(1+|\mu|)$$

for any $\varepsilon > 0$ and for all sufficiently large k . Hence

$$\log\left|p_\mu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\mu\right| \le -c_6 r_k(1+|\mu|) ,$$

whenever k is sufficiently large compared to ρ and $p_\mu\left(\tau^{(k)}\right) \ne 0$. Thus the series for $E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)$ is convergent for all k sufficiently large compared to ρ and, by the construction of Lemma 2,

$$\log\left|E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)\right| \le -c_4 r_k \rho^{1+1/n} ,$$

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whenever $\rho \geq c_3$ and k is in $N\left(\rho^{1+1/n}\right)$ and sufficiently large compared to ρ.

6. AN UPPER BOUND FOR $\left\|E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)\right\|$

Let $p(z) = \sum p_\mu z^\mu$ be a polynomial with coefficients in the field K. We say the polynomial $q(z) = \sum q_\mu z^\mu$ dominates $p(z)$, written $p(z) \prec q(z)$, if all the coefficients of $q(z)$ are rational integers and $\|p_\mu\| \leq q_\mu$ for each μ.

LEMMA 4. *Let* S_k *and* r_k *be the quantities defined in* (1) *and* (2). *Then*

$$\log\|f_k(S_k\alpha)\| \leq c_7 r_k \quad (k \geq 0) .$$

Proof. From the functional equations (4), we can write $f_k(z) = G_k(z)/H_k(z)$, where $G_k(z)$ and $H_k(z)$ are regular in some neighbourhood of the origin and satisfy

$$\begin{aligned} G_k\left(T_k z\right) &= \sum_{j=0}^{g_k} a_{kj}(z) G_{k-1}(z)^j H_{k-1}(z)^{g_k-j} , \\ H_k\left(T_k z\right) &= \sum_{j=0}^{g_k} b_{kj}(z) G_{k-1}(z)^j H_{k-1}(z)^{g_k-j} , \end{aligned} \tag{13}$$

and in addition, $G_0(\alpha)$ and $H_0(\alpha)$ are in K. From (6), we obtain a similar pair of equations for $G_k\left(S_k z\right)$ and $H_k\left(S_k z\right)$ as forms in $G_0(z)$ and $H_0(z)$, whose resultant is $\Delta_1(z)\Delta_2\left(S_1 z\right) \ldots \Delta_k\left(S_{k-1} z\right)$. Since α is an admissible point, it follows that $G_k\left(S_k\alpha\right)$ and $H_k\left(S_k\alpha\right)$ cannot both be zero. Set

$$P(z) = \prod_{j=1}^{n} \left(1+z_j\right) .$$

We note that $P\left(S_k z\right)$ is a polynomial of degree at most $c_8 r_k$, so

$$(14) \qquad P(S_k z) < P(z)^{c_8 r_k} .$$

Now, from (10) and (13),

$$G_k(S_k z),\ H_k(S_k z) < \tfrac{1}{2} c_9^{r_{k-1}} P(S_{k-1} z)^{d_k} \{G_{k-1}(S_{k-1} z) + H_{k-1}(S_{k-1} z)\}^{g_k} ,$$

where we regard $G_k(S_k z)$ and $H_k(S_k z)$ as polynomials in z, $G_{k-1}(S_{k-1} z)$ and $H_{k-1}(S_{k-1} z)$. By induction on k and repeated use of the inequality (14), we obtain

$$G_k(S_k z),\ H_k(S_k z) < \tfrac{1}{2} c_9^{s_k} P(z)^{c_8 e_k} \{G_0(z) + H_0(z)\}^{h_k} ,$$

where we have written

$$s_k = h_k \left\{ \frac{1}{h_1} + \frac{r_1}{h_2} + \ldots + \frac{r_{k-1}}{h_k} \right\} .$$

Using (10) and recalling the assumption $d_k \geq 1$, we get

$$G_k(S_k z),\ H_k(S_k z) < \tfrac{1}{2} \{c_9 P(z) (G_0(z) + H_0(z))\}^{c_{10} r_k}$$

and finally,

$$\log \| f_k(S_k \alpha) \| \leq \log \| G_k(S_k \alpha) \| + \log \| H_k(S_k \alpha) \| \leq c_7 r_k .$$

LEMMA 5. *Let* S_k, r_k, *and* $\tau^{(k)}$ *be the quantities defined in* (1), (2), *and* (5), *and let* $E_\rho(z;\, t)$ *be the function constructed in Lemma* 2. *For each* $\varepsilon > 0$,

$$\log \left\| E_\rho \left(S_k \alpha;\ \tau^{(k)} \right) \right\| \leq \varepsilon r_k \rho^{1+1/n} ,$$

whenever $\rho \geq C_2(\varepsilon)$ *and* k *is sufficiently large compared to* ρ .

Proof. Let $P_j(z;\, t)$ $(0 \leq j \leq \rho)$ be the polynomials constructed in Lemma 2 and set $m = \rho^{1+1/n}$. From Lemma 2, each $P_j(z;\, t)$ is a polynomial in z of degree at most ρ and has coefficients which are polynomials of degrees at most $\sigma = \sigma(\rho)$, say, in the variables t_μ with

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$|\mu| \leq m$. Thus

$$P_j(z;\ t) \prec \|P_j\| \prod_{j=1}^{n} (1+z_j)^{\rho} \prod_{|\mu|\leq m} (1+t_\mu)^{\sigma} \quad (0 \leq j \leq \rho) .$$

On using (11) and (14), we see that, for each $\varepsilon > 0$,

(15) $$\log\left\|P_j\left(S_k\alpha;\ \tau^{(k)}\right)\right\| \leq \varepsilon r_k \rho^{1+1/n} \quad (0 \leq j \leq \rho) ,$$

whenever $\rho \geq C_3(\varepsilon)$ and k is sufficiently large compared to ρ . The assertion of the lemma follows at once from (15) and Lemma 4.

7. COMPLETION OF THE PROOF OF THEOREM 1

Let d be the degree of the algebraic number field K described in Section 4. If β is a non-zero algebraic number in K , we have the fundamental inequality

(16) $$\log|\beta| \geq -2d \log\|\beta\| ,$$

which follows easily from the observation that the norm of the algebraic integer $(\text{den}\ \beta)\beta$ has absolute value at least 1 (see, for example, [*22*], page 6). We shall apply the fundamental inequality to the number $E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)$, but before doing so, we need the following lemma.

LEMMA 6. *Let* S_k, r_k , *and* $\tau^{(k)}$ *be the quantities defined in* (1), (2), *and* (5), *and let* $E_\rho(z;\ t)$ *be the function constructed in Lemma* 2. *Then* $E_\rho\left(S_k\alpha;\ \tau^{(k)}\right) \neq 0$ *for infinitely many* k *in* $N\left(\rho^{1+1/n}\right)$.

Proof. From (12), we have

(17) $$E_\rho\left(S_k\alpha;\ \tau^{(k)}\right) = \sum_\mu p_\mu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\mu$$

and, as in the proof of Lemma 3, the series is convergent whenever k is sufficiently large compared to ρ . Choose such a k which is also in the sequence $N\left(\rho^{1+1/n}\right)$. Then, from *(ii)* of Lemma 2, the $p_\mu\left(\tau^{(k)}\right)$ are not all zero. Since α is an admissible point, it satisfies (3) for some suitable positive n-tuple η whose coordinates are linearly independent over $\mathbb{Q}$. We can therefore pick a non-zero term $p_\nu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\nu$ of the

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series (17) such that

$$\langle \nu, \eta \rangle = \min_{\mu} \left\{ \langle \mu, \eta \rangle : p_\mu\left(\tau^{(k)}\right) \neq 0 \right\} .$$

Here ν depends on k ; but by the hypothesis of strong transcendence, ν takes only finitely many different values as k runs through $N\left(\rho^{1+1/n}\right)$. By restricting k to a suitable infinite subsequence, N say, we can suppose that the index ν defined above is independent of k . From the construction of Lemma 2, $p_\nu(t)$ is a polynomial in the variables t_μ with $|\mu| \le |\nu| + \rho^{1+1/n}$ and its degree in each t_μ can be bounded in terms of ρ alone. So by (11), for each $\varepsilon > 0$,

$$\log\left\|p_\nu\left(\tau^{(k)}\right)\right\| \le \varepsilon r_k |\nu|$$

providing $\rho \ge C_4(\varepsilon)$ and k is sufficiently large compared to ρ . From this estimate, together with (16) and (3), we obtain, for each $\varepsilon > 0$,

$$-(1+\varepsilon) r_k \langle \nu, \eta \rangle < \log\left|p_\nu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\nu\right| < -(1-\varepsilon) r_k \langle \nu, \eta \rangle$$

providing $\rho \ge C_5(\varepsilon)$ and k is in N and is sufficiently large compared to ρ . Similarly, if $p_\mu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\mu$ is any other non-zero term of the series (17), then, under the same conditions,

$$\log\left|p_\mu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\mu / p_\nu\left(\tau^{(k)}\right)\left(S_k\alpha\right)^\nu\right| < -(1-\varepsilon) r_k \langle \mu-\nu, \eta \rangle .$$

Now $\langle \mu-\nu, \eta \rangle > 0$ by the choice of ν , so it follows that for all sufficiently large k in N , the series (17) has a single dominant term, and this establishes the lemma.

We can now complete the proof of Theorem 1. In our previous construction, we choose $\varepsilon < c_4/2d$ and $\rho > \max\{c_3, C_2(\varepsilon)\}$, where c_3, c_4 , and $C_2(\varepsilon)$ are the constants appearing in Lemmas 3 and 5. By combining the results of these lemmas, we then obtain

$$\log\left|E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)\right| < -2d \log\left\|E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)\right\| \tag{18}$$

for all sufficiently large k in $N\left(\rho^{1+1/n}\right)$. On the other hand,

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$E_\rho\left(S_k\alpha;\ \tau^{(k)}\right)$ is an algebraic number in K and it is non-zero for infinitely many k in $N\left(\rho^{1+1/n}\right)$, by Lemma 6, so the inequalities (16) and (18) are incompatible. This contradiction shows that $f_0(\alpha)$ is transcendental and establishes the theorem.

3. Applications of the transcendence theorem

8. LACUNARY POWER SERIES

Let $f(z) = \sum a_h z^{u_h}$ be a power series in the complex variable z with a sufficiently rapidly increasing sequence of exponents u_h. Cijsouw and Tijdeman [3], generalising results of several previous authors, show that if $u_{h+1}/u_h \to \infty$ and the coefficients a_h satisfy certain reasonable conditions, then $f(z)$ is transcendental for any algebraic number z inside the circle of convergence. Analogous results involving series for which the ratio u_{h+1}/u_h of successive exponents does not tend to ∞ are much harder to find and depend on special properties of the series. For example, Schneider ([18], page 35) applies the Thue-Siegel-Roth Theorem to show that the Fredholm series

$$g(z) = \sum_{h=0}^{\infty} z^{2^h}$$

is transcendental if $z = a/b$ is a non-zero rational number with $|z| < b^{-\frac{1}{2}-\varepsilon}$ $(\varepsilon > 0)$. However, as described in the introduction, Mahler [9] proved much more, showing that $g(z)$ is transcendental for any algebraic z with $0 < |z| < 1$. Recently, Mahler [13, 14] has extended his method to establish the transcendency of such series as

$$h(z) = \sum_{h=0}^{\infty} (h!)^{-1} z^{2^h}$$

for any algebraic z with $0 < |z| < 1$, but the allowable coefficients are still heavily restricted. Our Theorem 1 yields a considerably wider class of lacunary series which includes all Mahler's examples and complements those of Cijsouw and Tijdeman [3].

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THEOREM 2. *Let* $\{u_0, u_1, \ldots\}$ *be a strictly increasing sequence of positive integers with* $u_h | u_{h+1}$ *for* $h \geq 0$. *Let* $\{a_0(z), a_1(z), \ldots\}$ *be a sequence of polynomials whose coefficients all lie in a fixed algebraic number field and denote the degree of* $a_h(z)$ *by* s_h . *Suppose that* $s_h + \log\|a_h\| = o(u_h)$ *as* $h \to \infty$. *If* α *is an algebraic number with* $0 < |\alpha| < 1$ *and* $a_h(\alpha)$ *is non-zero for infinitely many* h , *then the number*

$$\sum_{h=0}^{\infty} a_h(\alpha)\alpha^{u_h}$$

is transcendental.

We deduce that the hypotheses of the theorem imply that the function $f(z) = \sum a_h(z)z^{u_h}$ is a transcendental function. Of course, this conclusion follows more directly from the Hadamard gap theorem (see [4], page 231), which shows that $f(z)$ has the circle $|z| = 1$ as a natural boundary.

Proof. By a suitable change of notation, we may suppose that $a_h(\alpha) \neq 0$ for all h and that $u_0 = 1$. If the quotients u_{h+1}/u_h are unbounded, the assertion of the theorem follows from the results of Cijsouw and Tijdeman (see [3], Section 5, Remark (v)), since the hypotheses of the theorem ensure that there is no way of introducing brackets into the series $\sum a_h(\alpha)\alpha^{u_h}$ which will reduce it to a finite sum. We can therefore assume that the quotients u_{h+1}/u_h are bounded. We apply Theorem 1 to the functions

$$f_k(z) = \sum_{h=k}^{\infty} a_h(\alpha)z^{u_h/u_k} \quad (k \geq 0)$$

which satisfy the system of functional equations

$$f_k\left(z^{u_k/u_{k-1}}\right) = f_{k-1}(z) - a_{k-1}(\alpha)z \quad (k \geq 1) .$$

To verify the strong transcendence of the functions $f_k(z)$, consider, in the notation of Section 2, a polynomial

$$E\left(z;\ \tau^{(k)}\right) = \sum_{j=0}^{\rho} P_j\left(z;\ \tau^{(k)}\right) f_k(z)^j$$

and suppose that the $P_j\left(z;\ \tau^{(k)}\right)$ are not all identically zero. Now $f_k(z)^j$ is a power series in which the exponents of the non-vanishing terms can be expressed uniquely in the shape

$$\left(\varepsilon_1 u_{h_1} + \ldots + \varepsilon_l u_{h_l}\right)/u_k \quad \left(1 \le l \le j,\ h_1 < \ldots < h_l,\ 1 \le \varepsilon_i < u_{h_{i+1}}/u_{h_i}\right) .$$

On taking $h_1 < h_2 < \ldots < h_\rho$ with $h_1 - k$ sufficiently large, we get a non-zero term of $f_k(z)^\rho$ with a large gap in both directions to the nearest term of any of the series $f_k(z)^j$ $(0 \le j \le \rho)$. From this remark and the boundedness of u_{h+1}/u_h , it follows that there is an integer m , independent of k , such that $E\left(z;\ \tau^{(k)}\right)$ has a non-zero term with exponent at most m . The remaining hypotheses of Theorem 1 are readily checked and so $f_0(\alpha)$ is transcendental.

By introducing functions of several complex variables, we can prove the transcendence of certain lacunary series of the shape $\sum a_h z^{u_h}$ in which the ratio u_{h+1}/u_h of successive exponents is arbitrarily close to 1 .

THEOREM 3. *Let* $\{u_0,\ u_1,\ \ldots\}$ *be a sequence of positive integers satisfying a linear recurrence* $u_{h+n} = t_1 u_{h+n-1} + \ldots + t_n u_h$ *with* $h \ge 0$, *where the* t_j $(1 \le j \le n)$ *are non-negative integers, and suppose that the characteristic polynomial* $\lambda^n - t_1\lambda^{n-1} - \ldots - t_n$ *is irreducible and that its largest root,* r *say, is greater than* 1 *and greater than the absolute values of all its other roots. Let* $\{a_0(z),\ a_1(z),\ \ldots\}$ *be a sequence of polynomials in* $z = \left(z_1,\ \ldots,\ z_n\right)$, *whose coefficients all lie in a fixed algebraic number field, and denote the degree of* $a_h(z)$ *by*

s_h. *Suppose that* $s_h + \log\|a_h\| = o\left(r^h\right)$ *as* $h \to \infty$. *If* $\alpha = \left(\alpha_1, \ldots, \alpha_n\right)$ *is an algebraic point of* $\mathbb{C}^{*n}$ *such that*

$$\log|\alpha_1| + r \log|\alpha_2| + \ldots + r^{n-1} \log|\alpha_n| < 0 ,$$

and $a_h(\alpha) \neq 0$ *for all* h, *then the number*

$$\sum_{h=0}^{\infty} a_h(\alpha)\alpha_1^{u_h} \ldots \alpha_n^{u_{h+n-1}}$$

is transcendental.

In particular, if α_1 *is an algebraic number such that* $0 < |\alpha_1| < 1$ *and* $a_h\left(\alpha_1, 1, \ldots, 1\right) \neq 0$ *for all* h, *then the number*

$$\sum_{h=0}^{\infty} a_h\left(\alpha_1, 1, \ldots, 1\right)\alpha_1^{u_h}$$

is transcendental.

Proof. Set $\mu = \left(u_0, u_1, \ldots, u_{n-1}\right)$ and

$$T = \begin{bmatrix} 0 & 0 & \ldots & 0 & t_n \\ 1 & 0 & \ldots & 0 & t_{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & t_2 \\ 0 & 0 & \ldots & 1 & t_1 \end{bmatrix} .$$

We apply Theorem 1 to the functions

$$f_k(z) = \sum_{h=k}^{\infty} a_h(\alpha) z^{\mu T^{h-k}} \quad (k \geq 0)$$

of the n complex variables $z = \left(z_1, \ldots, z_n\right)$, which satisfy the system of functional equations

$$f_k(Tz) = f_{k-1}(z) - a_{k-1}(\alpha) z^{\mu} \quad (k \geq 1) .$$

As shown in the proof of Theorem 3 of [7], the sequence of matrices

$\mathsf{T} = \{T, T, \ldots\}$ is coherent and it is easy to see that the neighbourhood $U(T)$ comprises the points z satisfying

$$\log|z_1| + r\log|z_2| + \ldots + r^{n-1}\log|z_n| < 0 ,$$

and that the functions $f_k(z)$ converge for z in $U(\mathsf{T})$. To show that the functions $f_k(z)$ are strongly transcendental, we proceed as in the proof of Theorem 2, observing that $f_k(z)$, considered as a power series in z_1, has large regular gaps. The application of Theorem 1 is now readily justified and the assertions of the theorem follow.

The special linear recurrence of Theorem 3 can be replaced by a recurrence of a much wider class. To see this, we note that Theorem 1 can be applied in a similar way to functions of the more general shape

$$g_k(z) = \sum_{h=k}^{\infty} a_h(z) z^{\mu_h S_h S_k^{-1}} ,$$

where the matrices S_h belong to the associated sequence of a coherent sequence of matrices. This example falls into the pattern of Theorem 4 below.

9. MORE GENERAL FUNCTIONS WITH GAPS

In the previous section, we applied Theorem 1 to lacunary power series. We now give applications to infinite products and continued fractions with "gaps". For this purpose, we introduce the following notation and hypotheses which are assumed in the statements of Theorems 4 to 6 below.

Let $\mathsf{T} = \{T_1, T_2, \ldots\}$ be a coherent sequence of $n \times n$ matrices. Let $\{\phi_0(z), \phi_1(z), \ldots\}$ be a sequence of rational functions of the n complex variables $z = (z_1, \ldots, z_n)$, each one being regular in some neighbourhood of the origin and with all their coefficients lying in a fixed algebraic number field. Denote by d_h the maximum of the degrees of the numerator and denominator of $\phi_h(z)$ and by $\|\phi_h\|$ the maximum size of the coefficients of $\phi_h(z)$. Let S_k and r_k be the quantities defined

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by (1) and (2). Assume that there is a positive constant c such that

$$d_1r_1 + d_2r_2 + \ldots + d_{h-1}r_{h-1} \le cr_h \quad (h \ge 1) ,$$

and that $\log\|\phi_h\| = o(r_h)$ as $h \to \infty$. Finally, let α be an algebraic point of $U(T)$ such that the coordinates of the vector η determined by (3) for $z = \alpha$ are linearly independent over $\mathbb{Q}$.

THEOREM 4. *Suppose* $\phi_h(0) = 0 \quad (h \ge 0)$ *and consider the functions*

$$f_k(z) = \sum_{h=k}^{\infty} \phi_h\left(S_hS_k^{-1}z\right) \quad (k \ge 0)$$

which satisfy the system of functional equations

$$f_k\left(T_kz\right) = f_{k-1}(z) - \phi_{k-1}(z) \quad (k \ge 1) .$$

If the $f_k(z)$ *are strongly transcendental, then the number* $f_0(\alpha)$ *is transcendental.*

THEOREM 5. *Suppose* $\phi_h(0) = 1$ *and* $\phi_h\left(S_h\alpha\right) \ne 0 \quad (h \ge 0)$ *and consider the functions*

$$f_k(z) = \prod_{h=k}^{\infty} \phi_h\left(S_hS_k^{-1}z\right) \quad (k \ge 0)$$

which satisfy the system of functional equations

$$f_k\left(T_kz\right) = \phi_{k-1}(z)^{-1}f_{k-1}(z) \quad (k \ge 1) .$$

If the $f_k(z)$ *are strongly transcendental, then the number* $f_0(\alpha)$ *is transcendental.*

THEOREM 6. *Suppose* $\phi_h(0) = 0$ *and* $\phi_h\left(S_h\alpha\right) \ne 0 \quad (h \ge 0)$ *and consider the functions*

$$f_k(z) = 1 + \frac{\phi_k(z)}{1+}\,\frac{\phi_{k+1}\left(T_{k+1}z\right)}{1+}\,\frac{\phi_{k+2}\left(T_{k+2}T_{k+1}z\right)}{1+\ldots} \qquad (k \ge 0)$$

which satisfy the system of functional equations

$$f_k\left(T_kz\right) = \phi_{k-1}(z)/\left\{f_{k-1}(z)-1\right\} \quad (k \ge 1) .$$

If the $f_k(z)$ *are strongly transcendental, then the number* $f_0(\alpha)$ *is transcendental.*

The proofs of the theorems are straightforward applications of Theorem 1. To estimate the size of the coefficients of the power series expansions of the continued fractions occurring in Theorem 6, we may use the observations in Wall [23], pages 40-43.

By specialising the schema of Theorems 4 and 5, we can obtain many interesting examples, including some of the examples of Mahler [9], [13, 14], Mignotte [15], and Schwarz [19]. We have not found any results in the literature in the pattern of Theorem 6, although Scott and Wall [21] have considered the problem of deciding when continued fractions of this type represent transcendental functions and have obtained some partial results.

10. THE SERIES $\sum [h\omega]z^h$ AND RELATED EXAMPLES

As a final application of Theorem 1, we shall prove the following theorem.

THEOREM 7. *Suppose* $p(x)$ *is a non-constant polynomial with algebraic coefficients and* ω *is a real irrational number. Then the power series*

$$\sum_{h=1}^{\infty} p([h\omega])z^h \tag{19}$$

takes a transcendental value for any algebraic number z *with* $0 < |z| < 1$.

Special cases of the series (19) have been treated by various authors. Thus Böhmer [1] showed that if g is a positive integer with $g \geq 2$ and the partial quotients of the simple continued fraction expansion of ω are unbounded, then

$$\sum_{h=1}^{\infty} [h\omega]g^{-h}$$

is a Liouville number and hence transcendental. His method has recently been extended by Wallisser [24] to cover the series

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$$\sum_{h=1}^{\infty} f([h\omega])g^{-h} ,$$

where $\{f(h)\}$ is a non-constant periodic sequence of rational integers and g is a positive integer with $g > \max|f(h)|$. As a counterpart to Böhmer's result, Mahler [9] proved that if ω is a quadratic irrational, then the series

$$\sum_{h=1}^{\infty} [h\omega]z^h \tag{20}$$

is transcendental for any algebraic z with $0 < |z| < 1$. Much more is known about the functional properties of these and related series. Such questions were apparently first investigated by Hecke [6] who showed that for any irrational ω the series (20) has the unit circle as a natural boundary. More generally, by combining the results of Newman [16] and Petersson [17], we find that under the hypotheses of Theorem 7, the series (19) has the unit circle as a natural boundary. The subsequent developments in this area can be traced from the survey article of Schwarz [20].

We follow Mahler in deducing Theorem 7 from the following result for a related function of 2 complex variables.

THEOREM 8. *Let* ω *be an irrational number with* $0 < \omega < 1$ *and let* $P(x, y)$ *be a polynomial with algebraic coefficients and not identically zero. Denote the convergents of the simple continued fraction expansion of* ω *by* p_h/q_h $(h \ge 0)$. *If* α *and* β *are algebraic numbers satisfying*

$$\alpha\beta \neq 0 , \quad \alpha^{q_h}\beta^{p_h} \neq 1 \quad (h \ge 1) , \text{ and } \log|\alpha| + \omega \log|\beta| < 0 ,$$

then

$$\sum_{h_1=1}^{\infty} \sum_{1 \le h_2 < h_1\omega} P(h_1, h_2)\alpha^{h_1}\beta^{h_2}$$

is transcendental.

We remark that the restriction $0 < \omega < 1$ is convenient, but clearly not essential. So Theorem 7 follows from Theorem 8 with $\beta = 1$ and the appropriate choice for the polynomial $P(x, y)$. We give the proof of

Theorem 8 in the next section.

11. PROOF OF THEOREM 8

We require some notation from the theory of continued fractions, which we normalise as in Cassels [2], Chapter 1. Suppose $0 < \omega < 1$ and denote by $\{a_1, a_2, \ldots\}$ the sequence of partial quotients of the simple continued fraction of ω. Set

(21) $$\omega_k = \frac{1}{a_k+}\,\frac{1}{a_{k+1}+\ldots} \qquad (k \geq 1),$$

so that $\omega_1 = \omega$ and $\omega_{k+1} = \omega_k^{-1} - a_k$. The convergents p_k/q_k of ω are determined by

$$p_0 = 1, \quad q_0 = 0; \quad p_1 = 0, \quad q_1 = 1;$$

$$p_{k+1} = a_k p_k + p_{k-1}, \quad q_{k+1} = a_k q_k + q_{k-1} \quad (k \geq 1),$$

and we have

(22) $$\frac{p_{k+1}}{q_{k+1}} = \frac{1}{a_1+}\,\frac{1}{a_2+\ldots}\,\frac{1}{a_k}, \quad \frac{q_k}{q_{k+1}} = \frac{1}{a_k+}\,\frac{1}{a_{k-1}+\ldots}\,\frac{1}{a_1}$$

We begin the proof by considering the function

$$F_\omega(z_1, z_2) = \sum_{h_1=1}^{\infty} \sum_{1 \leq h_2 < h_1\omega} z_1^{h_1} z_2^{h_2},$$

which clearly converges in the domain

(23) $$\log|z_1| + \omega \log|z_2| < 0 .$$

Since ω is irrational, any pair of positive integers (h_1, h_2) satisfies just one of the inequalities $1 \leq h_2 < h_1\omega$ or $1 \leq h_1 < h_2\omega^{-1}$, so we have

(24) $$F_\omega(z_1, z_2) + F_{1/\omega}(z_2, z_1) = \sum_{h_1, h_2=1}^{\infty} z_1^{h_1} z_2^{h_2} = \frac{z_1 z_2}{(1-z_1)(1-z_2)} .$$

Next, for any integer a with $0 \leq a < \omega$, we have

$$(25)\qquad F_\omega(z_1, z_2) = \sum_{h_1=1}^{\infty}\left\{\sum_{h_2=1}^{h_1 a} z_1^{h_1} z_2^{h_2} + \sum_{1\le h_2<h_1(\omega-a)} z_1^{h_1} z_2^{h_2+h_1 a}\right\}$$

$$= \frac{z_1 z_2}{(1-z_1)(1-z_2)} - \frac{z_1 z_2^{a+1}}{\left(1-z_1 z_2^a\right)(1-z_2)} + F_{\omega-a}\left(z_1 z_2^a, z_2\right)$$

Now (24), (25), and a little rearrangement gives

$$(26)\qquad F_{\omega_{k+1}}\left(z_1^{a_k} z_2, z_1\right) = -F_{\omega_k}(z_1, z_2) + \frac{z_1^{a_k+1} z_2}{\left(1-z_1^{a_k} z_2\right)(1-z_1)} \quad (k \ge 1) ,$$

and by telescoping these functional equations together, we get the formula

$$(27)\qquad F_\omega(z_1, z_2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z_1^{q_{k+1}+q_k} z_2^{p_{k+1}+p_k}}{\left(1-z_1^{q_{k+1}} z_2^{p_{k+1}}\right)\left(1-z_1^{q_k} z_2^{p_k}\right)} ,$$

for all (z_1, z_2) satisfying (23). Formulae analogous to (26) and (27) hold for the more general function

$$F_\omega(z_1, z_2; P) = \sum_{h_1=1}^{\infty} \sum_{1\le h_2<h_1\omega} P(h_1, h_2) z_1^{h_1} z_2^{h_2}$$

of Theorem 8, the extension being immediately effected by applying the differential operator

$$D(P)\ (\text{say}) = P\left(z_1 \frac{\partial}{\partial z_1}, z_2 \frac{\partial}{\partial z_2}\right)$$

to both sides of the preceding equations.

We must now consider 2 cases according as the partial quotients a_k are bounded or unbounded.

First Case. Suppose that the a_k are bounded. From (22), this implies that λ (say) $= \liminf q_{k+1}/q_k$ is finite and, in fact, λ must be irrational. Let N be a sequence of positive integers such that $q_{k+1}/q_k \to \lambda$ as $k \to \infty$ through N .

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We introduce the matrices

$$T_k = \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix}, \quad S_k = T_k T_{k-1} \ldots T_1 = \begin{bmatrix} q_{k+1} & p_{k+1} \\ q_k & p_k \end{bmatrix} \quad (k \geq 1) .$$

In order to apply Theorem 1, we must restrict k to the sequence N. First, it is easy to check that the matrices S_k for k in N arise from a coherent sequence of matrices T', say, whose elements are products of suitable blocks of the T_k, and that $U(T')$ is the set of points defined by (23). By combining the equations (26) in blocks corresponding to the derivation of the sequence T' and applying the operator $D(P)$ defined above, we obtain a recursive system of functional equations linking the functions $F_{\omega_k}(z_1, z_2; P)$ for k in N. Since $r(S_k) \sim (\omega+\lambda)q_k$ as $k \to \infty$ through N, these functional equations satisfy the requirements (10) and (11) of Theorem 1. The first 2 conditions of the definition of an admissible point assert that (z_1, z_2) satisfies (23) and that $z_1^{q_k} z_2^{p_k} \neq 1$ for $k \geq 1$. The third condition is implied by the first in this case, since the vector η in (3) is proportional to $(\lambda, 1)$. Finally, to verify the strong transcendence of the functions $f_k(z) = F_{\omega_k}(z_1, z_2; P)$, consider, in the notation of Section 2, the polynomial

$$E\left(z; \tau^{(k)}\right) = \sum_{j=0}^{\rho} P_j\left(z; \tau^{(k)}\right) f_k(z)^j .$$

The remarks of Section 10 show that $f_k(z)$ cannot be an algebraic function, so $E\left(z; \tau^{(k)}\right)$ is not identically zero unless all the $P_j\left(z; \tau^{(k)}\right)$ are identically zero. From the definition of $f_k(z)$, we see that

$$\tau_\mu^{(k)} = \begin{cases} P\left(h_1, h_2\right) & \text{if } \mu = \left(h_1, h_2\right) \text{ and } 1 \leq h_2 < h_1\omega_k , \\ 0 & \text{otherwise.} \end{cases}$$

It follows that if $E\left(z; \tau^{(k)}\right)$ is not identically zero and ω_l is

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sufficiently close to ω_k, then the power series for $E(z; \tau^{(l)})$ and $E(z; \tau^{(k)})$ agree up to the first non-vanishing term of $E(z; \tau^{(k)})$. By the hypothesis of this case, the ω_k are bounded away from 0 and so the usual sort of compactness argument gives the required strong transcendence property. Thus all the hypotheses of Theorem 1 are fulfilled and the desired conclusion follows.

Second Case. Suppose that the a_k are unbounded. In this case, Theorem 1 is inapplicable, but we can instead proceed more directly. We suppose that $F_\omega(\alpha, \beta; P)$ is algebraic and we aim to reach a contradiction, showing that this assumption is false. Let K be an algebraic number field of finite degree, d say, over $\mathbb{Q}$ containing the numbers α, β, and $F_\omega(\alpha, \beta; P)$, and all the coefficients of the polynomial $P(x, y)$. In the following, we denote by $c_1, c_2, \ldots$ positive constants depending only on the quantities just mentioned.

After applying the operator $D(P)$ to (27), we obtain a representation for $F_\omega(z_1, z_2; P)$ of the shape

$$F_\omega(z_1, z_2; P) = \sum_{h=1}^{\infty} \phi_h(z_1, z_2) ,$$

where the $\phi_h(z_1, z_2)$ are certain rational functions of $z_1^{q_h} z_2^{p_h}$ and $z_1^{q_{h+1}} z_2^{p_{h+1}}$ in which only the coefficients vary with the index h. We now write

$$\gamma_k = F_\omega(\alpha, \beta; P) - \sum_{h=1}^{k-1} \phi_h(\alpha, \beta) = \sum_{h=k}^{\infty} \phi_h(\alpha, \beta) \quad (k \geq 1) . \tag{28}$$

The first equality in (28) shows that the γ_k are algebraic numbers in K and that

$$\log\|\gamma_k\| \leq c_1 q_k . \tag{29}$$

On the other hand, the second equality in (28) shows that

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(30) $$\log|\gamma_k| \le c_2\left(q_{k+1} \log|\alpha|+p_{k+1} \log|\beta|\right) \le -c_3 q_{k+1} ,$$

whenever k is sufficiently large, since $\log|\alpha| + \omega \log|\beta| < 0$. We can express the right side of (28) as a power series in the variables $\alpha^{q_h}\beta^{p_h}$ with $h \ge k$. Again, since $F_\omega(z_1, z_2; P)$ is not an algebraic function, the coefficients of each of the power series arising in this way do not all vanish and so, for all sufficiently large k , each of these power series has a single dominant term. It therefore follows, just as in Lemma 6, that $\gamma_k \ne 0$ for all sufficiently large k . Now by (22) and the hypothesis of this case, there is an infinite sequence, N say, such that $q_{k+1}/q_k \to \infty$ as $k \to \infty$ through N . From (29) and (30), we have

$$\log|\gamma_k| < -2d \log\|\gamma_k\|$$

for all sufficiently large k in N . But the last inequality clashes with the fundamental inequality (16), giving the desired contradiction.

This completes the proof of Theorem 8.

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