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Variations of Mixed Hodge Structures of Multiple Polylogarithms

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Abstract. It is well known that multiple polylogarithms give rise to good unipotent variations of mixed Hodge-Tate structures. In this paper we shall *explicitly* determine these structures related to multiple logarithms and some other multiple polylogarithms of lower weights. The purpose of this explicit construction is to give some important applications. First we study the limit of mixed Hodge-Tate structures and make a conjecture relating the variations of mixed Hodge-Tate structures of multiple logarithms to those of general multiple *poly*logarithms. Then following Deligne and Beilinson we describe an approach to defining the single-valued real analytic version of the multiple polylogarithms which generalizes the well-known result of Zagier on classical polylogarithms. In the process we find some interesting identities relating single-valued multiple polylogarithms of the same weight k when $k = 2$ and 3. At the end of this paper, motivated by Zagier's conjecture we pose a problem which relates the special values of multiple Dedekind zeta functions of a number field to the single-valued version of multiple polylogarithms.

1 Introduction

In early 1980s Deligne [5] discovered that the dilogarithm gives rise to good variations of mixed Hodge-Tate structures. This has been generalized to polylogarithms (*cf.* [10]) following Ramakrishnan's computation of the monodromy of the polylogarithms. The monodromy computation also yields the single-valued variant $\mathcal{L}_n(z)$ of the polylogarithms (*cf.* [1, 17]). These functions in turn have significant applications in arithmetic such as Zagier's conjecture [17, p. 622]. On the other hand, as pointed out in [9], "higher cyclotomy theory" should study the multiple polylogarithm motives at roots of unity, not only those of the polylogarithms. For this reason we want to look at the variations of mixed Hodge structures associated with the multiple polylogarithms and see how far we can generalize the classical results. In theory such variations of mixed Hodge structures are well known to the experts. The purpose of our explicit construction is to give some important applications.

For any positive integer $m_1, \dots, m_n$, the multiple polylogarithm is defined as follows:

$$\mathrm{Li}_{m_1,\dots,m_n}(x_1,\dots,x_n) = \sum_{0<k_1<k_2<\cdots<k_n} \frac{x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}}{k_1^{m_1} k_2^{m_2} \dots k_n^{m_n}}, \qquad |x_i| < 1. \tag{1}$$

We call n the *depth* and $K := m_1 + \cdots + m_n$ the *weight*. When the depth $n = 1$ the function is nothing but the classical polylogarithm. More than a century ago

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H. Poincaré [14] already knew that hyperlogarithms

$$F_n\left(\begin{matrix} a_1,\dots,a_n \\ b_1,\dots,b_n \end{matrix}\middle| z\right) = \int_{b_n}^{z}\cdots\int_{b_2}^{t_3}\int_{b_1}^{t_2} \frac{dt_1}{t_1-a_1}\frac{dt_2}{t_2-a_2}\cdots\frac{dt_n}{t_n-a_n}$$

are important for solving differential equations. We observe that although the multiple polylogarithm can be represented by the iterated path integral in the sense of Chen [4]

(2)

$$\mathrm{Li}_{m_1,\dots,m_n}(x_1,\dots,x_n) = (-1)^n F_K\left(\begin{matrix} a_1, & \overbrace{0,\dots,0}^{m_1-1\text{ times}}, & a_2, & \overbrace{0,\dots,0}^{m_2-1\text{ times}}, & \dots, & a_n, & \overbrace{0,\dots,0}^{m_n-1\text{ times}} \\ 0\ , & 0,\dots,0, & 0\ , & 0,\dots,0, & \dots, & 0\ , & 0,\dots,0 \end{matrix}\middle| 1\right),$$

where $a_i = 1/(x_i\cdots x_n)$ for $1 \le i \le n$, it is not obvious that this actually yields a genuine analytic continuation in the usual sense when $n \ge 2$.

According to the theory of framed mixed Hodge-Tate structures, the multiple polylogarithms are period functions of some variations of mixed Hodge-Tate structures (see [2], [8, §12] and [8, §3.5]). Wojtkowiak [16] studied mixed Hodge structures of iterated integrals over $\mathbb{C}P^1 \setminus \{0,1,\infty\}$ and investigated functional equations arising from there. In this paper we adopt a different approach and compute *explicitly* the variations of mixed Hodge-Tate structures related to the multiple logarithms

$$\mathfrak{L}_n(x_1,\dots,x_n) := \mathrm{Li}_{\underbrace{1,\dots,1}_{n\text{ times}}}(x_1,\dots,x_n).$$

This work relies on our new definition of analytic continuation of the multiple polylogarithms given in another paper [19], by using Chen's iterated path integrals over $\mathbb{C}P^n \setminus D_n$ with some non-normal crossing divisor D_n. In order to have reasonable variations we should be able to control their behavior at "infinity". This requires us to deal with the natural extension of the variations to the infinity using the classical result of Deligne [6, Proposition 5.2]. By the same idea we are able to treat all the weight-three multiple polylogarithms and present a result for the double polylogarithms. From the examples we make the following

Conjecture 1.1 *The variations of mixed Hodge-Tate structures related to every multiple polylogarithm can be produced as the variations of some limit mixed Hodge-Tate structures related to some suitable choice of multiple logarithm.*

We point out that the old form (2) of multiple polylogarithms is *not* suitable for the investigation of the MHS at the infinity because it is even not obvious from this form what the "infinity" is exactly.

As another important application of our explicit computation, in the last section of this paper we describe an approach to computing the single-valued real analytic version of the multiple polylogarithms following an idea of Beilinson and Deligne [1]. We find some interesting identities relating single-valued multiple polylogarithms of the same weight k when $k = 2$ and 3. For example, we find the single-valued real analytic double logarithm (see Eqs. (24) and (25))

$$\begin{aligned}\mathcal{L}_{1,1}(x,y) =& \operatorname{Im}\big(\operatorname{Li}_{1,1}(x,y)\big) - \arg(1-y)\log|1-x| - \arg(1-xy)\log\Big|\frac{x(1-y)}{x-1}\Big| \\ =& \mathcal{L}_2\Big(\frac{xy-y}{1-y}\Big) - \mathcal{L}_2\Big(\frac{y}{y-1}\Big) - \mathcal{L}_2(xy)\end{aligned}$$

where $\mathcal{L}_2(z)$ is the famous single-valued dilogarithm.

The motivation for this paper comes from [9, §2,3] where the Hodge-Tate structures associated with the double logarithms are discussed, and from [1] where an elegant construction of the single-valued real analytic version of classical polylogarithms are given.

As usual HS stands for "Hodge structure" and MHS for "mixed Hodge structure(s)".

2 Multiple Logarithms

First we define the index set

$$\mathfrak{S}_n = \{\mathbf{i} = (i_1, \dots, i_n) : \ i_t = 0 \text{ or } 1 \text{ for } t = 1, \dots, n\}$$

which is equipped with a weight function

$$|(i_1, \dots, i_n)| = i_1 + \cdots + i_n$$

and two different orders: a complete order $<$ and a partial order $\prec$. If $|\mathbf{i}| < |\mathbf{j}|$ then $\mathbf{i} < \mathbf{j}$ (or, equivalently, $\mathbf{j} > \mathbf{i}$). If $|\mathbf{i}| = |\mathbf{j}|$ then the usual lexicographic order from left to right is in force with $0 < 1 < \cdots$. The partial order is defined as follows. Let $\mathbf{i} = (i_1, \dots, i_n)$ and $\mathbf{j} = (j_1, \dots, j_n)$. We set $\mathbf{j} \prec \mathbf{i}$ (or, equivalently, $\mathbf{i} \succ \mathbf{j}$) if $j_t \le i_t$ for every $1 \le t \le n$. For example $(0,0,1,0) \prec (0,1,1,0)$ in $\mathfrak{S}_4$ but $(1,0,0,0) \not\prec (0,1,1,0)$ and $(1,0,0,0) \not\succ (0,1,1,0)$. Clearly $\mathbf{j} \prec \mathbf{i}$ implies $\mathbf{j} < \mathbf{i}$ but not vice versa.

Let $\mathbf{u}_s = (0, \dots, 1, \dots, 0)$ where the only 1 sits at the sth position. For any $\mathbf{i} = (i_1, \dots, i_n) \in \mathfrak{S}_n$ with $i_s = 0$ we define $\operatorname{pos}(\mathbf{i}, \mathbf{i} + \mathbf{u}_s) = s$ as the position where the component is increased by 1. For example $\operatorname{pos}\big((1,0),(1,1)\big) = 2$. We define the position functions $f_n^1, \dots, f_n^n$ on $\vec{\mathbf{j}} \in \mathfrak{S}_n^n$ as follows:

$$f_n^1(\vec{\mathbf{j}}) = 1, \quad f_n^t(\vec{\mathbf{j}}) = \operatorname{pos}\big(\mathbf{j}_{t-1}, \mathbf{j}_t\big), \text{ for } 2 \le t \le n.$$

These functions tell us the places where the increments occur in the queue of $\vec{\mathbf{j}}$.

We know the n-tuple logarithm $\mathcal{L}_n(\mathbf{x})$ ($\mathbf{x} = (x_1, \dots, x_n)$) is related to multiple logarithms of lower weights. This can be seen easily, for instance, when we take the derivatives. To clarify this we need the following notation. Suppose $\mathbf{i}$ has weight k and suppose the 1's occur exactly at the positions $\tau_1, \dots, \tau_k$. Set $\tau_{k+1} = n + 1$ and define

$$\mathbf{x}(\mathbf{i}) = \mathbf{y} = (y_1, \dots, y_k), \qquad y_m = \prod_{\alpha=\tau_m}^{\tau_{m+1}-1} x_\alpha, \quad 1 \le m \le k. \tag{3}$$

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Now we may roughly say that the n-tuple logarithm $\mathcal{L}_n(\mathbf{x})$ is related to the k-tuple logarithm $\mathcal{L}_k(\mathbf{x}(\mathbf{i}))$ for every $\mathbf{i}$. How these functions are exactly interlocked together to provide an MHS is the central theme of our paper.

To express $\mathcal{L}_n(\mathbf{x})$ explicitly we set

$$w_1(\mathbf{x}) := d\log\Big(\frac{1}{1-x_1}\Big); \quad w_t(\mathbf{x}) := d\log\Big(\frac{1-x_{t-1}^{-1}}{1-x_t}\Big), \text{ for } 2 \le t \le n.$$

Then we have

Proposition 2.1 ([19, Proposition 5.1]) *The multiple logarithm $\mathfrak{L}_n(\mathbf{x})$ is a multi-valued holomorphic function on*

$$S'_n = \mathbb{C}^n \setminus \Big\{(x_1,\dots,x_n) : \prod_{1\le j\le n}(1-x_j) \prod_{1\le j<k\le n}\big(1-x_j\dots x_k\big) = 0\Big\}.$$

Moreover, it can be expressed by

$$\mathfrak{L}_n(\mathbf{x}) = \sum_{\vec{\mathbf{j}}=(\mathbf{j}_1,\dots,\mathbf{j}_n)\in\mathfrak{S}_n^n} \int_{\mathbf{0}}^{\mathbf{x}} w_{f_n^1(\vec{\mathbf{j}})}(\mathbf{x}(\mathbf{j}_1))w_{f_n^2(\vec{\mathbf{j}})}(\mathbf{x}(\mathbf{j}_2))\cdots w_{f_n^n(\vec{\mathbf{j}})}(\mathbf{x}(\mathbf{j}_n)), \tag{4}$$

where the path from $\mathbf{0}$ to $\mathbf{x}$ lies in S'_n.

3 Multiple Logarithm Variations of MHS

In this section we will define the variation matrix $\mathcal{M}_{[n]}(\mathbf{x})$ coming from the multiple logarithms of depths up to n. We will show that it is a $2^n \times 2^n$ multi-valued matrix which defines a good variation of an MHS over some Zariski open set S_n in $\mathbb{C}^n$.

3.1 Definition of Variations of MHS: A Review

In this section we briefly review the theory of variations of MHS.

A pure ($\mathbb{Z}$-)HS of weight k consists of a finitely generated abelian group $H(\mathbb{Z})$ and a decreasing Hodge filtration $\mathcal{F}^\bullet$ on $H(\mathbb{C}) := H(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}$ such that $H(\mathbb{C}) = \mathcal{F}^p \oplus \overline{\mathcal{F}^{k-p+1}}$ for all integers p. Here the "bar" is the complex conjugation on the second factor of the tensor product. A special example is the Tate structure $\mathbb{Z}(-k)$ of weight $2k$ consists of $H(\mathbb{Z}) = \mathbb{Z}$ and the filtration $\mathcal{F}^p = 0$ for $p > k$ and $\mathcal{F}^p = H(\mathbb{C})$ for $p \le k$. If we replace $\mathbb{Z}$ by $\mathbb{Q}$ in the above then we get a pure ($\mathbb{Q}$-)HS of weight k.

An MHS consists of a finitely generated abelian group $H(\mathbb{Z})$ and two filtrations: an increasing weight filtration $W_\bullet$ on $H(\mathbb{Q}) := H(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ and a decreasing filtration $\mathcal{F}^\bullet$ on $H(\mathbb{C})$, which are compatible in the following sense. On each graded piece of the weight filtration $\mathrm{gr}_k^W = W_k/W_{k-1}$ the induced Hodge filtration determined by

$$\mathcal{F}^p(\mathrm{gr}_k^W)(\mathbb{C}) = (\mathcal{F}^p \cap W_k(\mathbb{C}) + W_{k-1}(\mathbb{C}))\big/ W_{k-1}(\mathbb{C})$$

is a pure Hodge structure of weight k where $W_k(\mathbb{C}) := W_k \otimes_{\mathbb{Z}} \mathbb{C}$. If all the pure Hodge structures induced as above are direct sums of Tate structures then we say the MHS is a Tate structure. For a mixed Hodge-Tate structure we can put a framing as in [2, §1.3.4, §1.4].

Following Steenbrink and Zucker [15, Definitions 3.1, 3.2 and 3.4] we have

Definition 3.1 A *variation of HS of weight k* defined over $\mathbb{Q}$ and a complex manifold S is a collection of data $(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^{\bullet})$ where:

(a) $\mathbb{V}_{\mathbb{Q}}$ is a locally constant sheaf (local system) of $\mathbb{Q}$-vector spaces on S.
(b) $\mathcal{F}^{\bullet}$ is a decreasing filtration by holomorphic subbundles of the locally free sheaf $\mathcal{V} = \mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{V}_{\mathbb{Q}}$.
(c) At each $s \in S$, $\mathcal{F}^{\bullet}$ induces the Hodge filtration $\mathcal{F}_s^{\bullet}$ of a Hodge structure of weight k on the fiber $\mathcal{V}_s$ of $\mathcal{V}$ such that
   (i) whenever $p + q = k$ one has $\mathcal{V}_s = \mathcal{F}_s^p \oplus \overline{\mathcal{F}_s^{q+1}}$, where the "bar" denotes the complex conjugation,
   (ii) equivalently, one has $\mathcal{V}_s = \bigoplus_{p+q=k} H_s^{p,q}$ where $H_s^{p,q} = \mathcal{F}_s^p \cap \overline{\mathcal{F}_s^q}$.
(d) (Griffiths transversality) Under the connection ∇ in $\mathcal{V}$,

$$\nabla \mathcal{F}^p \subset \Omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{F}^{p-1} \quad \text{for all } p.$$

Definition 3.2 *A polarization over* $\mathbb{Q}$ of a variation of Hodge structure of weight k over $\mathbb{Q}$ is a non-degenerated and flat bilinear pairing:

$$\beta \colon \mathbb{V}_{\mathbb{Q}} \times \mathbb{V}_{\mathbb{Q}} \longrightarrow \mathbb{Q},$$

such that β is $(-1)^k$-symmetric, and the Hermitian form $\beta_s(C_s v, \bar{w})$ is positive on each fiber. Here C_s denotes the Weil operator with respect to $\mathcal{F}_s$, namely the direct sum of multiplication by i^{p-1} on $H_s^{p,q}$. A variation is called *polarizable* (over $\mathbb{Q}$) if it admits a polarization (over $\mathbb{Q}$).

Definition 3.3 A *variation of MHS* defined over $\mathbb{Q}$ and a complex manifold S is a collection of data $(\mathbb{V}_{\mathbb{Q}}, W_{\bullet}, \mathcal{F}^{\bullet})$ where:

(a) $\mathbb{V}_{\mathbb{Q}}$ is a local system of $\mathbb{Q}$-vector spaces on S.
(b) $W_{\bullet}$ is an increasing filtration of the $\mathbb{V}_{\mathbb{Q}}$ by local subsystems,
(c) $\mathcal{F}^{\bullet}$ is a decreasing filtration by holomorphic subbundles of $\mathcal{V} = \mathcal{O}_S \otimes_{\mathbb{Q}} \mathbb{V}_{\mathbb{Q}}$.
(d) $\nabla \mathcal{F}^p \subset \Omega_S^1 \otimes_{\mathcal{O}_S} \mathcal{F}^{p-1}$ for all p.
(e) The data

$$\Big(\operatorname{gr}_k^W \mathbb{V}_{\mathbb{Q}}, \mathcal{F}^{\bullet}\big(\mathcal{O}_S \otimes_E W_k \big/ \mathcal{O}_S \otimes_E W_{k-1} \big) \Big) \tag{5}$$

   is a variation of HS of weight k defined over $\mathbb{Q}$; or equivalently, on the fiber over $s \in S$, $(V_s, W_s, \mathcal{F}_s)$ is a MHS defined over $\mathbb{Q}$.
(f) If the induced collection of variations of HS (5) are all polarizable then the MHS is called *graded-polarizable.*

Remark 3.4 By extension of scalars in $\mathbb{V}_{\mathbb{Q}}$ one can define $\mathbb{V}_{\mathbb{F}}$ for any field $\mathbb{F}$ such that $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{R}$.

Giving a local system $\mathbb{V}_{\mathbb{Q}}$ is equivalent to specifying its monodromy representation

$$\rho_x\colon \pi_1(S,x) \longrightarrow \mathrm{Aut}_{\mathbb{Q}}\,\mathcal{V}_x.$$

A variation is called *unipotent* if this representation is unipotent. From [13, Proposition 1.3] we know that a variation of MHS $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ is unipotent if and only if each of the variations of Hodge structure $\mathrm{gr}_k^W \mathbb{V}_{\mathbb{Q}}$ is constant.

In general, the behavior of a variation of MHS over a non-compact base S at "infinity" is very hard to control. Steenbrink and Zucker [15] consider the case when S is a curve and define the admissibility condition at infinity. For higher dimensional S, Kashiwara, M. Saito, and others define a variation over S to be admissible if its restriction to every curve is admissible in the sense of Steenbrink-Zucker.

However, the behavior of *unipotent* variations of MHS at infinity can be controlled rather easily. We have the classical result of Deligne [6, Proposition 5.2] which defines the *canonical extension* $\tilde{\mathcal{V}}$ of $\mathcal{V}$.

Theorem 3.5 (Deligne) *Let $\tilde{S}$ be a normalization of S. Let $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ be a unipotent variation of MHS over S. Then*

(a) *There is a unique extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ over $\tilde{S}$ satisfying the following equivalent conditions:*
   - (i) *Inside every section of $\tilde{\mathcal{V}}$, every flat section of $\mathcal{V}$ increases at most at the rate of $O(\log^k \|x\|)$ (k large enough) on every compact set of $D = \tilde{S} - S$.*
   - (ii) *Similarly, every flat section of $\mathcal{V}^\vee$ (the dual) increases at most at the rate of $O(\log^k \|x\|)$ (k large enough).*

(b) *The combination of the two conditions* (i) *and* (ii) *is equivalent to the combination of the following two conditions:*
   - (iii) *In terms of any local basis of $\tilde{\mathcal{V}}$ the connection matrix ω of $\mathcal{V}$ has at most logarithmic singularities along D.*
   - (iv) *The residue of ω along any irreducible component of D is nilpotent.*

We will verify conditions (iii) and (iv) by Proposition 3.14 for the multiple logarithm variations of MHS. They are unipotent variations by Theorem 3.16.

Definition 3.6 Let $\tilde{S}$ be a compactification of S. Then a unipotent variation of MHS $(\mathbb{V}_{\mathbb{Q}}, W_\bullet, \mathcal{F}^\bullet)$ over S is said to be *good* if it satisfies the following conditions at infinity:

(1) the Hodge filtration bundles $\mathcal{F}^\bullet$ extend over $\tilde{S}$ to sub-bundles $\tilde{\mathcal{F}}^\bullet$ of the canonical extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ such that they induce the corresponding thing for each pure subquotient $\mathrm{gr}_k^W \mathbb{V}_{\mathbb{Q}}$;
(2) for the nilpotent logarithm N_j of a local monodromy transformation about a component D_j of D, the weight filtration of N_j relative to $W_\bullet$ exists.

A slightly different definition first appeared in [12, 13] with the extra assumption that $D = \tilde{S} - S$ is a normal crossing divisor. In these papers Hain and Zucker classified good unipotent variations of MHS on algebraic manifolds. With constant pure weight subquotients these variations behave well at infinity.

3.2 The Variation Matrix

The double logarithm was treated by Goncharov [9, §2]. We noticed an apparent typo on page 620: the term $2\pi i \log x$ in the matrix $A_{1,1}(x, y)$ should be replaced by $2\pi i \log(1 - x)$. We first rewrite $A_{1,1}(x, y)$ as $\mathcal{M}_{1,1}(x, y)$ below because we will use induction starting from this form of double logarithm variation of MHS later in several proofs.

$$\mathcal{M}_{1,1}(x, y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathfrak{L}_1(y) & 2\pi i & 0 & 0 \\ \mathfrak{L}_1(xy) & 0 & 2\pi i & 0 \\ \mathfrak{L}_2(x, y) & 2\pi i \mathfrak{L}_1(x) & 2\pi i \mathfrak{L}_1\left(\frac{1-xy}{1-x}\right) & (2\pi i)^2 \end{bmatrix}. \tag{6}$$

This is essentially the same as defined in [9] up to signs.

Let us consider the columns of this matrix. Let $\mathcal{D} = \{(x, y) \in \mathbb{C}^2 : x(1 - x)(1 - y)(1 - xy) = 0\}$. For every fixed pair $(x, y) \notin \mathcal{D}$ the second and third columns of (6) are not well defined since we know the logarithmic function is a multi-valued function. However, if we consider the abelian group generated by the last two columns, then it is well defined and has rank 2. Similarly, the rank 4 abelian group generated by all the columns is well defined because the monodromy of a double logarithm is given by logarithms. This is the primary reason why the matrix has a mixed Hodge structure: the columns provide the weight filtration, the right ranks show the Hodge filtration. Furthermore, if we let (x, y) vary, then we will get variations of MHS. If we consider the limit case when (x, y) approaches to the singularities in $\mathcal{D}$, then we will find the limit MHS.

We now generalize this idea to multiple logarithms. We begin by defining the variation matrix $\mathcal{M}_{[n]}(\mathbf{x})$ for every $\mathbf{x} \in S_n = \mathbb{C}^n \setminus D_n$ where D_n is the divisor defined by

$$\prod_{1\le j\le n} x_j(1 - x_j) \prod_{1\le j<k\le n} (1 - x_j \cdots x_k) = 0.$$

Remark 3.7 In fact, the irreducible component $x_n = 0$ in D_n is not needed in the case of multiple logarithms. But the variation matrix corresponding to general multiple polylogarithms may have singularities along this component, for example, $\mathcal{M}_{1,2}(x_1, x_2)$ of the double polylogarithm $\mathrm{Li}_{1,2}(x_1, x_2)$. See §5.

Definition 3.8 For $1 \le s \le n$ write $a_s = a_s(\mathbf{x}) =: x_s \cdots x_n$ and

$$\theta_s = \theta_s(\mathbf{x}) = \frac{dt}{t - a_s} = \frac{dt}{t - a_s(\mathbf{x})}.$$

Suppose $\mathbf{i} \succ \mathbf{j}$ has weight k and l, respectively. Further suppose the 1's occur at the positions $\tau_1, \dots, \tau_k$ in $\mathbf{i}$ and $t_1, \dots, t_l$ in $\mathbf{j}$, respectively.

(1) If $\mathbf{j} \not\prec \mathbf{i}$, we define the $(\mathbf{i}, \mathbf{j})$-th entry of $\mathcal{M}_{[n]}(\mathbf{x})$ to be 0.

(2) If $\mathbf{j} \prec \mathbf{i}$ then $\{t_1, \dots, t_l\}$ is a subset of $\{\tau_1, \dots, \tau_k\}$ so we can put $t_r = \tau_{\alpha_r}$ for $1 \le r \le l$. Recall from equation (3) that we have

$$\mathbf{x}(\mathbf{i}) = \mathbf{y} = (y_1, \dots, y_k), \quad y_m = \prod_{\alpha=\tau_m}^{\tau_{m+1}-1} x_\alpha = \frac{a_{\tau_{m+1}}(\mathbf{x})}{a_{\tau_m}(\mathbf{x})}, \quad 1 \le m \le k,$$

with $\tau_{k+1} = n+1$ and $a_{n+1} = 1$. Set $t_0 = \alpha_0 = 0$, $t_{l+1} = n+1$, $\alpha_{l+1} = k+1$, $a_0(\mathbf{x}) = a_0(\mathbf{y}) = 0$. Define the $(\mathbf{i}, \mathbf{j})$-th entry of $\mathcal{M}_{[n]}(\mathbf{x})$ as $(2\pi i)^l E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ where

$$\begin{aligned} E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{y}) :=& (-1)^{k-l} \prod_{r=0}^{l} \int_{a_{\alpha_r}(\mathbf{y})}^{a_{\alpha_{r+1}}(\mathbf{y})} \theta_{\alpha_r+1}(\mathbf{y}) \cdots \theta_{\alpha_{r+1}-1}(\mathbf{y}) \\ =& (-1)^{k-l} \prod_{r=0}^{l} \int_{p_r} \theta_{\tau_{\alpha_r+1}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}). \end{aligned} \tag{7}$$

Here the $l+1$ paths $p_0, \dots, p_l$ for the $l+1$ integrals are *independent of* $\mathbf{i}$ where p_r is any fixed contractible path from a_{t_r} to $a_{t_{r+1}}$ in the punctured complex plane $\mathbb{C} \setminus \bigcup_{t_r < s < t_{r+1}} \{a_s\}$, and the integral $\int_{p_r} = 1$ if $\alpha_r + 1 = \alpha_{r+1}$. We get the second equality by observing that

$$a_m(\mathbf{y}) = (y_m \cdots y_k)^{-1} = a_{\tau_m}(\mathbf{x}) \Longrightarrow a_{\alpha_r}(\mathbf{y}) = a_{\tau_{\alpha_r}}(\mathbf{x}) = a_{t_r}(\mathbf{x}).$$

Proposition 3.9 *Suppose* $\mathbf{i}$ *and* $\mathbf{j}$ *are given as in Definition* 3.8(2). *As multi-valued functions*

$$E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \prod_{r=0}^{l} \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{a_{\tau_{\alpha_r+2}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}, \dots, \frac{a_{t_{r+1}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) - a_{t_r}(\mathbf{x})} \right) \tag{8}$$

$$\begin{aligned} =& \mathfrak{L}_{\alpha_1-1}\big(x_{\tau_1} \cdots x_{\tau_2-1}, x_{\tau_2} \cdots x_{\tau_3-1}, \dots, x_{\tau_{\alpha_1-1}} \cdots x_{t_1-1} \big) \times \\ & \prod_{r=1}^{l} \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{1 - x_{t_r} \cdots x_{\tau_{\alpha_r+2}-1}}{1 - x_{t_r} \cdots x_{\tau_{\alpha_r+1}-1}}, \dots, \frac{1 - x_{t_r} \cdots x_{t_{r+1}-1}}{1 - x_{t_r} \cdots x_{\tau_{\alpha_{r+1}-1}-1}} \right). \end{aligned} \tag{9}$$

Here $\mathfrak{L}_0 = 1$ *and* $a_0 = 0$.

Proof By direct and simple calculation we get

$$\begin{aligned} &(-1)^{\alpha_{r+1}-\alpha_r-1} \int_{p_r} \theta_{\tau_{\alpha_r+1}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) \\ &= \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{a_{\tau_{\alpha_r+2}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}, \frac{a_{\tau_{\alpha_r+3}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}{a_{\tau_{\alpha_r+2}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}, \dots, \frac{a_{t_{r+1}}(\mathbf{x}) - a_{t_r}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) - a_{t_r}(\mathbf{x})} \right). \end{aligned}$$

The proposition follows immediately. ■

Example 3.10 On the last row of $\mathcal{M}_{[n]}(\mathbf{x})$ one has

$$E_{\mathbf{1},\mathbf{j}}(\mathbf{x}) = \gamma^n_{\mathbf{j}}(\mathbf{x}) = \prod_{r=0}^{l} \mathfrak{L}_{t_{r+1}-t_r-1}\Big(\frac{a_{t_r+2}-a_{t_r}}{a_{t_r+1}-a_{t_r}},\dots,\frac{a_{t_{r+1}}-a_{t_r}}{a_{t_{r+1}-1}-a_{t_r}}\Big)$$
$$= \prod_{r=0}^{l} \mathfrak{L}_{t_{r+1}-t_r-1}\Big(\frac{1-x_{t_r}x_{t_r+1}}{1-x_{t_r}},\dots,\frac{1-x_{t_r}\cdots x_{t_{r+1}-1}}{1-x_{t_r}\cdots x_{t_{r+1}-2}}\Big) \tag{10}$$

where $\mathfrak{L}_0 = 1$ and $x_0 = \infty$. In particular, $E_{\mathbf{1},\mathbf{0}} = \gamma^n_{\mathbf{0}}(\mathbf{x}) = \mathfrak{L}_n(\mathbf{x})$ and $E_{\mathbf{1},\mathbf{1}} = \gamma^n_{\mathbf{1}}(\mathbf{x}) = 1$.

We now fix a standard basis $\{e_{\mathbf{i}} : \mathbf{i} \in \mathfrak{S}_n\}$ of $\mathbb{C}^{2^n}$ consisting of column vectors. Suppose $|\mathbf{i}| = k$. It follows from definition that the $\mathbf{i}$-th row is

$$R_{\mathbf{i}} := \sum_{\mathbf{j}\prec\mathbf{i}} (2\pi i)^{|\mathbf{j}|}\gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)\, e^T_{\mathbf{j}} = (2\pi i)^k e^T_{\mathbf{i}} + \sum_{\mathbf{j}\precneqq\mathbf{i}} (2\pi i)^{|\mathbf{j}|}\gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)\, e^T_{\mathbf{j}} \tag{11}$$

where $e^T_{\mathbf{j}}$ are now row vectors. Note that $\gamma^k_{\rho_{\mathbf{i}}(\mathbf{i})} = \gamma^k_{\mathbf{1}_k} = 1$ by definition. It is clear that the first entry (*i.e.*, $\mathbf{j} = \mathbf{0}$) of this row is $\mathfrak{L}_k\big(\mathbf{x}(\mathbf{i})\big)$.

Let us call the minor of $\mathcal{M}_{[n]}(\mathbf{x})$ consisting of rows beginning with k-tuple logarithms *the k-th block*. It has $\binom{n}{k}$ rows with row indices $|\mathbf{i}| = k$.

Lemma 3.11 *The matrix $\mathcal{M}_{[n]}(\mathbf{x})$ is a lower triangular matrix. Moreover, the columns with $|\mathbf{j}| = k$ of the k-th block of $\mathcal{M}_{[n]}(\mathbf{x})$ is $(2\pi i)^k$ times the identity matrix of rank $\binom{n}{k}$.*

Proof The lemma follows directly from (11) because if $\mathbf{j} \precneqq \mathbf{i}$, then $\mathbf{j} < \mathbf{i}$. ∎

Lemma 3.12 *The $\mathbf{j}$-th column of $\mathcal{M}_{[n]}(\mathbf{x})$ is*

$$(2\pi i)^{|\mathbf{j}|} C_{\mathbf{j}} = (2\pi i)^{|\mathbf{j}|} \sum_{\mathbf{i}\succ\mathbf{j}} \gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)\, e_{\mathbf{i}}$$

where $\mathbf{x}(\mathbf{i})$ are defined by (3) *depending on $\mathbf{i}$.*

Proof Use (11). ∎

Example 3.13 By definition or the above proposition the first column

$$C_{\mathbf{0}}(\mathbf{x}) = \big[\,\mathfrak{L}_{|\mathbf{i}|}(\mathbf{x}(\mathbf{i})) : \mathbf{i} \in \mathfrak{S}_n\big]^T$$

where $\mathfrak{L}_0 = 1$.

Proposition 3.14 *The columns of* $\mathcal{M}_{[n]}(\mathbf{x})$ *form the set of the fundamental solutions of the following system of differential equations*

$$\begin{cases} dX_{\mathbf{0}} = 0, \\ dX_{\mathbf{i}} = \sum_{|\mathbf{k}|=|\mathbf{i}|-1,\, \mathbf{k}\prec\mathbf{i}} X_{\mathbf{k}}\, d\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{k})}\big(\mathbf{x}(\mathbf{i})\big) \quad \text{for all } 1 \le |\mathbf{i}| \le n \end{cases} \tag{12}$$

where $\mathbf{x}(\mathbf{i})$ *is determined as in equation* (3).

Proof We prove the proposition by induction on n. It is easy to see the proposition is valid for $n = 1$ and $n = 2$. We assume that $n \ge 3$ and the proposition is true for $\le n-1$. Let us now look at the $\mathbf{j}$-th column as expressed in Lemma 3.12. The cases $|\mathbf{i}| = 1$ or $\mathbf{j} > \mathbf{i}$ are obvious. Suppose (1) $1 < |\mathbf{i}| < n$ and $\mathbf{j} \le \mathbf{i}$. There are two cases.

(i) $\mathbf{j} \not\prec \mathbf{i}$. This is trivial because each term of both sides is zero. (ii) $\mathbf{j} \prec \mathbf{i}$. Then there is a t such that $i_t = j_t = 0$. We denote $\mathbf{i}' \in \mathfrak{S}_{n-1}$ the corresponding index after deleting the i_t-th component. By induction

$$\sum_{\substack{|\mathbf{k}'|=|\mathbf{i}'|-1\\ \mathbf{j}'\prec\mathbf{k}'\prec\mathbf{i}'}} \gamma^{|\mathbf{k}'|}_{\rho_{\mathbf{k}'}(\mathbf{j}')}\big(\mathbf{x}'(\mathbf{k}')\big)\, d\gamma^{|\mathbf{i}'|}_{\rho_{\mathbf{i}'}(\mathbf{k}')}\big(\mathbf{x}'(\mathbf{i}')\big) = d\gamma^{|\mathbf{i}'|}_{\rho_{\mathbf{i}'}(\mathbf{j}')}\big(\mathbf{x}'(\mathbf{i}')\big)$$

where we set $\mathbf{x}' = (x_1, \dots, x_{i_t-1}, x_{i_t}x_{i_t+1}, x_{i_t+2}, \dots, x_n)$. Since $|\mathbf{i}'| = |\mathbf{i}|$ and $|\mathbf{k}'| = |\mathbf{k}|$ we can get the desired equation by inserting 0 before the i_t-th components of $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$, *i.e.*, using the embedding ι_{i_t}.

(2) $\mathbf{i} = \mathbf{1}$ and $|\mathbf{j}| = l$. We need to show

$$d\gamma^n_{\mathbf{j}}(\mathbf{x}) = \sum_{|\mathbf{k}|=n-1,\, \mathbf{j}\prec\mathbf{k}} \gamma^{n-1}_{\rho_{\mathbf{k}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{k})\big)\, d\gamma^n_{\mathbf{k}}(\mathbf{x}). \tag{13}$$

This is trivial when $l = n$. The case $l = 0$ follows from

$$d\mathfrak{L}_n(\mathbf{x}) = \sum_{t=1}^{n} \mathfrak{L}_{n-1}(x_1, \dots, x_{t-2}, x_{t-1}x_t, x_{t+1}, \dots, x_n)\, d\log\frac{1 - x_{t-1}^{-1}}{1 - x_t}.$$

So we may assume $0 < l < n$, $j_{t_1} = \cdots = j_{t_l} = 1$ and $j_t = 0$ for all other indices t. By definition (10) we have

$$\gamma^n_{\mathbf{j}}(\mathbf{x}) = \sum_{r=0}^{l} \sum_{t_r<s<t_{r+1}} \gamma^{n-1}_{\rho_{\mathbf{v}_s}(\mathbf{j})}\big(\mathbf{x}(\mathbf{v}_s)\big)\, d\gamma^n_{\mathbf{v}_s}(\mathbf{x})$$

where $t_0 = 0$, $t_{l+1} = n+1$ and

$$\begin{array}{l} \mathbf{v}_s = (1, \dots, \underset{\substack{\uparrow\\ t_r\text{-th place}}}{1}, \dots\dots\dots, 1, \underset{\substack{\uparrow\\ s\text{-th place}}}{0}, 1, \dots\dots\dots, \underset{\substack{\uparrow\\ t_{r+1}\text{-th place}}}{1}, \dots, 1). \end{array}$$

Under the retraction map $\rho_{\mathbf{v}_s}$ the numbering of the indices changes as follows: $t \rightsquigarrow t$ if $t < s$ and $t \rightsquigarrow t-1$ if $t > s$. We also have

$$a_t\big(\mathbf{x}(\mathbf{v}_s)\big) = \begin{cases} a_t(\mathbf{x}) & \text{if } t < s, \\ a_{t+1}(\mathbf{x}) & \text{if } t > s. \end{cases}$$

Hence for each s such that $t_r < s < t_{r+1}$ the integral expression of $\gamma^{n-1}_{\rho_{\mathbf{k}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{k})\big)$ is unchanged under $\rho_{\mathbf{k}}$ ($\mathbf{j} \prec \mathbf{k}$) except the $\mathbf{v}_s$-term. Equation (13) now follows immediately from Leibniz rule and so the proposition is proved. ■

3.3 Monodromy of $\mathcal{M}_{[n]}(\mathbf{x})$

Fix an embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}P^n$. Let $\mathcal{D}_n = D_n \cup (\mathbb{C}P^n \setminus \mathbb{C}^n)$. Let $M_r(\mathbb{C})$ be the set of $r \times r$ matrices over $\mathbb{C}$. Put

$$\omega = \big(c_{\mathbf{i},\mathbf{j}}\big)_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n} \in H^0\big(\mathbb{C}P^n, \Omega^1_{\mathbb{C}P^n}(\log(\mathcal{D}_n))\big) \otimes M_{2^n}(\mathbb{C}) \tag{14}$$

where

$$c_{\mathbf{i},\mathbf{j}} = \begin{cases} d\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big) & \text{if } |\mathbf{j}| = |\mathbf{i}| - 1,\ \mathbf{j} \prec \mathbf{i}, \\ 0 & \text{otherwise.} \end{cases}$$

All of the 1-forms in ω have logarithmic singularity on $\mathcal{D}_n$ because of the following. Let $|\mathbf{i}| = l$ and $i_{t_1} = \cdots = i_{t_l} = 1$. Let $j_{t_s} = 0$ so that $|\mathbf{j}| = l-1$ and $\mathbf{j} \prec \mathbf{i}$. Let $\mathbf{x}(\mathbf{i}) = \mathbf{y} = (y_1, \dots, y_l)$. By definition (7)

$$\begin{aligned}
\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big) &= -\int_{a_{s-1}(\mathbf{y})}^{a_{s+1}(\mathbf{y})} \theta_s(\mathbf{y}) = -\log\left(\frac{a_{s+1}(\mathbf{y}) - a_s(\mathbf{y})}{a_{s-1}(\mathbf{y}) - a_s(\mathbf{y})}\right) \\
&= \begin{cases} -\log(1-y_1) & \text{if } s = 1 \\ -\log\Big(\frac{y_{s-1}(y_s-1)}{1-y_{s-1}}\Big) & \text{if } s \geq 2 \end{cases} \\
&= \begin{cases} -\log(1-x_1\dots x_{t_1}) & \text{if } s = 1 \\ -\log\Big(\frac{x_{t_{s-1}}\dots x_{t_s-1}(x_{t_s}\dots x_{t_{s+1}-1}-1)}{1-x_{t_{s-1}}\dots x_{t_s-1}}\Big) & \text{if } s \geq 2. \end{cases}
\end{aligned}$$

Example 3.15 When $n = 2$ we have

$$\omega = \begin{bmatrix} 0 & & & \\ -d\log(1-y) & 0 & & \\ -d\log(1-xy) & 0 & 0 & \\ 0 & -d\log(1-x) & -d\log\frac{x(1-y)}{x-1} & 0 \end{bmatrix}.$$

We proved in Proposition 3.14 that $\mathcal{M}_{[n]}(\mathbf{x})$ is a fundamental solution of first order linear partial differential equation

$$d\Lambda = \omega\Lambda \tag{15}$$

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where Λ is a possibly multi-valued function $S \longrightarrow M_{2^n}(\mathbb{C})$. Moreover $\mathcal{M}_{[n]}(\mathbf{x})$ is a unipotent matrix for very $\mathbf{x} \in S$. Applying d on equation (15) and plugging in $\Lambda = \mathcal{M}_{[n]}(\mathbf{x})$ we get

$$0 = d\omega\mathcal{M}_{[n]}(\mathbf{x}) - \omega \wedge d\mathcal{M}_{[n]}(\mathbf{x}) = (d\omega - \omega \wedge \omega)\mathcal{M}_{[n]}(\mathbf{x}).$$

Because $\mathcal{M}_{[n]}(\mathbf{x})$ is invertible and ω is closed we get

$$d\omega = 0, \quad \omega \wedge \omega = 0. \tag{16}$$

This shows that ω is integrable.

The main goal of this chapter is to show that if we analytically continue every integral entry of $\mathcal{M}_{[n]}(\mathbf{x})$ along a same loop $q \in \pi_1(S_n, \mathbf{x})$, the resulting matrix will still be a fundamental solution $\mathcal{M}_{[n]}(\mathbf{x})M(q)$ of (15) where $M(q) \in \mathrm{GL}_{2^n}(\mathbb{Z})$. In the following we also denote this action of q by $\Theta(q)$ operating on the left. We then define the monodromy representation

$$\rho_{\mathbf{x}} \colon \pi_1(S_n, \mathbf{x}) \longrightarrow \mathrm{GL}_{2^n}(\mathbb{Z})$$
$$q \longmapsto M(q)^T.$$

Here we take the transpose to ensure $\rho_{\mathbf{x}}$ to be a homomorphism because $M(pq) = M(q)M(p)$ by our convention. From the explicit computation in Theorem 3.16 we will see that $\rho_{\mathbf{x}}$ is a unipotent representation.

Theorem 3.16 *Let $\mathcal{M}_{[n]}(\mathbf{x}) = [E_{\mathbf{i},\mathbf{j}}(\mathbf{x})]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}$ where $E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ are defined by Proposition* 3.9. *Let $1 \le i \le j \le n$ and $q_{ij} \in \pi_1(S_n, \mathbf{x})$ (resp., $1 \le j < n$ and q_{j0}) enclose $\mathcal{D}_{ij} = \{x_i \dots x_j = 1\}$, (resp., $\mathcal{D}_{j0} = \{x_j = 0\}$) only once but no other irreducible component of D_n such that $\int_{q_{ij}} d\log(1 - x_i \dots x_j) = 2\pi i$ (resp., $\int_{q_{j0}} d\log x_j = 2\pi i$). Then*

$$M(q_{j0}) = I + [n_{\mathbf{i},\mathbf{j}}]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}, \quad M(q_{ij}) = I + [m_{\mathbf{i},\mathbf{j}}]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}$$

where I is the identity matrix of rank 2^n,

$$n_{\mathbf{i},\mathbf{j}} = \begin{cases} -1 & \text{if } t_r \le j \le t_{r+1} - 2,\ r \ge 1, \mathbf{i} = \mathbf{j} + \mathbf{u}_{s+1} \text{ and } j \le s \le t_{r+1} - 2, \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

and

$$m_{\mathbf{i},\mathbf{j}} = \begin{cases} 1 & \text{if } t_r = i \le j \le t_{r+1} - 2,\ r \ge 1, \mathbf{i} = \mathbf{j} + \mathbf{u}_{j+1}, \\ -1 & \text{if } t_r + 1 \le i \le j = t_{r+1} - 1,\ r \ge 0,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Here $\mathbf{i}$ and $\mathbf{j}$ in the case of $m_{\mathbf{i},\mathbf{j}} = \pm 1$ and $n_{\mathbf{i},\mathbf{j}} = -1$ satisfy the condition in Definition 3.8(2).

Proof By definition it is clear that if $\mathbf{i} \not\succ \mathbf{j}$ then $\Theta(q)E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ which is either 0 or 1. Thus we are only concerned with $E_{\mathbf{i},\mathbf{j}}$ with $\mathbf{i} \succ \mathbf{j}$.

We now fix some $\mathbf{j}$. If $|\mathbf{j}| = n$ then clearly $(\Theta(q) - I)C_{\mathbf{1}} = [0, \dots, 0]^T$ for any loop q. This proves the proposition for $|\mathbf{j}| = n$. We now assume $|\mathbf{j}| < n$. Let $\mathbf{i}$ and $\mathbf{j}$ be given as in Definition 3.8(2). By equation (8)

$$E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \mathfrak{L}_{\alpha_1 - 1}\big(x_{\tau_1} \cdots x_{\tau_2 - 1}, x_{\tau_2} \cdots x_{\tau_3 - 1}, \dots, x_{\tau_{\alpha_1 - 1}} \cdots x_{t_1 - 1}\big) \times \prod_{r=1}^{l} \mathfrak{L}_{\alpha_{r+1} - \alpha_r - 1}\Big(\frac{1 - x_{t_r} \cdots x_{\tau_{\alpha_r + 2} - 1}}{1 - x_{t_r} \cdots x_{\tau_{\alpha_r + 1} - 1}}, \dots, \frac{1 - x_{t_r} \cdots x_{t_{r+1} - 1}}{1 - x_{t_r} \cdots x_{\tau_{\alpha_{r+1} - 1} - 1}}\Big).$$

By Theorem 4.4 and Proposition 5.5 of [19] $E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ has monodromy along $\mathcal{D}_{j0}$ if and only if $t_r \le j \le t_{r+1} - 2$ for some $r \ge 1$. According to the computation in Proposition 5.5 we further have that

$$(\Theta(q_{j0}) - \mathrm{id})E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = -2\pi i \sum_{s=j+1}^{t_{r+1}-1} E_{\mathbf{i},\mathbf{j}+\mathbf{u}_s}(\mathbf{x})$$

which involves only the entries on the $\mathbf{i}$-th row. Hence

$$(\Theta(q_{j0}) - \mathrm{id})C_{\mathbf{j}}(\mathbf{x}) = -\sum_{s=j}^{t_{r+1}-1} C_{\mathbf{j}+\mathbf{u}_s}(\mathbf{x}).$$

By similar argument, using Proposition 5.4 and 5.5 of [19], we see that if $t_r = i \le j \le t_{r+1} - 2$, $r \ge 1$, then

$$(\Theta(q_{ij}) - \mathrm{id})E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = 2\pi i E_{\mathbf{i},\mathbf{j}+\mathbf{u}_{j+1}}(\mathbf{x})$$

and therefore

$$(\Theta(q_{ij}) - \mathrm{id})C_{\mathbf{j}}(\mathbf{x}) = -C_{\mathbf{j}+\mathbf{u}_{j+1}}(\mathbf{x}).$$

Similarly, thanks to Theorem 5.3 and Proposition 5.5 of [19] if $t_r + 1 \le i \le j = t_{r+1} - 1$, $r \ge 0$, then

$$(\Theta(q_{ij}) - \mathrm{id})E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = -2\pi i E_{\mathbf{i},\mathbf{j}+\mathbf{u}_i}(\mathbf{x}).$$

Hence

$$(\Theta(q_{ij}) - \mathrm{id})C_{\mathbf{j}} = -C_{\mathbf{j}+\mathbf{u}_i}(\mathbf{x}).$$

This completes the proof of the proposition. ■

Corollary 3.17 *The monodromy representation of* $\mathcal{M}_{[n]}(\mathbf{x})$

$$\rho_{\mathbf{x}} \colon \pi_1(S_n, \mathbf{x}) \longrightarrow \mathrm{GL}_{2^n}(\mathbb{Z})$$

is unipotent.

Proof Clear. ■

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3.4 MHS of Multiple Logarithms

Define a meromorphic connection ∇ on the trivial bundle

$$\mathbb{C}P^n \times \mathbb{C}^{2^n} \longrightarrow \mathbb{C}P^n \tag{19}$$

by

$$\nabla f = df - \omega f$$

where $f\colon S_n \to \mathbb{C}^{2^n}$ is a section. This connection has regular singularities along $\mathcal{D}_n$ because ω is integrable by (16) and all the 1-forms in ω are logarithmic in any compactification of S_n. By the explicit construction of ω we see immediately that the conditions (iii) and (iv) of Theorem 3.5 are satisfied. Proposition 3.14 further implies that the columns $(2\pi i)^{|\mathbf{j}|}C_{\mathbf{j}}(\mathbf{x})$ of $\mathcal{M}_{[n]}(\mathbf{x})$ satisfy $\nabla f = 0$ and are therefore flat sections of (19). Even though they are multi-valued, their $\mathbb{Z}$-linear span is well defined thanks to Theorem 3.16. Hence $V_{[n]}(\mathbf{x})$ forms a local system over S_n.

Definition 3.18 The local system $V_{[n]}(\mathbf{x})$ is called the *n-tuple logarithm local system.*

To define the MHS on $V_{[n]}$ we can define the weight filtration by putting $W_{2k+1} = W_{2k}$ and

$$W_{-2k}V_{[n]}(\mathbf{x}) = \langle (2\pi i)^{|\mathbf{i}|}C_{\mathbf{i}} : |\mathbf{i}| \ge k\rangle_{\mathbb{Q}}$$

which is the $\mathbb{Q}$ vector space with basis $\{(2\pi i)^{|\mathbf{i}|}C_{\mathbf{i}} : |\mathbf{i}| \ge k\}$. In particular, $W_{-2k}V_{[n]}(\mathbf{x}) = 0$ if $k > n$ and $W_{-2k}V_{[n]}(\mathbf{x}) = V_{[n]}(\mathbf{x})$ if $k \le 0$. By regarding $e_{\mathbf{i}}$'s as column vectors one can define the Hodge filtration on $V_{[n]}(\mathbf{x}) \otimes \mathbb{C} = V_{[n],\mathbb{C}}$ as follows:

$$\mathcal{F}^{-k}V_{[n],\mathbb{C}} := \langle e_{\mathbf{i}} : |\mathbf{i}| \le k\rangle_{\mathbb{C}}.$$

So in particular, $\mathcal{F}^{-k}V_{[n],\mathbb{C}} = 0$ for $k < 0$ and $\mathcal{F}^{-k}V_{[n],\mathbb{C}} = V_{[n],\mathbb{C}}$ for $k \ge n$.

By induction on n and using Lemma 3.11 it is easy to show that

$$\mathcal{F}^{-p} \cap W_{-2k}V_{[n],\mathbb{C}} = \begin{cases} 0 & \text{if } p \le k-1, \\ \langle (2\pi i)^{|\mathbf{i}|}e_{\mathbf{i}} : k \le |\mathbf{i}| \le p\rangle & \text{if } k \le p \le n, \\ \langle (2\pi i)^{|\mathbf{i}|}e_{\mathbf{i}} : k \le |\mathbf{i}| \le n\rangle & \text{if } p \ge n. \end{cases}$$

This implies that

$$\mathcal{F}^{-p}\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = \begin{cases} 0 & \text{if } p \le k-1, \\ W_{-2k}V_{[n],\mathbb{C}}/W_{-2k-1}V_{[n],\mathbb{C}} & \text{if p} \ge k. \end{cases}$$

In other words, $\mathcal{F}^q\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = 0$ for $q \ge -k+1$ and $\mathcal{F}^q\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}}$ for $q \le -k$. This means that the Hodge filtration induces a pure HS of weight $-2k$ on each weight graded piece. Furthermore, it is not hard to see by checking the powers of $2\pi i$ appearing on the diagonal of $\mathcal{M}_{[n]}(\mathbf{x})$ that this induced structure on $\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}}$ is isomorphic to the direct sum of $\binom{n}{k}$ copies of the Tate structure $\mathbb{Z}(k)$ by Lemma 3.11.

4 Limit MHS of Multiple Logarithms

Let the monodromy of $\mathcal{M}_{[n]}(\mathbf{x})$ at any subvariety $\mathcal{D}$ of $\mathbb{C}P^n$ be given by the matrix $T_{\mathcal{D}}$ and the local monodromy logarithm by $N_{\mathcal{D}} = \log T_{\mathcal{D}}/2\pi i$. Note that $T_{\mathcal{D}}$ is unipotent, so $N_{\mathcal{D}}$ is well-defined.

Now let us recall the construction of the unipotent variations of limit MHS at the "infinity" with *normal crossing*. Let S be a complex manifold of dimension d. Suppose that S is embedded in $\tilde{S}$, via the mapping j, such that $D = \tilde{S} - S$ is a divisor with normal crossings. Let $\mathbb{V}$ be any unipotent local system of complex vector spaces on S, and $\mathcal{V}$ the corresponding vector bundle. According to Theorem 3.5 by Deligne there is a canonical extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ over $\tilde{S}$. Moreover, when the local monodromy is nilpotent $\tilde{\mathcal{V}}$ is a subsheaf of $j_*\mathcal{V}$. The local picture of $S \subset \tilde{S}$ is $(\Delta^*)^r \times \Delta^{d-r} \subset \Delta^d$ where Δ is the unit disk and Δ^* is the punctured one. We let $t_1, \dots, t_r$ denote the variables on $(\Delta^*)^r$, and $N_1, \dots, N_r$ the (commuting) local nilpotent logarithms of the associated monodromy transformations of the fibre. For $z_1, \dots, z_r$ in the upper half-plane, the universal covering mapping for $(\Delta^*)^r$ is given by

$$t_j = \exp(2\pi i z_j), \quad j = 1, \dots, r.$$

Let $v_1, \dots, v_m$ be a basis of the multi-valued sections of $\mathbb{V}$ over $(\Delta^*)^r \times \Delta^{d-r}$, the formula

$$[\tilde{v}_1, \dots, \tilde{v}_m] = [v_1, \dots, v_m] \exp\Big(-\sum_{j=1}^{r} 2\pi i z_j N_j\Big) = [v_1, \dots, v_m] \prod_{j=1}^{r} t_j^{-N_j}$$

determines a basis of the sections of $\mathcal{V}$ over Δ^d and these provide, by definition, the generators of $\tilde{\mathcal{V}}$ over Δ^d.

In our situation, although the divisor D_n is not normal crossing, Theorem 3.5 is still valid because the image of the global holomorphic logarithmic forms in the complex of smooth forms on S is independent of the normal crossings compactification (see [11, Proposition 3.2]). In fact, the forms we are considering lie in the subcomplex generated by 1-forms of the type df/f where f is a rational function. Such forms are automatically logarithmic in any compactification and therefore our connection is automatically regular. Hence the admissibility and the existence of the limit MHS is an automatic consequence of the admissibility of our variations restricted to every curve in S_n. Moreover, the pullback to $\tilde{S}_n$ of our trivial bundle (19) restricted to S_n is exactly Deligne's canonical extension of (19), and the pullbacks of the subbundles $\mathcal{F}^\bullet$ and $W_\bullet$ are the correct extended Hodge and weight subbundles. Therefore we have

Theorem 4.1 *The n-tuple logarithm underlies a good unipotent graded-polarizable variation of mixed Hodge-Tate structures $(V_{[n]}, W_\bullet, \mathcal{F}^\bullet)$ over*

$$S_n = \mathbb{C}^n \setminus \Big\{ \prod_{1 \le j \le n} x_j(1 - x_j) \prod_{1 \le i < j \le n} (1 - x_i \cdots x_j) = 0 \Big\}.$$

with the weight-graded quotients gr^W_{-2k} being given by $\binom{n}{k}$ copies of the Tate structure $\mathbb{Z}(k)$.

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Proof It is clear that all the odd graded weight quotients are zero so that we can let the polarizations on the weight graded quotients gr^W_{-2k} be the ones that give each vector $2\pi i e_{\mathbf{j}}$ ($|\mathbf{j}| = k$) length 1. Then everything is clear except the Griffiths transversality condition. But this condition is also satisfied because $dC_{\mathbf{j}} = \omega C_{\mathbf{j}}$ for every $\mathbf{j} \in \mathfrak{S}_n$ by Proposition 3.14. ■

If we want to determine the limit MHS of multiple logarithms explicitly we can still apply the techniques used in the normal crossing case. We will carry this out only for the depth two and three cases. The general picture is similar but much more complicated.

4.1 Limit MHS of Double Logarithm

First we look at the double logarithm variation of MHS. We have

$$\mathcal{M}_{1,1}(x,y) = \begin{bmatrix} 1 & & & \\ \mathfrak{L}_1(y) & 2\pi i & & \\ \mathfrak{L}_1(xy) & 0 & 2\pi i & \\ \mathfrak{L}_2(x,y) & 2\pi i \mathfrak{L}_1(x) & 2\pi i H(x,y) & (2\pi i)^2 \end{bmatrix}$$

where $H(x,y) = \mathfrak{L}_1(y) - \mathfrak{L}_1(x) - \log x$.

(i) Let us first try to extend the MHS to the divisor $\mathcal{D}_{10} = \{x = 0\}$ along the tangent vector $\partial/\partial x$. We have

$$T_{\{x=0\}} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad N_{\{x=0\}} = \frac{\log T_{\{x=0\}}}{2\pi i} = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & -\frac{1}{2\pi i} & 0 \end{bmatrix}.$$

Let $\mathcal{M}_{1,1}(x,y) = [C_0(x,y) \cdots C_3(x,y)]$. Define

$$\begin{aligned} \begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} &= \lim_{t \to 0} \mathcal{M}_{1,1}(t,y) \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & \log t/(2\pi i) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & \\ \mathfrak{L}_1(y) & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ 0 & 0 & 2\pi i \mathfrak{L}_1(y) & (2\pi i)^2 \end{bmatrix}. \end{aligned}$$

Let $V_{\mathbb{Q},\{x=0\}}$ be the $\mathbb{Q}$-linear span of s_0, s_1, s_2, s_3, and $V_{\mathbb{C},\{x=0\}} = \mathbb{C} \otimes V_{\mathbb{Q},\{x=0\}}$. Let $\{e_j : j = 0, \dots, 3\}$ be the standard basis of $\mathbb{C}^4$ where the only nonzero entry of e_j is at the $(j+1)$st component. Then the limit MHS on $\{(x,y) : x = 0, y \neq 1\}$ along $\partial/\partial x$ is given by

$$\big((V_{\mathbb{Q},\{x=0\}}, W_\bullet), (V_{\mathbb{C},\{x=0\}}, F^\bullet) \big)$$

where for $k = 0, \dots, 3$

$$W_{-2k} V_{\mathbb{Q},\{x=0\}} = \langle s_k, \dots, s_3 \rangle, \quad W_{-2k} = W_{-2k+1} \tag{20}$$

and

$$F^{-k} V_{\mathbb{C},\{x=0\}} = \langle e_0, \dots, e_k \rangle. \tag{21}$$

(ii) A similar calculation shows that along the tangent vector $\partial/\partial x$ the limit MHS on the divisor $\mathcal{D}_{11} = \{(1, y) : y \neq 1\}$ is the $\mathbb{Q}$-linear span of $s_0, \dots, s_3$ where

$$\begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ \mathfrak{L}_1(y) & 2\pi i & & \\ \mathfrak{L}_1(y) & 0 & 2\pi i & \\ \mathfrak{L}_2(1, y) & 0 & 2\pi i \mathfrak{L}_1(y) & (2\pi i)^2 \end{bmatrix}.$$

It is easy to see by differentiation that $\mathfrak{L}_2(1, y) = \big(\mathfrak{L}_1(y)\big)^2/2$.

(iii) The extension of MHS to $\mathcal{D}_{22} = \{(x, 1) : x \neq 0, 1\}$ along the tangent vector $\partial/\partial y$ is given by the $\mathbb{Q}$-linear span of $s_0, \dots, s_3$ where

$$\begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 2\pi i & & \\ -\operatorname{Li}_1(\frac{x}{x-1}) & 0 & 2\pi i & \\ \operatorname{Li}_2(\frac{x}{x-1}) & 2\pi i \operatorname{Li}_1(x) & -2\pi i \log \frac{x}{x-1} & (2\pi i)^2 \end{bmatrix}.$$

(iv) Limit MHS on $\mathcal{D}_{12} = \{(1/y, y) : y \neq 0, 1\}$ along the tangent vector $\partial/\partial x$ is given by the $\mathbb{Q}$-linear span of $s_0, \dots, s_3$ where

$$\begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -\operatorname{Li}_1(\frac{y}{y-1}) & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ -\operatorname{Li}_2(\frac{y}{y-1}) & 2\pi i \log \frac{y}{y-1} & 0 & (2\pi i)^2 \end{bmatrix}.$$

(v) $\mathcal{D}_{10} \cap \mathcal{D}_{22} = (0, 1)$. From (i) we see that there are limit MHS on the open set $\mathcal{D}_{10} \setminus \{(0, 1)\}$ of $\mathcal{D}_{10}$. We now can easily extend these MHS to $(0, 1)$ along the vector $\partial/\partial y$ and find the limit MHS at $(0, 1)$ to be the $\mathbb{Q}$-linear span of $s_0, \dots, s_3$ where

$$\begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ 0 & 0 & 0 & (2\pi i)^2 \end{bmatrix}.$$

If we start from (iii) and then extend the MHS to $(0, 1)$ along tangent vector $\partial/\partial x$ we will get the same limit MHS.

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(vi) $\mathcal{D}_{11}\cap\mathcal{D}_{12}=\mathcal{D}_{12}\cap\mathcal{D}_{22}=\mathcal{D}_{11}\cap\mathcal{D}_{22}=(1,1)$. We can start from either case (ii), (iii) or (iv). Extending the limit MHS of case (ii) we see immediately that the along the tangent vector $\partial/\partial y$ the limit MHS at $(1,1)$ is given by the $\mathbb{Q}$-linear span of

$$\begin{bmatrix} s_0 & s_1 & s_2 & s_3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 2\pi i & & \\ 0 & 0 & 2\pi i & \\ E_{4,1} & 0 & 0 & (2\pi i)^2 \end{bmatrix}. \tag{22}$$

If we extend the limit MHS of case (iii) to $(1,1)$ along tangent vector $\partial/\partial x$ we find that only the lower left corner entry is different from the above. Instead of 0, it is

$$E_{4,1} = \lim_{x\to 1} \mathrm{Li}_2(\frac{x}{x-1}) + \frac{1}{2}\log^2(1-x) - \log x \log(1-x) = -\mathrm{Li}_2(1) = -\frac{\pi^2}{12},$$

since

$$\mathrm{Li}_2(1-t) + \mathrm{Li}_2(1-1/t) + \log^2 t/2 = 0 \quad \forall t \neq 0. \tag{23}$$

But if we take $s_0' = s_0 - s_3/48$ we get the same basis as in (22). The same phenomenon occurs if we start from case (iv) and then use tangent vector $\partial/\partial y$.

If we extend the limit MHS of (iv) to the point $(1,1)$ along the tangent vector $\partial/\partial y$, then we find that

$$E_{4,1} = \lim_{y\to 1} -\mathrm{Li}_2(\frac{y}{y-1}) - \frac{1}{2}\log^2(1-y) = \mathrm{Li}_2(1) = \frac{\pi^2}{12}$$

by taking $t = 1 - y$ in (23). Now if we let $s_0' = s_0 + \frac{1}{48}s_3$, then we get the same basis as in (22). This phenomenon happens in higher logarithm cases too.

4.2 Limit MHS of Triple Logarithm

The triple logarithm function $\mathfrak{L}_3(x,y,z)$ is defined by [19, Example 5.2]

$$\begin{aligned}\mathrm{Li}_{1,1,1}(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} & \frac{dz}{1-z}\frac{dy}{1-y}\frac{dx}{1-x} + \frac{d(yz)}{1-yz}\left(\frac{dz}{1-z}+\frac{dy}{y(y-1)}\right)\frac{dx}{1-x} \\ & + \frac{d(yz)}{1-yz}\frac{dx}{1-x}\left(\frac{dz}{1-z}+\frac{dy}{y(y-1)}\right) \\ & + \frac{dz}{1-z}\frac{d(xy)}{1-xy}\left(\frac{dy}{1-y}+\frac{dx}{x(x-1)}\right) \\ & + \frac{d(xyz)}{1-xyz}\left(\frac{dz}{1-z}+\frac{d(xy)}{xy(xy-1)}\right)\left(\frac{dy}{1-y}+\frac{dx}{x(x-1)}\right) \\ & + \frac{d(xyz)}{1-xyz}\left(\frac{d(yz)}{1-yz}+\frac{dx}{x(x-1)}\right)\left(\frac{dz}{1-z}+\frac{dy}{y(y-1)}\right).\end{aligned}$$

Set

$$\tau_{[3]}(2\pi i) = \operatorname{diag}[1, 2\pi i, 2\pi i, 2\pi i, (2\pi i)^2, (2\pi i)^2, (2\pi i)^2, (2\pi i)^3],$$

and define the matrix $\mathcal{M}_{[3]}(x, y, z)\tau_{[3]}(2\pi i)^{-1}$ by

$$\begin{pmatrix}
1 \\
\mathfrak{L}_1(z) & 1 \\
\mathfrak{L}_1(yz) & 0 & 1 \\
\mathfrak{L}_1(xyz) & 0 & 0 & 1 \\
\mathfrak{L}_2(y,z) & \mathfrak{L}_1(y) & H(y,z) & 0 & 1 \\
\mathfrak{L}_2(xy,z) & \mathfrak{L}_1(xy) & 0 & H(xy,z) & 0 & 1 \\
\mathfrak{L}_2(x,yz) & 0 & \mathfrak{L}_1(x) & H(x,yz) & 0 & 0 & 1 \\
\mathfrak{L}_3(x,y,z) & \mathfrak{L}_2(x,y) & H(y,z)\mathfrak{L}_1(x) & E_{8,4} & \mathfrak{L}_1(x) & H(x,y) & H(y,z) & 1
\end{pmatrix}$$

where

$$E_{8,4} = \mathfrak{L}_2\Big(\frac{1-xy}{1-x}, \frac{1-xyz}{1-xy}\Big) = \mathfrak{L}_2(y^{-1}, x^{-1}) - \mathfrak{L}_2(y^{-1}, yz) + \log\frac{x-1}{x(1-yz)}\mathfrak{L}_1(z).$$

(i) Extension to an open set of $\mathcal{D}_{10} = \{x = 0\}$ along the vector $\partial/\partial x$. By Theorem 3.16 or direct computation, $T_{\{x=0\}} = I_8 - e_{64} - e_{74} - e_{86}$. So

$$t^{-N_{\{x=0\}}} = I_8 + \frac{\log t}{2\pi i}(e_{64} + e_{74} + e_{86}) + \log^2 t/2(2\pi i)^2 e_{84}.$$

To determine the limit MHS along $\{x = 0\}$ we need to find $g(y, z) = \lim_{t\to 0} I(t)$, where

$$I(t) = \mathfrak{L}_2\Big(\frac{1-ty}{1-t}, \frac{1-tyz}{1-ty}\Big) - \log t \log\big(ty(z-1)\big) + \frac{\log^2 t}{2},$$

because $\lim_{t\to 0}\log t\log(1-t) = 0$. We see that $I'(t) = f'(t)$ where

$$f(t) = \operatorname{Li}_2(1-t) - \log(1-t)\log\frac{y(1-z)}{y-1} - \operatorname{Li}_2\Big(\frac{(1-t)y}{y-1}\Big) + \operatorname{Li}_2(ty).$$

Thus

$$\begin{aligned}
g(y,z) &= I(1/yz) + f(0) - f(1/yz) \\
&= \operatorname{Li}_2(1-yz) + \operatorname{Li}_2(1) + \log\frac{y(1-z)}{y-1}\log(1-yz) \\
&\qquad - \operatorname{Li}_2\Big(\frac{y}{y-1}\Big) + \operatorname{Li}_2\Big(\frac{1-yz}{z(1-y)}\Big) - \operatorname{Li}_2\Big(\frac{1}{z}\Big) + \log(1-y)\log(yz).
\end{aligned}$$

It is easy to see by differentiation with respect to y and $g(0, z) = 2\operatorname{Li}_2(1)$ that

$$g(y,z) = \operatorname{Li}_{1,1}(y,z) - \operatorname{Li}_2(1-y) + 3\operatorname{Li}_2(1).$$

Hence the local system $V_{\mathbb{Q},\{x=0\}}$ of the limit MHS over $\{(0,y,z): y(1-y)(1-z)(1-yz)\neq 0\}$ is the $\mathbb{Q}$-linear span of $s_0,\dots,s_7$ where $[s_0\cdots s_7]$ is given by

$$\begin{pmatrix}
1 \\
\mathfrak{L}_1(z) & 1 \\
\mathfrak{L}_1(yz) & 0 & 1 \\
0 & 0 & 0 & 1 \\
\mathfrak{L}_2(y,z) & \mathfrak{L}_1(y) & H(y,z) & 0 & 1 \\
0 & 0 & 0 & \mathfrak{L}_1(z)-\log y & 0 & 1 \\
0 & 0 & 0 & \mathfrak{L}_1(yz) & 0 & 0 & 1 \\
0 & 0 & 0 & g(y,z) & 0 & \mathfrak{L}_1(y) & H(y,z) & 1
\end{pmatrix}\tau_{[3]}(2\pi i).$$

(ii) On $\mathcal{D}_{20}=\{y=0\}$. Similar computation as above shows that the local system $V_{\mathbb{Q},\{y=0\}}$ of the limit MHS over $\{(x,0,z): x(1-x)(1-z)\neq 0\}$ along the vector $\partial/\partial y$ is the $\mathbb{Q}$-linear span of $s_0,\dots,s_7$ where $[s_0\cdots s_7]$ is given by

$$\begin{pmatrix}
1 \\
\mathfrak{L}_1(z) & 1 \\
0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & \mathfrak{L}_1(z) & 0 & 1 \\
0 & 0 & 0 & \mathfrak{L}_1(z)-\log x & 0 & 1 \\
0 & 0 & \mathfrak{L}_1(x) & -\mathfrak{L}_1(x)-\log x & 0 & 0 & 1 \\
0 & 0 & \mathfrak{L}_1(z)\mathfrak{L}_1(x) & g(x,z) & \mathfrak{L}_1(x) & -\mathfrak{L}_1(x)-\log x & \mathfrak{L}_1(z) & 1
\end{pmatrix}\tau_{[3]}(2\pi i),$$

where

$$g(x,z)=\mathrm{Li}_2(1)-\mathrm{Li}_1(z)(\mathrm{Li}_1(x)+\log x)-\mathrm{Li}_2(1-x^{-1}).$$

(iii) On $\mathcal{D}_{11}=\{x=1\}$. Then the local system $V_{\mathbb{Q},\{x=1\}}$ of the limit MHS over $\{(1,y,z): y(1-y)(1-z)(1-yz)\neq 0\}$ along the vector $\partial/\partial x$ is the $\mathbb{Q}$-linear span of $s_0,\dots,s_7$ where $[s_0\cdots s_7]$ is given by

$$\begin{pmatrix}
1 \\
\mathfrak{L}_1(z) & 1 \\
\mathfrak{L}_1(yz) & 0 & 1 \\
\mathfrak{L}_1(yz) & 0 & 0 & 1 \\
\mathfrak{L}_2(y,z) & \mathfrak{L}_1(y) & H(y,z) & 0 & 1 \\
\mathfrak{L}_2(y,z) & \mathfrak{L}_1(y) & 0 & H(y,z) & 0 & 1 \\
\mathfrak{L}_2(1,yz) & 0 & 0 & \mathfrak{L}_1(yz) & 0 & 0 & 1 \\
\mathfrak{L}_3(1,y,z) & \mathfrak{L}_2(1,y) & 0 & g(y,z) & 0 & \mathfrak{L}_1(y) & H(y,z) & 1
\end{pmatrix}\tau_{[3]}(2\pi i),$$

where

$$g(y,z)=\mathfrak{L}_2(y,z)+\mathrm{Li}_2\big(1/(1-y)\big).$$

(iv) On $\mathcal{D}_{22}=\{y=1\}$. The local system $V_{\mathbb{Q},\{y=1\}}$ of the limit MHS over $\{(x,1,z): x(1-x)(1-z)(1-xz)\neq 0\}$ along the vector $\partial/\partial y$ is the $\mathbb{Q}$-linear span of $s_0,\dots,s_7$

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where $[s_0 \cdots s_7]$ is given by

$$\begin{pmatrix} 1 &&&&&&& \\ \mathfrak{L}_1(z) & 1 &&&&&& \\ \mathfrak{L}_1(z) & 0 & 1 &&&&& \\ \mathfrak{L}_1(xz) & 0 & 0 & 1 &&&& \\ \mathfrak{L}_2(1,z) & 0 & \mathfrak{L}_1(z) & 0 & 1 &&& \\ \mathfrak{L}_2(x,z) & \mathfrak{L}_1(x) & 0 & H(x,z) & 0 & 1 && \\ \mathfrak{L}_2(x,z) & 0 & \mathfrak{L}_1(x) & H(x,z) & 0 & 0 & 1 & \\ \mathfrak{L}_3(x,1,z) & \mathfrak{L}_2(\frac{x}{x-1}) & \mathfrak{L}_1(x)\mathfrak{L}_1(z) & \mathfrak{L}_2(1,\frac{1-xz}{1-x}) & \mathfrak{L}_1(x) & -\mathfrak{L}_1(x)-\log x & \mathfrak{L}_1(z) & 1 \end{pmatrix} \tau_{[3]}(2\pi i).$$

(v) On $\mathcal{D}_{33} = \{z = 1\}$. This case is the most interesting because the variation of MHS for $\mathrm{Li}_{2,1}$ appears implicitly. The local system $V_{\mathbb{Q},\{z=1\}}$ of the limit MHS over $\{(x, y, 1) : xy(1-x)(1-y)(1-xy) \neq 0\}$ along the vector $\partial/\partial z$ is the $\mathbb{Q}$-linear span of $s_0, \dots, s_7$ where $[s_0 \cdots s_7]$ is given by

$$\begin{pmatrix} 1 &&&&&&& \\ 0 & 1 &&&&&& \\ \mathfrak{L}_1(y) & 0 & 1 &&&&& \\ \mathfrak{L}_1(xy) & 0 & 0 & 1 &&&& \\ \mathrm{Li}_2(\frac{y}{y-1}) & \mathfrak{L}_1(y) & \log\frac{y-1}{y} & 0 & 1 &&& \\ \mathrm{Li}_2(\frac{xy}{xy-1}) & \mathfrak{L}_1(xy) & 0 & \log\frac{xy-1}{xy} & 0 & 1 && \\ \mathfrak{L}_2(x,y) & 0 & \mathfrak{L}_1(x) & H(x,y) & 0 & 0 & 1 & \\ g(x,y) & \mathfrak{L}_2(x,y) & \log\frac{y-1}{y}\mathfrak{L}_1(x) & h(x,y) & \mathfrak{L}_1(x) & H(x,y) & \log\frac{y-1}{y} & 1 \end{pmatrix} \tau_{[3]}(2\pi i),$$

where

$$g(x,y) = \mathrm{Li}_{1,2}\Big(\frac{x(y-1)}{xy-1}, \frac{y}{y-1}\Big) + \log(1-xy)\,\mathrm{Li}_2\Big(\frac{y}{y-1}\Big)$$

and

$$h(x,y) = \mathrm{Li}_2\Big(\frac{1-xy}{x(1-y)}\Big) + H(x,y)\log\frac{xy-1}{xy}.$$

We observe that this is *essentially* the variation matrix $\mathcal{M}_{1,2}\Big(\frac{x(y-1)}{xy-1}, \frac{y}{y-1}\Big)$.

We omit the following similar cases:

(vi) On $\mathcal{D}_{12} = \{xy = 1\}$. Extend along the vector $\partial/\partial x$ or $\partial/\partial y$.
(vii) On $\mathcal{D}_{23} = \{yz = 1\}$. Extend along the vector $\partial/\partial y$ or $\partial/\partial z$.
(viii) On $\mathcal{D}_{13} = \{xyz = 1\}$. Extend along the vector $\partial/\partial x$, or $\partial/\partial y$, or $\partial/\partial z$.

(ix) $\mathcal{D}_{10} \cap \mathcal{D}_{20}$. We may start from either case (i) or case (ii). Straightforward calculation starting from case (i) shows that the extension of the MHS on $\mathcal{D}_{10}$ to $\mathcal{D}_{10} \cap \mathcal{D}_{20}$ along the vector $\partial/\partial y$ is the $\mathbb{Q}$-linear span of $[s_0 \cdots s_7]$ given by

$$\begin{pmatrix} 1 &&&&&&& \\ \mathfrak{L}_1(z) & 1 &&&&&& \\ 0 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 1 &&&& \\ 0 & 0 & \mathfrak{L}_1(z) & 0 & 1 &&& \\ 0 & 0 & 0 & \mathfrak{L}_1(z) & 0 & 1 && \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 2\,\mathrm{Li}_2(1) & 0 & 0 & \mathfrak{L}_1(z) & 1 \end{pmatrix} \tau_{[3]}(2\pi i).$$

If we start from case (ii) and take the vector $\partial/\partial x$ then we will get the same result.

(x) $\mathcal{D}_{11} \cap \mathcal{D}_{22}$. We may start from either case (iii) or case (iv). Straightforward calculation starting from case (iii) shows that along the vector $\partial/\partial y$ the limit MHS on $\mathcal{D}_{11} \cap \mathcal{D}_{22}$ is the $\mathbb{Q}$-linear span of $[s_0 \cdots s_7]$ given by

$$\begin{pmatrix} 1 &&&&&&& \\ \mathfrak{L}_1(z) & 1 &&&&&& \\ \mathfrak{L}_1(z) & 0 & 1 &&&&& \\ \mathfrak{L}_1(z) & 0 & 0 & 1 &&&& \\ \mathfrak{L}_2(1,z) & 0 & \mathfrak{L}_1(z) & 0 & 1 &&& \\ \mathfrak{L}_2(1,z) & 0 & 0 & \mathfrak{L}_1(z) & 0 & 1 && \\ \mathfrak{L}_2(1,z) & 0 & 0 & \mathfrak{L}_1(z) & 0 & 0 & 1 & \\ \mathfrak{L}_3(1,1,z) & 0 & 0 & E_{8,4} & 0 & 0 & \mathfrak{L}_1(z) & 1 \end{pmatrix} \tau_{[3]}(2\pi i),$$

where $E_{8,4} = \mathfrak{L}_2(z) + 2\,\mathrm{Li}_2(1)$. If we start from case (iv) then we find that $E_{8,4} = \mathfrak{L}_2(z)$ and therefore we get the same limit MHS on $\mathcal{D}_{11} \cap \mathcal{D}_{22}$ along vector $\partial/\partial x$.

By similar computation we can determine the limit MHS on the intersections of any two of the irreducible components $\mathcal{D}_{ij}$ along any vector. Finally, at all the of the following four points: $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$ we find without much difficulty that the columns of the matrix $\tau_{[3]}(2\pi i)$ provide us $s_0, \ldots, s_7$ for the limit MHS along vectors $\partial/\partial x$, or $\partial/\partial y$, or $\partial/\partial z$.

From all the above examples we want to make the following

Conjecture 4.2 *The variations of mixed Hodge-Tate structures related to every multiple polylogarithm can be produced as the variations of some limit mixed Hodge-Tate structures related to some suitable choice of multiple logarithm.*

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5 Double Polylogarithm Variations of MHS

One can similarly generalize the above theory to multiple polylogarithms. One knows that on $\mathbb{C}^\times \setminus \{1\}$ the matrix $\mathcal{M}_n(x)$

$$\begin{bmatrix} 1 & & & & & & \\ \mathrm{Li}_1(x) & 1 & & & & & \\ \mathrm{Li}_2(x) & \log x & 1 & & & & \\ \mathrm{Li}_3(x) & \frac{\log^2 x}{2} & \log x & 1 & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ \mathrm{Li}_{n-1}(x) & \frac{\log^{n-2} x}{(n-2)!} & \frac{\log^{n-3} x}{(n-3)!} & \cdots & \log x & 1 & \\ \mathrm{Li}_n(x) & \frac{\log^{n-1} x}{(n-1)!} & \frac{\log^{n-2} x}{(n-2)!} & \cdots & \frac{\log^2 x}{2} & \log x & 1 \end{bmatrix} \mathrm{diag}\big[1, 2\pi i, \dots, (2\pi i)^n\big]$$

provides a variation of mixed Hodge-Tate structures related to the classical n-logarithm (*cf.* [10]). To be more precise, in the definition of $\mathrm{Li}_m(x)$ and $\log^m(x)/m!$ above we actually fixed a path p from 0 to x and a path q from 1 to x (both independent of m) and set

$$\mathrm{Li}_m(x) = \int_p \frac{dt}{1-t} \underbrace{\frac{dt}{t} \cdots \frac{dt}{t}}_{m-1 \text{ times}}, \qquad \frac{\log^m(x)}{m!} = \int_q \underbrace{\frac{dt}{t} \cdots \frac{dt}{t}}_{m \text{ times}}.$$

5.1 Double Polylogarithms of Weight 3

There are only four multiple polylogarithms of weight 3. Having dealt with Li_3 and $\mathrm{Li}_{1,1,1}$ we now turn to $\mathrm{Li}_{1,2}$ and $\mathrm{Li}_{2,1}$. The referee points out that the following theorem is an immediate consequence of the motivic construction of Deligne and Goncharov. It seems to me that the general theory can only tell us that the monodromy representations lie in $\mathrm{GL}(\mathbb{Q})$ while in our case we can show that for both $\mathrm{Li}_{1,2}$ and $\mathrm{Li}_{2,1}$ we can even use $\mathrm{GL}(\mathbb{Z})$.

Theorem 5.1 *Each of the weight three depth two multiple polylogarithms underlies a good variation of mixed Hodge-Tate structures over $S_2 = \mathbb{C}^2 \setminus \{xy(1-x)(1-y)(1-xy) = 0\}$. For $\mathrm{Li}_{2,1}$ the graded weight quotients are $\mathbb{Z}(0)$, $\mathbb{Z}(1)\oplus\mathbb{Z}(1)$, $\mathbb{Z}(2)\oplus\mathbb{Z}(2)$, and $\mathbb{Z}(3)$. For $\mathrm{Li}_{1,2}$ they are $\mathbb{Z}(0)$, $\mathbb{Z}(1)\oplus\mathbb{Z}(1)$, $\mathbb{Z}(2)\oplus\mathbb{Z}(2)\oplus\mathbb{Z}(2)$, and $\mathbb{Z}(3)$.*

Proof Let $\tau_{2,1}(\lambda) = \mathrm{diag}\big[1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^3\big]$. We define the multi-valued matrix function over S_2,

$$\mathcal{M}_{2,1}(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(y) & 1 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(xy) & 0 & 1 & 0 & 0 & 0 \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 & 0 & 0 \\ \mathrm{Li}_2(xy) & 0 & \log(xy) & 0 & 1 & 0 \\ \mathrm{Li}_{2,1}(x,y) & \mathrm{Li}_2(x) & f(x,y) & \log x & \mathrm{Li}_1(y) & 1 \end{bmatrix} \tau_{2,1}(2\pi i),$$

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where

$$f(x,y) = -\int_{a_1}^{1} \frac{dt}{t}\frac{dt}{t-a_2} = \mathrm{Li}_2(x^{-1}) - \mathrm{Li}_2(y) + \log(xy)\,\mathrm{Li}_1(y).$$

The columns of $\mathcal{M}_{2,1}(x,y)$ form the fundamental solutions of the differential equation over S_2,

$$d\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ d\,\mathrm{Li}_1(y) & 0 & 0 & 0 & 0 & 0 \\ d\,\mathrm{Li}_1(xy) & 0 & 0 & 0 & 0 & 0 \\ 0 & d\,\mathrm{Li}_1(x) & d\,\mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 0 & 0 & 0 \\ 0 & 0 & d\log(xy) & 0 & 0 & 0 \\ 0 & 0 & 0 & d\log x & d\,\mathrm{Li}_1(y) & 0 \end{bmatrix} \lambda.$$

Let $1 \le i \le j \le 2$ and $q_{ij} \in \pi_1(S_2, \mathbf{x})$ (resp., $1 \le j \le 2$ and q_{j0}) be a loop in S_2 turning around the irreducible component $\mathcal{D}_{ij}$ counterclockwise only once such that $\int_{q_{ij}} d\log(1 - x_i \cdots x_j) = -2\pi\sqrt{-1}$ (resp., $\int_{q_{j0}} d\log x_j = 2\pi i$). Let e_{st} be the matrix with 1 at (s,t)-th entry and 0 elsewhere. Observe that if $q_{i\infty}$ is a loop in S_2 turning around $x_i = \infty$ only once then $q_{i\infty} = -q_{i0} + q_{ii}$. By simple computation we see that the monodromy representation $\rho\colon \pi_1(S_2, \mathbf{x}) \to \mathrm{GL}_6(\mathbb{Q})$ is given as follows:

$$\begin{aligned} M(q_{10}) &= I - e_{43} + e_{53} + e_{64} \\ M(q_{20}) &= I + e_{63} \\ M(q_{11}) &= I + e_{42} - e_{43} \\ M(q_{22}) &= I + e_{21} + e_{43} + e_{65} \\ M(q_{12}) &= I + e_{31}. \end{aligned}$$

We can now easily define the weight and Hodge filtrations, determine the MHS over S_2 and compute the limit MHS at the "infinity". This proves the theorem for $\mathrm{Li}_{2,1}$.

To deal with the multiple polylogarithm $\mathrm{Li}_{1,2}(x,y)$ we set

$$\tau_{1,2}(\lambda) = \mathrm{diag}[1, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2, \lambda^3]$$

and define the multi-valued matrix function $\mathcal{M}_{1,2}(x,y)$ over S_2 as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(y) & 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_1(xy) & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 & 0 & 0 & 0 \\ \mathrm{Li}_2(y) & \log(y) & 0 & 0 & 1 & 0 & 0 \\ \mathrm{Li}_2(xy) & 0 & \log(xy) & 0 & 0 & 1 & 0 \\ \mathrm{Li}_{1,2}(x,y) & \mathrm{Li}_1(x)\log(y) & g(x,y) & \log y & \mathrm{Li}_1(x) & -\mathrm{Li}_1(x^{-1}) & 1 \end{bmatrix} \tau_{1,2}(2\pi i)$$

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where

$$g(x,y) = -\int_{a_1}^{1} \frac{dt}{t-a_2}\frac{dt}{t} = \mathrm{Li}_2(y) - \mathrm{Li}_2(x^{-1}) - \log(xy)\,\mathrm{Li}_1(x^{-1}).$$

The columns of $\mathcal{M}_{2,1}(x,y)$ form the fundamental solutions of the differential equation over S

$$d\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d\,\mathrm{Li}_1(y) & 0 & 0 & 0 & 0 & 0 & 0 \\ d\,\mathrm{Li}_1(xy) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d\,\mathrm{Li}_1(x) & d\,\mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 0 & 0 & 0 & 0 \\ 0 & d\log(y) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d\log(xy) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d\log(y) & d\,\mathrm{Li}_1(x) & -d\,\mathrm{Li}_1(x^{-1}) & 0 \end{bmatrix} \lambda.$$

The monodromy representation $\rho\colon \pi_1(S_2,\mathbf{x}) \to \mathrm{GL}_7(\mathbb{Q})$ is given as follows:

$$M(q_{10}) = I - e_{43} + e_{63} - e_{76}$$
$$M(q_{20}) = I + e_{52} + e_{63} + e_{74}$$
$$M(q_{11}) = I + e_{42} - e_{43} + e_{75} - e_{76}$$
$$M(q_{22}) = I + e_{21} + e_{43}$$
$$M(q_{12}) = I + e_{31}.$$

We can now determine the MHS over S_2 and compute the limit MHS at the "infinity" as before. This proves the theorem for $\mathrm{Li}_{2,1}$. ■

5.2 Some Open Problems

It seems very difficult to write down explicitly the variation matrix associated with the general multiple polylogarithm $\mathrm{Li}_{m_1,\dots,m_n}(\mathbf{x})$. However, the following general result has been proved by Deligne and Goncharov [7]:

The multiple polylogarithm $\mathrm{Li}_{m_1,\dots,m_n}(\mathbf{x})$ underlies a good unipotent graded-polarizable variation of mixed Hodge-Tate structures $(V_{m_1,\dots,m_n}, W_\bullet, \mathcal{F}^\bullet)$ over

$$S_n = \mathbb{C}^n \setminus \Big\{ \prod_{i=1}^{n} x_i(1-x_i) \prod_{1\le i<j\le n} (1-x_i\cdots x_j) = 0 \Big\}$$

with the weight-graded quotients gr^W_{-2k} being given by c_k copies of the Tate structure $\mathbb{Z}(k)$ which are nonzero only for $0 \le k \le K$.

Here c_k is the number of different ways to pick ordered $(k+2)$-tuples $(b_{\alpha_0},\dots,b_{\alpha_{k+1}})$ from the ordered numbers $(b_0,\dots,b_{K+1})$ in the following tableau, where $a_1,\dots,a_n$ are nonzero

$$(*) \qquad |b_0|\cdots|b_{K+1}| = |\,0\,|a_1 \underbrace{|\,0\,|\cdots|\,0\,|}_{a_1-1 \text{ times}} a_2 \cdots\cdots a_n \underbrace{|\,0\,|\cdots|\,0\,|}_{a_n-1 \text{ times}} 1\,|$$

such that all of the following conditions are satisfied:

(i) $\alpha_0 = 0$,
(ii) $\alpha_{k+1} = K + 1$,
(iii) For all $0 \le i \le k$, either $\alpha_{i+1} = \alpha_i + 1$ or at least one of b_{α_i} and $b_{\alpha_{i+1}}$ is nonzero,

It is apparent that

$$c_k \ge d_k(m_1, \dots, m_n) = \sum_{\substack{k_1 + \cdots + k_n = k \\ 0 \le k_i \le m_i}} 1.$$

Each term in the sum corresponds to the following choice: for every $i = 1, \dots, n$, choose k_i 0's immediately after a_i.

Example 5.2 By the definition, we always have $c_0 = c_K = 1$. When $m_1 = \cdots = m_n = 1$, tableau $(*)$ becomes

$$\big| b_0 \big| \cdots \big| b_{n+1} \big| = \big| 0 \big| \underbrace{1 \big| \cdots \big| 1}_{n+1 \text{ times}} \big| .$$

Because b_0 and the last b_{n+1} is always picked, c_k is the number of ways to choose k elements from the set $\{b_1, \dots, b_n\}$, *i.e.*, $c_k = \binom{n}{k}$.

For ease of statement let us put a box $\boxed{}$ on a number whenever we choose it.

Example 5.3 Let's look at $\mathrm{Li}_{1,2}$. We have the following six nontrivial ways to put boxes on $\big| 0 \big| a_1 \big| a_2 \big| 0 \big| 1 \big|$:

$$(1)\big|\boxed{0}\big|\boxed{a_1}\big| a_2 \big| 0 \big|\boxed{1}\big| \quad (2)\big|\boxed{0}\big| a_1 \big|\boxed{a_2}\big| 0 \big|\boxed{1}\big| \quad (3)\big|\boxed{0}\big|\boxed{a_1}\big|\boxed{a_2}\big| 0 \big|\boxed{1}\big|$$

$$(4)\big|\boxed{0}\big|\boxed{a_1}\big| a_2 \big|\boxed{0}\big|\boxed{1}\big| \quad (5)\big|\boxed{0}\big| a_1 \big|\boxed{a_2}\big|\boxed{0}\big|\boxed{1}\big| \quad (6)\big|\boxed{0}\big|\boxed{a_1}\big|\boxed{a_2}\big|\boxed{0}\big|\boxed{1}\big|$$

Thus $c_0 = c_3 = 1$, $c_1 = 2$ and $c_2 = 3$.
However, for $\mathrm{Li}_{2,1}$ we have altogether only six ways to do this:

$$(1)\big|\boxed{0}\big| a_1 \big| 0 \big| a_2 \big|\boxed{1}\big| \quad (2)\big|\boxed{0}\big|\boxed{a_1}\big| 0 \big| a_2 \big|\boxed{1}\big| \quad (3)\big|\boxed{0}\big| a_1 \big| 0 \big|\boxed{a_2}\big|\boxed{1}\big|$$

$$(4)\big|\boxed{0}\big|\boxed{a_1}\big|\boxed{0}\big| a_2 \big|\boxed{1}\big| \quad (5)\big|\boxed{0}\big|\boxed{a_1}\big| 0 \big|\boxed{a_2}\big|\boxed{1}\big| \quad (6)\big|\boxed{0}\big|\boxed{a_1}\big|\boxed{0}\big|\boxed{a_2}\big|\boxed{1}\big|$$

Thus $c_0 = c_3 = 1$, $c_1 = c_2 = 2$.

We now can generalize Theorem 5.1 to

Theorem 5.4 *The double polylogarithm $\mathrm{Li}_{r,s}$ underlies a good unipotent graded-polarizable variation of a mixed Hodge-Tate structure with the graded weight piece gr^W_{-2k} being direct sums of c_k copies of $\mathbb{Z}(k)$ where*

$$c_k = \begin{cases} d_k(r,s) + 1 & \text{if } r \ne k = s, \\ d_k(r,s) & \text{otherwise,} \end{cases}$$

and

$$d_k(r,s) = \begin{cases} 0 & \text{if } k < 0 \text{ or } k > r+s, \\ k+1 & \text{if } 0 \le k \le \min\{r,s\}, \\ \min\{r,s\}+1 & \text{if } \min\{r,s\} \le k \le \max\{r,s\}, \\ r+s+1-k & \text{if } \max\{r,s\} \le k \le r+s. \end{cases}$$

Among all the double polylogarithms the homogeneous one $\mathrm{Li}_{r,r}(x,y)$ behaves most regularly. It satisfies $c_0 = c_{2r} = 1$, $c_1 = c_{2r-1} = 2, \ldots, c_{r-1} = c_{r+1} = r$, $c_r = r+1$.

In general, as we remarked at the beginning of this section, the multiple polylogarithm $\mathrm{Li}_{m_1,\ldots,m_n}(\mathbf{x})$ underlies a good variation of mixed Hodge-Tate structures with the graded weight piece gr^W_{-2k} being direct sums of c_k copies of $\mathbb{Z}(k)$ for some positive integer c_k. It is clear that $c_k \ge d_k(m_1,\ldots,m_n)$ and $c_k(1,\ldots,1,m_n) = d_k(m_1,\ldots,m_n)$. It would be very interesting to solve the following problem.

Problem 5.5

(1) Find a closed formula for c_k depending only on $m_1,\ldots,m_n$ and k.
(2) Determine the variation matrix $\mathcal{M}_{m_1,\ldots,m_n}(\mathbf{x})$ explicitly.
(3) Determine the connection matrix ω explicitly.
(4) Determine the monodromy actions explicitly.

6 Single-Valued Version of Multiple Polylogarithms

If Problem 5.5(2) is solved then following an idea of Beilinson and Deligne [1] as given in [3] one can easily discover the single-valued version of $\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$ which we denote by $\mathcal{L}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$ and which should be a real analytic function. In what follows we outline the procedure for multiple logarithms only.

6.1 General Procedure for Producing Single-Valued Multiple Logarithms

For any $n \ge 2$ let $L_{[n]} = L_{[n]}(\mathbf{x}) = [C_{\mathbf{0}} \cdots C_{\mathbf{1}}]$ be the matrix with 2^n columns $C_{\mathbf{j}}$ ($\mathbf{j} \in \mathcal{S}_n$) as before and $\mathcal{M}_{[n]} = \mathcal{M}_{[n]}(\mathbf{x}) = L_{[n]}(\mathbf{x})\tau_{[n]}(2\pi i)$ where

$$\tau_{[n]}(\lambda) = \mathrm{diag}\big[\,\lambda^{|\mathbf{j}|}\big]_{\mathbf{j}\in\mathcal{S}_n}.$$

Define the matrix

$$B_{[n]} = \tau_{[n]}(i)\mathcal{M}_{[n]}\overline{\mathcal{M}}_{[n]}^{-1}\tau_{[n]}(i)$$

where $\overline{\mathcal{M}}_{[n]}$ is the complex conjugation of $\mathcal{M}_{[n]}$. From our calculation of the monodromy we see that B is a single-valued matrix function defined over S_n. Moreover

$$\overline{B}_{[n]} = B_{[n]}^{-1}$$

since $\overline{\tau_{[n]}(i)} = \tau_{[n]}(i)^{-1}$. Now that $B_{[n]} = I + N$ with I the identity matrix and N a nilpotent matrix, we see that $\log B$ is well defined and satisfies

$$\overline{\log B_{[n]}} = -\log B_{[n]},$$

namely, $\log B_{[n]}$ is a pure imaginary matrix. Then we define $-1/(2i)$ times the lower left corner entry of $\log B$ to be $\mathcal{L}_{[n]}(\mathbf{x})$ which is a single-valued real analytic version of the multiple logarithm $\mathfrak{L}_n(\mathbf{x})$.

Remark 6.1 Our method is slightly different from that in [1]. In fact when we are in the polylogarithm case the matrix B constructed as above is the conjugate of the one in [1] by $\tau(i)$.

6.2 Single-Valued Double Logarithms

We have seen that

$$L_{1,1}(x,y) = \begin{bmatrix} 1 & & & \\ \mathrm{Li}_1(y) & 1 & & \\ \mathrm{Li}_1(xy) & 0 & 1 & \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \log\frac{x-1}{x(1-y)} & 1 \end{bmatrix} \text{ and } \tau_{1,1}(\lambda) = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda & \\ & & & \lambda^2 \end{bmatrix}.$$

Let $B_{1,1}(x,y) = \tau_{1,1}(i)L_{1,1}(x,y)\tau_{1,1}(-1)\overline{L_{1,1}(x,y)}^{-1}\tau_{1,1}(i)$. Then $B_{1,1}(x,y)$ is unipotent and single-valued. An easy calculation shows

$$\log B_{1,1}(x,y) = \begin{bmatrix} 0 & & & \\ -2i\log|1-y| & 0 & & \\ -2i\log|1-xy| & 0 & 0 & \\ -2i\mathcal{L}_{1,1}(x,y) & -2i\log|1-x| & 2i\log\left|\frac{x-1}{x(1-y)}\right| & 0 \end{bmatrix}$$

where

(24)

$$\mathcal{L}_{1,1}(x,y) = \mathrm{Im}\left(\mathrm{Li}_{1,1}(x,y)\right) - \arg(1-y)\log|1-x| + \arg(1-xy)\log\left|\frac{x-1}{x(1-y)}\right|$$

is the single-valued real analytic version of $\mathrm{Li}_{1,1}(x,y)$.

By differentiation it is easy to check that

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\left(\frac{xy-y}{1-y}\right) - \mathrm{Li}_2\left(\frac{y}{y-1}\right) - \mathrm{Li}_2(xy).$$

So by using the single-valued dilogarithm function

$$\mathcal{L}_2(z) = \mathrm{Im}\left(\mathrm{Li}_2(z)\right) + \arg(1-z)\log|z|$$

we can also recover (24) as

$$\mathcal{L}_{1,1}(x,y) = \mathcal{L}_2\left(\frac{xy-y}{1-y}\right) - \mathcal{L}_2\left(\frac{y}{y-1}\right) - \mathcal{L}_2(xy). \tag{25}$$

This function satisfies the functional equations

$$\mathcal{L}_{1,1}(x,y) = -\mathcal{L}_{1,1}\left(1-x, \frac{y}{y-1}\right)$$

by the functional equations $\mathcal{L}_2(x) = -\mathcal{L}_2(1-x) = -\mathcal{L}_2(1/x)$.

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6.3 Single-Valued Double Polylogarithms $\mathcal{L}_{1,2}$ and $\mathcal{L}_{2,1}$

By [17] a single-valued version of $\mathrm{Li}_3(x)$ can be defined as

$$\mathcal{L}_3(z) = \mathrm{Re}\big(\mathrm{Li}_3(z)\big) - \log|z|\,\mathrm{Re}\big(\mathrm{Li}_2(z)\big) - \frac{1}{3}(\log|z|)^2\log|1-z|. \tag{26}$$

We now look at $\mathrm{Li}_{2,1}(x,y)$ and $\mathrm{Li}_{1,2}(x,y)$. By the procedure outlined in the first section of this chapter we find that the single-valued version of $\mathrm{Li}_{1,2}(x,y)$ is

$$\begin{aligned}\mathcal{L}_{1,2}(x,y) = {} & \mathrm{Re}\,\mathrm{Li}_{1,2}(x,y) - \arg(1-xy)\big[\mathcal{L}_2(x)+\mathcal{L}_2(y)\big] + \log|1-x|\,\mathrm{Re}\,\mathrm{Li}_2(y)\\ & - \log|y|\,\mathrm{Re}\,\mathrm{Li}_{1,1}(x,y) - \log|1-x^{-1}|\,\mathrm{Re}\,\mathrm{Li}_2(xy)\\ & - \frac{1}{3}\log|xy^2|\log|1-xy|\log\big|1-x^{-1}\big|\\ & + \frac{1}{3}\log|y|\big(2\log|1-y|\log|1-x| + \log|1-xy|\log|x(1-y)|\big).\end{aligned}$$

The single-valued version of $\mathrm{Li}_{2,1}(x,y)$ is

$$\begin{aligned}\mathcal{L}_{2,1}(x,y) = {} & \mathrm{Re}\,\mathrm{Li}_{2,1}(x,y) + \arg(1-xy)\big[\mathcal{L}_2(x)+\mathcal{L}_2(y)\big] - \arg(1-y)\mathcal{L}_2(x)\\ & + \log|1-y|\,\mathrm{Re}\,\mathrm{Li}_2(xy) - \log|x|\,\mathrm{Re}\,\mathrm{Li}_{1,1}(x,y)\\ & + \frac{1}{3}\log|1-y|\log|xy|\log|1-xy|\\ & + \frac{1}{3}\log|x|\Big[\log|1-y|\log|1-x| + \log|1-xy|\log\Big|\frac{x(1-y)}{1-x}\Big|\Big].\end{aligned}$$

Using the single-valued versions of dilogarithm $\mathcal{L}_2(z)$ and trilogarithm $\mathcal{L}_3(z)$ we can express $\mathcal{L}_{2,1}(y,x)$ by the trilogarithms

$$\begin{aligned}\mathcal{L}_{2,1}(y,x) = {} & \mathcal{L}_3(1-xy) + \mathcal{L}_3(1-x) - \mathcal{L}_3\Big(\frac{1-x}{1-xy}\Big)\\ & - \mathcal{L}_3(y) + \mathcal{L}_3\Big(\frac{y-xy}{1-xy}\Big) - \mathcal{L}_3(1),\end{aligned}$$

where $\mathcal{L}_3$ is the single-valued trilogarithm given by (26). This follows from the relation (see [18]) first discovered by Zagier after Goncharov's conviction that such identity should exist:

$$\begin{aligned}\mathrm{Li}_{2,1}(y,x) = {} & \mathrm{Li}_3(1-xy) + \mathrm{Li}_3(1-x) - \mathrm{Li}_3\Big(\frac{1-x}{1-xy}\Big) - \mathrm{Li}_3(y) + \mathrm{Li}_3\Big(\frac{y-xy}{1-xy}\Big)\\ & - \mathrm{Li}_3(1) - \log(1-xy)\big(\mathrm{Li}_2(1) + \mathrm{Li}_2(1-x)\big)\\ & - \log\Big(\frac{1-x}{1-xy}\Big)\mathrm{Li}_2(y) + \frac{1}{2}\log(y)\log^2(1-xy).\end{aligned}$$

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By straightforward computation we further discover the following interesting formula:

$$\mathcal{L}_{1,2}(x,y)+\mathcal{L}_{2,1}(y,x)+\mathcal{L}_3(xy)=0.$$

One should compare this with

$$\mathrm{Li}_{1,2}(x,y)+\mathrm{Li}_{2,1}(y,x)+\mathrm{Li}_3(xy)=-\log(1-x)\,\mathrm{Li}_2(y).$$

Finally we find the interesting identity

$$\mathcal{L}_{1,1,1}(x,y,z)=\mathcal{L}_3\Big(\frac{(y-1)(1-xyz)}{y(1-x)(1-z)}\Big)+\mathcal{L}_3\Big(\frac{y}{y-1}\Big)+\mathcal{L}_3(xy)-\mathcal{L}_3\Big(\frac{1-xyz}{1-x}\Big)$$
$$-\mathcal{L}_3\Big(\frac{1-xyz}{xy(1-z)}\Big)-\mathcal{L}_3\Big(\frac{y-yz}{y-1}\Big)-\mathcal{L}_3\Big(\frac{y-xy}{y-1}\Big)+\mathcal{L}_3(1-x).$$

We remind the readers that such identities in higher weight cases do not exist in general. For example, $\mathcal{L}_{2,2}(x,y)$ cannot be expressed by only tetralogarithms $\mathcal{L}_4$.

6.4 A Problem of Multiple Dedekind Zeta Values

In general there should exist a single-valued real analytic version of the multiple polylogarithm $\mathrm{Li}_{m_1,\dots,m_n}(\mathbf{x})$ which we denote by $\mathcal{L}_{m_1,\dots,m_n}(\mathbf{x})$. For $m_n\geq 2$ the value of this function when $|x_i|\leq 1$ is given by the power series expansion (1). We end our paper by stating a generalized Zagier conjecture about special values of Dedekind zeta function over number fields.

Denote by O_F the ring of integers of a number field F and I_F the set of integral ideals of O_F. Let N be the norm from F to Q. Then we define the multiple Dedekind zeta function of depth d over F as

$$\zeta_F(s_1,\dots,s_d)=\sum_{\substack{\mathfrak{n}_1,\dots,\mathfrak{n}_d\in O_F\\ N(\mathfrak{n}_1)<\cdots<N(\mathfrak{n}_d)}} N(\mathfrak{n}_1)^{-s_1}\cdots N(\mathfrak{n}_d)^{-s_d}.$$

This function is well defined for $\mathrm{Re}(s_1)>0,\dots,\mathrm{Re}(s_{d-1})>0,\mathrm{Re}(s_d)>1$.

Problem 6.2 For any integers $m_1,\dots,m_{d-1}\geq 1$ and $m_d\geq 2$, is there an expression of $\zeta_F(m_1,\dots,m_d)$ in terms of a determinant of $\mathcal{L}_{m_1,\dots,m_d}$ evaluated at F rational points up to some factors determined only by the number field F (such as the discriminant, the number of real and complex embeddings, etc.)?

When $F=\mathbb{Q}$ the problem has an easy answer:

$$\zeta_{\mathbb{Q}}(m_1,\dots,m_d)=\mathcal{L}_{m_1,\dots,m_d}(1,\dots,1).$$

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