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GENERALISED CONVEXITY AND DUALITY IN MULTIPLE OBJECTIVE PROGRAMMING

T. WEIR AND B. MOND

By considering the concept of weak minima, different scalar duality results are extended to multiple objective programming problems. A number of weak, strong and converse duality theorems are given under a variety of generalised convexity conditions.

1. INTRODUCTION

Consider the multiple objective optimisation problem:

(P) minimise $f(x)$ subject to $g(x) \leqq 0$

where $f: \mathbb{R}^n \to \mathbb{R}^k$ and $g: \mathbb{R}^n \to \mathbb{R}^m$. Several approaches to duality for the multiple objective optimisation problem may be found in the literature. These include the use of vector valued Lagrangians [13, 16, 17] and Lagrangians incorporating matrix Lagrange multipliers [3, 4, 7 and 9].

In Weir [16], using a vector-valued Lagrangian, the well-known duality results of Wolfe [18] for scalar valued convex programs were extended to the multiple objective optimisation problem. The approach there was to consider the properly efficient solutions [8] of (P) and to relate primal and dual properly efficient solutions. Some restricted results on duality for non-convex multiple objective optimisation problems were also given.

In this paper, by considering the concept of a weak minimum, a complete generalisation of the scalar duality results of Wolfe [18] and those of Mond and Weir [12] and Bector and Bector [2] will be described for the multiple objective optimisation problem. The results also include generalisations of the converse duality results of Mond and Weir [12] and Bector and Bector [2] and the strict converse duality results of Weir [14], [15] and Bector and Bector [2].

2. PRELIMINARIES

Throughout this paper the following conventions for vectors in $\mathbb{R}^n$ will be followed:

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$$x > y \text{ if and only if } x_i > y_i\,, i = 1,2,\ldots,n,$$
$$x \geqq y \text{ if and only if } x_i \geqq y_i\,, i = 1,2,\ldots,n,$$
$$x \geqslant y \text{ if and only if } x_i \geqq y_i\,, i = 1,2,\ldots,n \text{ but } x \neq y,$$
$$x \not> y \text{ is the negation of } x > y.$$

For the problem (P), a point x_0 is said to be a weak minimum if there exists no other feasible point x for which $f(x_0) > f(x)$.

The following Fritz John and Kuhn-Tucker theorems [6] will be needed:

THEOREM 2.1. *Let (P) have a weak minimum at $x = x_0$. Then there exists $\lambda \in \mathbf{R}^k, y \in \mathbf{R}^m$ such that*

$$\nabla\lambda^t f(x_0) + \nabla y^t g(x_0) = 0, \tag{1}$$
$$y^t g(x_0) = 0, \tag{2}$$
$$(\lambda, y) \geqslant 0. \tag{3}$$

THEOREM 2.2. *Let x_0 be a weak minimum for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists $\lambda \in \mathbf{R}^k, y \in \mathbf{R}^m$ such that*

$$\nabla\lambda^t f(x_0) + \nabla y^t g(x_0) = 0, \tag{4}$$
$$y^t g(x_0) = 0, \tag{5}$$
$$y \geqq 0, \tag{6}$$
$$\lambda \geqslant 0, \lambda^t e = 1, \tag{7}$$

where $e = (1,1,\ldots,1) \in \mathbf{R}^k$.

3. DUALITY

In relation to (P) consider the problem

(D) *maximise* $f(u) + y^t g(u)e$

$$\text{subject to } \nabla\lambda^t f(u) + \nabla y^t g(u) = 0, \tag{8}$$
$$y \geqq 0, \tag{9}$$
$$\lambda \in \Lambda, \tag{10}$$

where $\Lambda = \left\{\lambda \in \mathbf{R}^k : \lambda \geqslant 0, \lambda^t e = 1\right\}$.

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The problem (D) may be regarded as a multiple objective Wolfe [18] *dual for (P).*

THEOREM 3.1. (Weak Duality). *If, for all feasible (x, u, y, λ),*

(a) *$f + y^t ge$ is pseudoconvex; or*

(b) *$\lambda^t f + y^t g$ is pseudoconvex, then $f(x) \nless f(u) + y^t g(u)e$.*

PROOF:

(a) Let x be feasible for (P) and (u, y, λ) feasible for (D). Suppose $f_i(x) < f_i(u) + y^t g(u)$ for all $i = 1, 2, \ldots, k$. Then $f_i(x) + y^t g(x) < f_i(u) + y^t g(u)$ for all $i = 1, 2, \ldots, k$. The pseudoconvexity of $f + y^t ge$ implies that

$$(x - u)^t \nabla \left\{ f_i(u) + y^t g(u) \right\} < 0$$

and hence

$$(x - u)^t \left\{ \nabla \lambda^t f(u) + \nabla y^t g(u) \right\} < 0$$

which contradicts the constraint (8) of (D).

(b) Let x be feasible for (P) and (u, y, λ) feasible for (D). Suppose $f_i(x) < f_i(u) + y^t g(u)$ for all $i = 1, 2, \ldots, k$. Then $f_i(x) + y^t g(x) < f_i(u) + y^t g(u)$ for all $i = 1, 2, \ldots, k$. Thus, $\lambda^t f(x) + y^t g(x) < \lambda^t f(u) + y^t g(u)$ and the pseudoconvexity of $\lambda^t f + y^t g$ implies that

$$(x - u)^t \left\{ \nabla \lambda^t f(u) + \nabla y^t g(u) \right\} < 0$$

which contradicts the constraint (8) of (D). ∎

THEOREM 3.2. (Strong Duality). *Let x_0 be a weak minimum for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist (y, λ) such that (x_0, y, λ) is feasible for (D) and the objective values of (P) and (D) are equal. If, also, (a) $f + y^t ge$ is pseudoconvex or (b) $\lambda^t f + y^t g$ is pseudoconvex then (x_0, y, λ) is a weak maximum for (D).*

PROOF: Since x_0 is a weak minimum for (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 2.2, there exist $y \geqq 0, \lambda \geqslant 0, \lambda^t e = 1$ such that $\nabla \lambda^t f(x_0) + \nabla y^t g(x_0) = 0, y^t g(x_0) = 0$.

Thus (x_0, y, λ) is feasible for (D) and clearly the objective values of (P) and (D) are equal.

If (x_0, y, λ) is not a weak maximum for (D) then there exists feasible (u^*, y^*, λ^*) for (D) such that

$$f_i(u^*) + y^{*t} g(u^*) > f_i(x_0) + y^{*t} g(x_0) \text{ for all } i = 1, 2, \ldots, k.$$

(a) Since $f + y^t ge$ is pseudoconvex, $(x_0 - u)^t \nabla (f_i(u^*) + y^{*t} g(u^*)) < 0$ for all $i = 1, 2, \ldots, k$. Thus $(x_0 - u^*)^t \{\nabla \lambda^{*t} f(u^*) + \nabla y^{*t} g(u^*)\} < 0$ contradicting the feasibility of (u^*, y^*, λ^*).

Thus (x_0, y, λ) is a weak maximum for (D).

(b) Since $f_i(u^*) + y^{*t}g(u^*) > f_i(x_0) + y^{*t}(x_0)$ for all $i = 1, 2, \ldots, k$, then $\lambda^{*t}f(u^*) + y^{*t}g(u^*) > \lambda^{*t}f(x_0) + y^{*t}g(x_0)$. Since $\lambda^{*t}f + y^{*t}g$ is pseudoconvex, it follows that

$$(x_0 - u^*)^t\{\nabla\lambda^{*t}f(u^*) + \nabla y^{*t}g(u^*)\} < 0$$

contradicting the feasibility of (u^*, y^*, λ^*).

Hence (x_0, y, λ) is a weak maximum for (D). ∎

Remark 3.3. Theorems 3.1 and 3.2 give multiple objective extensions of the results of Bector et al [**1**] and Mahajan and Vartak [**10**] for scalar valued minimisation problems.

Mond and Weir [**12**] proposed a number of different duals to the scalar valued minimisation problem. Here it is shown, as for the Wolfe dual, that there are analogous results for the multiple objective optimisation problem (P).

In relation to (P) consider the problem

(D1) maximise $f(u)$

$$\text{subject to } \nabla\lambda^t f(u) + \nabla y^t g(u) = 0, \tag{11}$$

$$y^t g(u) \geqq 0, \tag{12}$$

$$y \geqq 0, \tag{13}$$

$$\lambda \in \Lambda, \tag{14}$$

where $\Lambda = \{\lambda \in \mathbf{R}^k : \lambda \geqslant 0, \lambda^t e = 1\}$.

THEOREM 3.4. (Weak Duality) *If, for all feasible* (x, u, y, λ)

(a) f *is pseudoconvex and* $y^t g$ *is quasiconvex; or*

(b) $\lambda^t f$ *is pseudoconvex and* $y^t g$ *is quasiconvex; or*

(c) f *is quasiconvex and* $y^t g$ *is strictly pseudoconvex; or*

(d) $\lambda^t f$ *is quasiconvex and* $y^t g$ *is strictly pseudoconvex,*

then $f(x) \nless f(u)$.

PROOF:

(a) Let x be feasible for (P) and (u, y, λ) feasible for $(D1)$. Suppose $f_i(x) < f_i(u)$ for all $i = 1, 2, \ldots, k$. By pseudoconvexity of $f_i, i = 1, 2, \ldots, k$

$$(x - u)^t \nabla f_i(u) < 0, i = 1, 2, \ldots, k.$$

Since $\lambda \geqslant 0$,

$$(x - u)^t \nabla\lambda^t f(u) < 0 \tag{15}$$

Since $y^t g(x) - y^t g(u) \leqq 0$, the quasiconvexity of $y^t g$ implies that

$$(x-u)^t \nabla y^t g(u) \leqq 0. \tag{16}$$

Combining (15) and (16) gives

$$(x-u)^t \{\nabla \lambda^t f(u) + \nabla y^t g(u)\} < 0$$

which contradicts the constraint (11) of $(D1)$.

(b) Let x be feasible for (P) and (u, y, λ) feasible for $(D1)$. Suppose $f_i(x) < f_i(u)$ for all $i = 1, 2, \dots, k$. Since $\lambda \geqslant 0$ it follows that

$$\lambda^t f(x) < \lambda^t f(u)$$

and pseudoconvexity of $\lambda^t f$ implies

$$(x-u)^t \nabla \lambda^t f(u) < 0. \tag{17}$$

Since $y^t g(x) - y^t g(u) \leqq 0$ the quasiconvexity of $y^t g$ implies that

$$(x-u)^t \nabla y^t g(u) \leqq 0. \tag{18}$$

Combining (17) and (18) gives

$$(x-u)^t \{\nabla \lambda^t f(u) + \nabla y^t g(u)\} < 0$$

which contradicts the constraint (11) of $(D1)$.

(c) Let x be feasible for (P) and (u, y, λ) feasible for $(D1)$. Suppose $f_i(x) < f_i(u)$ for all $i = 1, 2, \dots, k$. The quasiconvexity of $f_i, i = 1, 2, \dots, k$ implies that

$$(x-u)^t \nabla f_i(u) \leqq 0$$

and since $\lambda \geqslant 0$

$$(x-u)^t \nabla \lambda^t f(u) \leqq 0.$$

By (11)

$$(x-u)^t \nabla y^t g(u) \geqq 0$$

and since $y^t g$ is strictly pseudoconvex

$$y^t g(x) > y^t g(u)$$

which is a contradiction since $y^t g(x) \leqq 0$ and $y^t g(u) \geqq 0$.

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(d) Let x be feasible for (P) and (u, y, λ) feasible for $(D1)$. Suppose $f_i(x) < f_i(u)$ for all $i = 1, 2, \ldots, k$. Since $\lambda \geqslant 0$, $\lambda^t f(x) < \lambda^t f(u)$, the quasiconvexity of $\lambda^t f$ implies that

$$(x-u)^t \nabla \lambda^t f(u) \leqq 0.$$

By (11)

$$(x-u)^t \nabla y^t g(u) \geqq 0$$

and since $y^t g$ is strictly pseudoconvex

$$y^t g(x) > y^t g(u)$$

which is a contradiction since $y^t g(x) \leqq 0$ and $y^t g(u) \geqq 0$. ∎

THEOREM 3.5. (Strong Duality) *Let x_0 be a weak minimum for (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist (y, λ) such that (x_0, y, λ) is feasible for $(D1)$ and the objective values of (P) and $(D1)$ are equal. If, also,*

(a) *f is pseudoconvex and $y^t g$ is quasiconvex; or*
(b) *$\lambda^t f$ is pseudoconvex and $y^t g$ is quasiconvex; or*
(c) *f is quasiconvex and $y^t g$ is strictly pseudoconvex; or*
(d) *$\lambda^t f$ is quasiconvex and $y^t g$ is strictly pseudoconvex,*
then (x_0, y, λ) is a weak maximum for $(D1)$.

PROOF: Since x_0 is a weak minimum for (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 2.2, there exists $y \geqq 0$, $\lambda \geqslant 0$, $\lambda^t e = 1$ such that

$$\nabla \lambda^t f(x_0) + \nabla y^t g(x_0) = 0, \; y^t g(x_0) = 0.$$

Thus (x_o, y, λ) is feasible for $(D1)$ and clearly the objective values of (P) and $(D1)$ are equal.

If (x_0, y, λ) is not a weak maximum for $(D1)$ there exists feasible (u^*, y^*, λ^*) for $(D1)$ such that $f_i(u^*) > f_i(x_0)$ for all $i = 1, 2, \ldots, k$.

(a) Since f is pseudoconvex $(x_0 - u^*)^t \nabla f_i(u^*) < 0$ for all $i = 1, 2, \ldots, k$. Thus, since $\lambda^* \geqslant 0$,

$$(x_0 - u^*)^t \nabla \lambda^{*t} f(u^*) < 0. \tag{19}$$

Also $y^{*t} g(x_0) - y^{*t} g(u^*) \leqq 0$ and since $y^{*t} g$ is quasiconvex

$$(x_0 - u^*)^t \nabla y^{*t} g(u^*) \leqq 0. \tag{20}$$

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Combining (19) and (20) gives

$$(x_0 - u^*)^t\{\nabla\lambda^{*t}f(u^*) + \nabla y^{*t}g(u^*)\} < 0$$

which contradicts the feasibility of (u^*, y^*, λ^*).

(b) Since $\lambda \geqslant 0$, then

$$\lambda^{*t}f(u^*) > \lambda^{*t}f(x_0).$$

Since $\lambda^{*t}f$ is pseudoconvex then

(21) $$(x_0 - u^*)^t\nabla\lambda^{*t}f(u^*) < 0$$

Also $y^{*t}g(x_0) - y^{*t}g(u^*) \leqq 0$ and since $y^{*t}g$ is quasiconvex

(22) $$(x_0 - u^*)^t\nabla y^{*t}g(u^*) \leqq 0$$

Combining (21) and (22) gives

$$(x_0 - u^*)^t\{\nabla\lambda^{*t}f(u^*) + \nabla y^{*t}g(u^*)\} < 0$$

which contradicts the feasibility of (u^*, y^*, λ^*).

(c) Since f is quasiconvex $(x_0 - u^*)^t\nabla f_i(u^*) \leqq 0$ for all $i = 1, 2, \dots, k$. Since $\lambda \geqslant 0$

$$(x_0 - u^*)^t\nabla\lambda^{*t}f(u^*) \leqq 0.$$

By (11)

$$(x_0 - u^*)^t\nabla y^{*t}g(u^*) \geqq 0$$

and since $y^{*t}g$ is strictly pseudoconvex

$$y^{*t}g(x_0) > y^{*t}g(u^*).$$

This is a contradiction as $y^{*t}g(x_0) \leqq 0$ and $y^{*t}g(u^*) \geqq 0$.

(d) Since $\lambda^* \geqslant 0, \lambda^{*t}f(u^*) > \lambda^{*t}f(x_0)$. Thus,since $\lambda^{*t}f$ is quasiconvex

$$(x_0 - u^*)^t\nabla\lambda^{*t}f(u^*) \leqq 0.$$

By (11)

$$(x_0 - u^*)^t\nabla y^{*t}g(u^*) \geqq 0$$

and since $y^{*t}g$ is strictly pseudoconvex

$$y^{*t}g(x_0) > y^{*t}g(u^*).$$

This is a contradiction as $y^{*t}g(x_0) \leqq 0$ and $y^{*t}g(u^*) \geqq 0$. ∎

In a similar manner to that given in [12] we state a general dual for the multiple objective optimisation problem. For completeness we shall consider the case where the primal problem has equality as well as inequality constraints.

Consider the problem:

(PE) minimise $f(x)$

$$\text{subject to } g(x) \leqq 0, \quad h(x) = 0,$$

where $f : \mathbb{R}^n \to \mathbb{R}^k, g : \mathbb{R}^n \to \mathbb{R}^m, h : \mathbb{R}^n \to \mathbb{R}^\ell$ are all differentiable.

Let $M = \{1, 2, \ldots, m\}, L = \{1, 2, \ldots, l\}, I_\alpha \subseteq M, \alpha = 0, 1, 2, \ldots, \nu$ with $I_\alpha \cap I_\beta = \phi, \alpha \neq \beta$ and $\bigcup_{\alpha=0}^{\nu} I_\alpha = M$ and $J_\alpha \subseteq L, \alpha = 0, 1, 2, \ldots, \nu$, with $J_\alpha \cap J_\beta = \phi, \alpha \neq \beta$ and $\bigcup_{\alpha=0}^{\nu} J_\alpha = L$.

Note that any particular I_α or J_α may be empty. Thus if M has ν_1 disjoint subsets and L has ν_2 disjoint subsets, $\nu =$ Max $[\nu_1, \nu_2]$. So that if $\nu_1 > \nu_2$, then $J_\alpha, \alpha > \nu_2$ is empty.

In relation to (PE) consider the problem:

(DE) maximise $f(u) + \sum_{i \in I_0} y_i g_i(u)e + \sum_{j \in J_0} z_j h_j(u)e$

$$\text{subject to } \nabla \lambda^t f(u) + \nabla y^t g(u) + \nabla z^t h(u) = 0,$$
$$\sum_{i \in I_\alpha} y_i g_i(u) + \sum_{j \in J_\alpha} z_j h_j(u) \geqq 0, \alpha = 1, 2, \ldots, \nu,$$
$$y \geqq 0,$$
$$\lambda \in \Lambda,$$

where $\Lambda = \{\lambda \in \mathbb{R}^k : \lambda \geqslant 0, \lambda^t e = 1\}$.

The following weak and strong duality theorems are stated without proof. They may be established in a manner similar to Theorems 3.1, 3.2, 3.4 and 3.5.

Theorem 3.6. (Weak Duality) *If, for all feasible* (x, u, y, z, λ)

(a) $f + \sum_{i \in I_0} y_i g_i e + \sum_{j \in J_0} z_j h_j e$ *is pseudoconvex and* $\sum_{i \in I_\alpha} y_i g_i + \sum_{j \in J_\alpha} z_j h_j, \alpha = 1, 2, \ldots, \nu$ *is quasiconvex; or*

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(b) $\lambda^t f + \sum_{i \in I_0} y_i g_i + \sum_{j \in J_0} z_j h_j$ *is pseudoconvex and* $\sum_{i \in I\alpha} y_i g_i + \sum_{j \in J\alpha} z_j h_j\,, \alpha = 1, 2, \ldots, \nu$ *is quasiconvex; or*

(c) $I_0 \neq M$ and $J_0 \neq L, f + \sum_{i \in I_0} y_i g_i e + \sum_{j \in J_0} z_j h_j e$ is quasiconvex and $\sum_{i \in I_\alpha} y_i g_i + \sum_{j \in J_\alpha} z_j h_j\,, \alpha = 1, 2, \ldots, \nu$ *is strictly pseudoconvex; or*

(d) $I_0 \neq M$ and $J_0 \neq L\,, \lambda^t f + \sum_{i \in I_0} y_i g_i + \sum_{j \in J_0} z_j h_j$ is quasiconvex and $\sum_{i \in I_\alpha} y_i g_i + \sum_{j \in J_\alpha} z_j h_j\,, \alpha = 1, 2, \ldots, \nu$ *is strictly pseudoconvex, then*

$$f(x) \not< f(u) + \sum_{i \in I_0} y_i g_i(u) e + \sum_{j \in J_0} z_j h_j(u) e.$$

THEOREM 3.7. (Strong Duality) *Let* x_0 *be a weak minimum for* (PE) *at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist* (y, z, λ) *such that* (x_0, y, z, λ) *is feasible for* (DE) *and the objective values of* (PE) *and* (DE) *are equal. If also the assumptions (a), (b), (c) or (d) of Theorem 3.6 are satisfied, then* (x_0, y, z, λ) *is a weak maximum for* (DE).

4. Converse Duality

THEOREM 4.1. *Let* (x_0, y_0, λ_0) *be a weak maximum of* $(D1)$. *Assume the* $n \times n$ *Hessian matrix*

$$\nabla^2 \lambda_0^t f(x_0) + \nabla^2 y_0^t g(x_0) \tag{23}$$

is positive or negative definite and the vectors $\nabla f_i(x_0)\,, i = 1, 2, \ldots, k$ *are linearly independent. If for all feasible* (x, u, y, λ)

(a) f *is pseudoconvex and* $y^t g$ *is quasiconvex; or*
(b) $\lambda^t f$ *is pseudoconvex and* $y^t g$ *is quasiconvex; or*
(c) f *is quasiconvex and* $y^t g$ *is strictly pseudoconvex; or*
(d) $\lambda^t f$ *is quasiconvex and* $y^t g$ *is strictly pseudoconvex, then* x_0 *is a weak minimum for* (P).

PROOF: Since (x_0, y_0, λ_0) is a weak maximum for (D), then by Theorem 2.1 there exist

$$\tau \in \mathbf{R}^k, \nu \in \mathbf{R}^n,\ p \in \mathbf{R}, s \in \mathbf{R}^m, w \in \mathbf{R}^k,$$

such that

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$$\nabla \tau^t f(x_0) + \nabla \nu^t [\nabla \lambda_0^t f(x_0) + \nabla y_0^t g(x_0)] + p \nabla y_0^t g(x_0) = 0, \tag{24}$$

$$(\nabla g(x_0))^t \nu + pg(x_0) + s = 0, \tag{25}$$

$$(\nabla f(x_0))^t \nu + w = 0, \tag{26}$$

$$py_0^t g(x_0) = 0, \tag{27}$$

$$s^t y_0 = 0, \tag{28}$$

$$w^t \lambda_0 = 0, \tag{29}$$

$$(\tau, s, p, w) \geqq 0, \tag{30}$$

$$(\tau, \nu, s, p, w) \neq 0. \tag{31}$$

Since $\lambda_0 \in \Lambda$, (29) gives $w = 0$; (26) then gives

$$\nu^t \nabla f(x_0) = 0. \tag{32}$$

Multiplying (25) by y_0 and using (27) and (28) gives

$$\nu^t \nabla y_0^t g(x_0) = 0. \tag{33}$$

Multiplying (24) by ν^t and using (32) and (33) gives

$$\nu^t [\nabla^2 \lambda_0^t f(x_0) + \nabla^2 y_0^t g(x_0)] \nu = 0.$$

Since (23) is assumed positive or negative definite, $\nu = 0$.

Since $\nu = 0$, (24) and the equality constraint (11) of $(D1)$ give

$$\nabla (\tau - p\lambda_0)^t f(x_0) = 0.$$

By the linear independence of $\nabla f_i(x_0), i = 1, 2, \ldots, k$ it follows that

$$\tau = p\lambda_0.$$

Since $\lambda_0 \geqslant 0$, $\tau = 0$ implies $p = 0$ and then, by (25), $s = 0$ giving $(\tau, \nu, s, p, w) = 0$ contradicting (31). Thus $\tau \neq 0$ and $p > 0$. Since $\nu = 0, p > 0$ and $s \geqq 0$, (25) gives $g(x_0) \leqq 0$, and (27) gives $y_0^t g(x_0) = 0$. Thus x_0 is feasible for (P). That x_0 is a weak minimum for (P) then follows under assumptions (a), (b), (c) or (d) from weak duality, Theorem 3.4. ∎

As in [**12**] a more general converse duality result may be established for (PE) and (DE). The proof follows in a manner similar to that of Theorem 4.1.

THEOREM 4.2. *Let* $(x_0, y_0, z_0, \lambda_0)$ *be a weak maximum of* (DE). *Assume the* $n \times n$ *Hessian matrix*

$$\nabla^2 \lambda_0^t f(x_0) + \nabla^2 y_0^t g(x_0) + \nabla^2 z_0^t h(x_0)$$

is positive or negative definite and that the set

$$\left\{ \nabla f_i(x_0), i = 1, 2, \ldots, k, \sum_{i \in I_\alpha} \nabla y_{0i} g_i(x_0) + \sum_{j \in J_\alpha} \nabla z_{0j} h_j(x_0), \alpha = 1, 2, \ldots, \nu \right\}$$

is linearly independent whenever $I_0 \neq M$ *or* $J_0 \neq L$.

If the assumptions (a), (b), (c) or (d) of Theorem 3.6 hold, then x_0 *is a weak minimum for* (PE).

In the case $I_0 = M$ and $L = \phi$ this result simplifies slightly.

THEOREM 4.3. *Let* (x_0, y_0, λ_0) *be a weak maximum of* (D). *Assume the* $n \times n$ *Hessian matrix*

$$\nabla^2 \lambda_0^t f(x_0) + \nabla^2 y_0^t g(x_0)$$

is positive or negative definite. If the assumptions (a) or (b) of Theorem 3.1 hold, then x_0 *is a weak minimum of* (P).

Theorems 4.1, 4.2 and 4.3 give multiple objective generalisations of the scalar valued programming results of Mond and Weir [**12**], Bector and Bector [**2**] and Craven and Mond [**5**].

We now turn our attention to strict converse duality.

THEOREM 4.4. *Let* x_0 *be a weak minimum for* (P) *and* (u_0, y_0, λ_0) *be a weak maximum for* $(D1)$ *such that* $\lambda_0^t f(x_0) \leqq \lambda_0^t f(u_0)$. *Assume that*

(a) $\lambda_0^t f$ *is strictly pseudoconvex at* u_0 *and* $y_0^t g$ *is quasiconvex at* u_0; *or*

(b) $\lambda_0^t f$ *is quasiconvex at* u_0 *and* $y_0^t g$ *is strictly pseudoconvex at* u_0;

then $x_0 = u_0$; *that is,* u_0 *is a weak minimum for* (P).

PROOF:

(a) We assume $x_0 \neq u_0$ and exhibit a contradiction. Since x_0 and (u_0, y_0, λ_0) are feasible for (P) and $(D1)$ respectively

$$y_0^t g(x_0) - y_0^t g(u_0) \leqq 0$$

and the quasiconvexity of $y_0^t g$ at u_0 implies

$$(x_0 - u_0)^t \nabla y_0^t g(u_0) \leqq 0.$$

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From (11), $(x_0 - u_0)^t \nabla \lambda_0^t f(u_0) \geqq 0$ and, by the strict pseudoconvexity of $\lambda_0^t f$ at u_0,

$$\lambda_0^t f(x_0) > \lambda_0^t f(u_0)$$

contradicting the assumption that $\lambda_0^t f(x_0) \leqq \lambda_0^t f(u_0)$.

(b) We assume $x_0 \neq u_0$ and exhibit a contradiction. Since x_0 and (u_0, y_0, λ_0) are feasible for (P) and $(D1)$ respectively

$$y_0^t g(x_0) - y_0^t g(u_0) \leqq 0$$

and the strict pseudoconvexity of $y_0^t g$ at u_0 implies

$$(x_0 - u_0)^t \nabla y_0^t g(u_0) < 0.$$

From (11), $(x_0 - u_0)^t \nabla \lambda_0^t f(u_0) > 0$ and, by quasiconvexity of $\lambda_0^t f$ at u_0,

$$\lambda_0^t f(x_0) > \lambda_0^t f(u_0)$$

contradicting the assumption that $\lambda_0^t f(x_0) \leqq \lambda_0^t f(u_0)$. ∎

THEOREM 4.5. *Let x_0 be a weak minimum for (PE) and $(u_0, y_0, z_0, \lambda_0)$ be a weak maximum for (DE) such that $\lambda_0^t f(x_0) \leqq \lambda_0^t f(u_0) + \sum_{i \in I_0} y_{0i} g_i(u_0) + \sum_{j \in J_0} z_{0j} h_j(u_0)$. If*

(a) *$\lambda_0^t f + \sum_{i \in I_0} y_{0i} g_i + \sum_{j \in J_0} z_{0j} h_j$ is strictly pseudoconvex at u_0 and each $\sum_{i \in I_\alpha} y_{0i} g_i + \sum_{j \in J_\alpha} z_{0j} h_j$, $\alpha = 1, 2, \ldots, \nu$ is quasiconvex at u_0; or*

(b) *$\lambda_0^t f + \sum_{i \in I_0} y_{0i} g_i + \sum_{j \in J_0} z_{0j} h_j$ is quasiconvex at u_0 and each*

$$\sum_{i \in I_\alpha} y_{0i} g_i + \sum_{j \in J_\alpha} z_{0j} h_j, \alpha = 1, 2, \ldots, \nu$$

is strictly pseudoconvex at u_0,

then $x_0 = u_0$; that is u_0 is a weak minimum for (P).

COROLLARY 4.6. *Let x_0 be a weak minimum for (P) and (u_0, y_0, λ_0) be a weak maximum for (D) such that $\lambda_0^t f(x_0) \leqq \lambda_0^t f(u_0) + y_0^t g(u_0)$. If $\lambda_0^t f + y_0^t g$ is strictly pseudoconvex at u_0, then $x_0 = u_0$; that is u_0 is a weak minimum for (P).*

These strict converse duality results give multiple objective analogues of the scalar programming theorems of Mond and Weir [**12**], Bector and Bector [**2**], Weir [**14, 15**] and Mahajan and Vartak [**10**].

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Dr T. Weir
7/35 Gaza Rd.,
West Ryde NSW 2114
Australia

Professor B. Mond
Department of Mathematics
La Trobe University
Bundoora, Victoria, 3083
Australia

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