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MATHEMATICAL NOTES

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ON APPROXIMATELY CONTINUOUS AND ALMOST CONTINUOUS FUNCTIONS

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The purpose of this note is to discuss the relationship between the concepts of approximate continuity and almost continuity of real functions of a real variable.

Contrary to the papers of T. Husain [1], [2], [3], we shall show that there is little relationship between these concepts, that approximate continuity does not imply almost continuity without additional restrictions and we shall give one such restriction.

Let $N_\epsilon(x)$ denote an ϵ-neighborhood of x. A function f is said to be *approximately continuous* at x_0 if for every $\epsilon>0$, x_0 is a point of metric density of $f^{-1}(N_\epsilon(f(x_0)))$. A function is said to be *almost continuous* at x_0 if for every $\epsilon>0$, $f^{-1}(N_\epsilon(f(x_0)))$ is dense in some neighborhood of x_0. $f|_A$ will denote the restriction of the function f to the set A.

A classic result is that a function f is approximately continuous at x_0 if and only if x_0 is a point of density of some set A and $f|_A$ is continuous at x_0. For this and many more results on approximate continuity and metric density the reader may consult [4] or [5].

Husain and Dwivedi [2] have shown that f is almost continuous at x_0 if and only if there is a set A dense in some neighborhood of x_0 such that $f|_A$ is continuous at x_0.

We say that an approximately continuous function is *strongly approximately continuous* on a set A if there is a set B so that for each x in A, x is a point of metric density of B and $f|_B$ is continuous at x.

A function is said to be a Darboux function if it has the intermediate value property.

We assert the following:

(1) approximate continuity a.e. $\not\Rightarrow$ almost continuity a.e.

(2) approximate continuity everywhere $\not\Rightarrow$ almost continuity everywhere

(3) almost continuity everywhere $\not\Rightarrow$ measurability

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(4) strong approximate continuity a.e. $\Longrightarrow$ almost continuity a.e.

(5) strong approximate continuity everywhere $\Longrightarrow$ continuity

(6) strong approximate continuity a.e. and the Darboux property $\not\Longrightarrow$ the existence of a point of continuity.

Let $\mathfrak{C}_E$ be the characteristic function of a set E.

If E is a nowhere dense perfect set of positive measure, then $\mathfrak{C}_E$ demonstrates assertion (1).

If E and its complement are dense nonmeasurable sets, then $\mathfrak{C}_E$ shows the truth of assertion (3). The reader may convince himself of the existence of such an E by modifying Proposition 18, p. 165, of [4].

Let E be a set of full measure, f strongly approximately continuous on E, G the set whose existence is assured by the definition of strong approximate continuity. Then G is dense since it has full measure. $f|_G$ is continuous on E. The above cited result of Husain and Dwivedi shows that f is almost continuous on E, proving assertion (4).

We now give an example of an approximately continuous function that is not almost continuous everywhere proving assertion (2).

Let $I_n=(c_n, d_n)$, $n=1, 2, 3, \ldots$, be a sequence of intervals contained in [0, 1] such that the distance between any two of them is positive, $\frac{1}{2}\notin I_n$, $n=1, 2, 3, \ldots$, $\lim_n c_n=\frac{1}{2}=\lim_n d_n$, and so that $\bigcup_1^\infty I_n$ has metric density one at $x=\frac{1}{2}$. Let $J_n=[a_n, b_n]$, $n=1, 2, 3, \ldots$, be the system of contiguous intervals, and $b_n-a_n=\lambda_n$. Define

$$f(x) = \begin{cases} 0 \text{ if } x\in(c_n, d_n) \\ 0 \text{ if } x=\frac{1}{2} \\ 1 \text{ if } x\in[a_n+\lambda_n/3,\ b_n-\lambda_n/3] \\ \text{linear in } [a_n, a_n+\lambda_n/3] \text{ and } [b_n-\lambda_n/3, b_n]. \end{cases}$$

Then f is not almost continuous at $x=\frac{1}{2}$ but is approximately continuous on [0, 1].

We turn to assertion (5). By hypothesis, there is a set A so that $f|_A$ is continuous and the density of A at each point is one. A has full measure and is dense. Suppose $\lim_n x_n=x$. Then there is a $y_k^n\in A$ so that $\lim_k y_k^n=x_n$ and $\lim_k f(y_k^n)=f(x_n)$, $n=1, 2, 3, \ldots$.

There exists a sequence of integers $k_1<k_2<k_3<\cdots$ so that $|y_{k_j}^j-x_j|<1/j$ and $|f(y_{k_j}^j)-f(x_j)|<1/j, j=1, 2, 3, \ldots$. The continuity of $f|_A$ with $y_{k_j}^j\in A$ and $\lim_j y_{k_j}^j=x$ imply that $\lim_j f(y_{k_j}^j)=f(x)$. But since

$$|f(x_j)-f(x)| \le |f(x_j)-f(y_{k_j}^j)|+|f(y_{k_j}^j)-f(x)|$$

we see that $\lim_j f(x_j)=f(x)$ and our assertion (5) is proved.

To prove assertion (6) we require additional notation.

On [0, 1] we construct the Cantor middle third set which we denote by C_0^0. This construction yields 3^0 intervals, I_1^1, of length $1/3^1$. On I_1^1, we construct a

Cantor middle third set which we denote by C_1^1. This last construction yields 3^1 intervals, I_1^2, I_2^2, I_3^2, of length $1/3^2$. On each of these intervals, we construct a Cantor middle third set which we denote by C_1^2, C_2^2, C_3^2. This latest construction yields 3^2 intervals, $I_1^3, I_2^3, \ldots, I_9^3$, of length $1/3^3$.

We continue this process. At the nth stage there will be 3^{n-1} intervals I_k^n, $k=1, 2, 3, \ldots, 3^{n-1}$, of length $1/3^n$. On each of these, we construct a Cantor middle third set which we denote by C_k^n, $k=1, 2, 3, \ldots, 3^{n-1}$, respectively.

For $n=1, 2, 3, \ldots$ and $k=1, 2, 3, \ldots, 3^{n-1}$, we call f_k^n the classical Lebesgue singular function on I_k^n having range [0, 1].

We set $F_k^n(x)=\complement_{C_k^n}(x)f_k^n(x)$ for $n=1, 2, 3, \ldots$, and $k=1, 2, 3, \ldots, 3^{n-1}$.

If

$$F(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{3^{n-1}} F_k^n(x),$$

then F has no points of continuity, is Darboux, and since it is zero a.e., it is strongly approximately continuous a.e.

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