From the theorem, the following proof of the nine-point circle is obtainable (Fig. 27).

Take any line CK, and draw rectangles as in the figure, we have

$$\angle \mathrm{MPK} = \angle \mathrm{PMC} - \angle \mathrm{DKC} = \angle \mathrm{ECM} - \angle \mathrm{DCK} = \angle \mathrm{EFD},$$

therefore F, P, D, E are concyclic. If KD is perpendicular to AC, PM is perpendicular to BC, and their intersection is the mid point of CO, where O is the orthocentre. If CK and CB coincide, P is the foot of the perpendicular from A on BC.

Seventh Meeting, May 12, 1893.

JOHN ALISON, Esq., M.A., F.R.S.E., President, in the Chair.

On the History of the Fourier Series.

By GEORGE A. GIBSON, M.A.

§ 1. The treatment of the Fourier Series, that is, of the series which proceeds according to sines and cosines of multiples of the variable, is in most English text-books very unsatisfactory; in many cases it shows almost no advance upon that of Poisson and, even where a more or less accurate reproduction of Dirichlet's investigations is given, there is no attempt at indicating the advantages it possesses over the so-called proof of Poisson. Nor is the *uniformity* of the convergence of the series so much as mentioned, not to say discussed. I have therefore thought it might be useful to give a fairly complete outline of the historical development of the series so far as the materials at my disposal allow. I do not think that any important contribution to the theory is omitted, but, as I indicate at one or two places, there are some memoirs to which I have not had access and which I only know at second hand.

Again it is to be understood that only series of the form

$$A_0 + \sum_{n=1}^{n=\infty}(A_n \cos nx + B_n \sin nx),$$

n being *an integer*, are dealt with, those cases in which n is not integral being omitted in the meantime.

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In many of the memoirs referred to in what follows historical notes of the work of predecessors will be found, but there are two writers to whose work I am deeply indebted. In fact these two have done their work so thoroughly as to leave practically nothing for later investigation. The first of these is Riemann, who devotes the introductory pages of his *Habilitationsschrift*, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (*Werke*, pp. 213–253) to a summary of the views of preceding mathematicians, that is, those prior to 1854. This summary is masterly though it is very curious when we consider the influence Poisson has had in this connection on English writers to note that nowhere does Riemann allude to his proof. The other writer referred to is Arnold Sachse who, in his *Versuch einer Geschichte der Darstellung willkürlicher Functionen einer Variabele durch trigonometrische Reihen* (Göttingen, 1879), has in a manner completed the summary of Riemann; this dissertation is also of very great value and contains some important additions to the theory due to Schwarz and derived from his lectures. Unfortunately the German text is out of print, but a translation appears in Darboux's *Bulletin* for 1880. It is this translation which I quote when referring to Sachse's Essay. I may also refer to Reiff's *Geschichte der unendlichen Reihen* (Tübingen, 1889) where the connection of the trigonometric series with the theory of infinite series in general is carefully discussed.

It may be useful to remark at the outset :—

That up till the appearance of Fourier's memoir on the "Analytical Theory of Heat" the possibility of the expansion of an *arbitrary* function in a trigonometric series was not admitted by any mathematician.

That Fourier had a thorough grasp of the nature of such expansions and gave in broad outline, though not in such detail as its importance demanded, a sound proof of the expansion, so that from the time his memoir became known the validity of the expansion has never been questioned.

That Dirichlet was the first to give a proof in which the restrictions on the function to be expanded, in other words the limits of its arbitrariness, are carefully stated.

That the work of subsequent writers has consisted largely in

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extending the limits given by Dirichlet, while following in the main his methods, though new ground was broken by Riemann.

And finally, that in comparatively recent times the series has been shown to be in general uniformly convergent. We have thus to keep before us these three points: first, the possibility of the expansion of an arbitrary function; second, the limits to the arbitrariness of the function in order that the series which represents it may converge to the value of the function; and third, the nature of the convergence, whether uniform or not.

§ 2. The controversy as to the possibility of expanding an arbitrary function of one variable in a series of sines and cosines of multiples of the variable arose about the middle of last century in connection with the problem of vibrating chords. To appreciate properly the difficulty which the expansion presented to the mathematicians of that day we must bear in mind that their conception of a function was much more limited than ours. In the *Introductio in Analysin Infinitorum*, vol. II., cap. I., § 9, Euler says that curves may be divided into *continuous* and *discontinuous* or *mixed*; a curve is continuous when its nature can be expressed by one definite function (*i.e.*, analytical expression) of the variable; if on the other hand different portions of the curve require different functions to express them the curves are called discontinuous or mixed or irregular as not following the same law through their whole course but being composed of portions of continuous curves. Curves which are discontinuous in this sense seem to have been considered to be beyond the scope of analysis; on this point reference may be made to Lagrange, *Oeuvres*, I., p. 68, and to D'Alembert, *Opuscules*, I., p. 7. As a consequence or accompaniment of this view it was supposed that if two functions of a variable were equal for any definite range of values of the variable they must be so for all values so that if the curves which represent them coincide for any interval they must do so entirely. Thus the objection was constantly urged that an algebraic function could not be represented by a trigonometric series for the latter gives a periodic curve while the former does not. Fourier was the first to see and state that when a function is defined for a given range of values of the argument its course outside that range is in no way determined. One obvious consequence of these views is that no one

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before Fourier could have properly understood the representation of an arbitrary function by a trigonometric series.

§ 3. D'Alembert in the *Mémoires de l'Académie de Berlin* for 1747, vol. III., page 214, discusses the problem of the vibrating chord. The origin of co-ordinates being at one end of the chord whose length is l, the axis of x in the direction of the chord and y the displacement at time t, he shows that y must satisfy the equation $\frac{\delta^2 y}{\delta t^2} = a^2 \frac{\delta^2 y}{\delta x^2}$ (In the memoir $a = 1$, but I keep the usual form). He obtains the solution $y = f(at + x) + \phi(at - x)$, and since $y = 0$ for $x = 0$ and $x = l$ he finds $y = f(at + x) - f(at - x)$ and shows that f represents such a function that $f(z) = f(z + 2l)$. In a memoir immediately following this one in the same volume (p. 220) he seeks to find functions which satisfy this relation of periodicity.

In the *Mémoires* for the following year (1748) vol. IV., p. 69, Euler discusses the same problem. He observes that the motion of the string will be completely determined if its form and the velocity of each point of it be known for any one position. He deduces the equation $y = \phi(x + at) + \phi(x - at)$ where ϕ is such that $\phi(at) + \phi(-at) = 0$ and $\phi(l + at) + \phi(l - at) = 0$ for every t; and from these equations which ϕ must satisfy he concludes that every curve *whether regular or irregular* which consists of repetitions alternately below and above the axis of any given curve which the string may be supposed to take (each point where the curve crosses the axis being a centre of the curve) is suitable for representing ϕ. He then shows how the ordinate of any point at any given time may be determined by a simple geometrical construction. He gives on p. 84 as a particular solution for $\phi(x)$ the equation

$$\phi(x) = a\sin\frac{\pi x}{l} + \beta\sin\frac{2\pi x}{l} + \gamma\sin\frac{3\pi x}{l} + \text{etc.}$$

Euler's solution is clearly more general than that of D'Alembert who always supposes the curve taken by the chord to be regular; but in the *Mémoires* for 1750, vol. VI., p. 355, the latter objects that Euler's solution is not more general than his own because the extension to *irregular* curves is illegitimate. He does not attack any special point in Euler's investigation, but seems rather to rest his objection on the illegitimacy of concluding from regular to irregular curves since the latter, not being expressible by one

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definite function through their whole course, cannot form the subject of analysis. Euler replies in the *Mémoires* for 1753, vol. IX., p. 156, by presenting his solution in great detail and asking where in his proof the law of continuity is assumed. D'Alembert does not seem to have answered Euler's challenge directly although repeating his previous objection (*Opuscules*, vol. I.). Lagrange while agreeing that Euler's solution is more general than that of D'Alembert still holds his proof to be unsatisfactory on what I suppose to be the same general grounds as D'Alembert. (Lagrange, *Oeuvres*, I., p. 68). If, as Lagrange seems to hold, and as Euler himself in the *Introd. in Anal. Inf.* leads us to think, an irregular curve cannot form the subject of mathematical investigation, there can be no question, I think, of the soundness of the objection to Euler's proof, and it was precisely because of his doubts that Lagrange undertook his investigation of the problem. Euler, however, seems always to have held to the accuracy of his solution and the other two to their objections, the one of these two to the generalisation and the mode of reaching it, the other not to the generalisation but only to the mode of reaching it; the difficulty was only explained by a better insight into the nature of functions and their mathematical treatment.

§ 4. The bearing of these memoirs and of the discussions as to the generality of the solution on the subject of this paper is fully seen when we consider an article by Daniel Bernoulli on the same subject which appeared in the Berlin *Mémoires* for 1753, vol. IX., p. 173. In that article Bernoulli approaches the consideration of the problem of the vibrating chord from the physical rather than from the mathematical side and proposes a synthetical solution of it. Basing his arguments on the expression given by Brook Taylor in his treatise *De Methodo Incrementorum* for a particular integral of the differential equation, namely, $y = A\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$, and on the principle of the Coexistence of Small Motions, he maintains that any position of the string may be given by the equation

$$y = \alpha\sin\frac{\pi x}{l} + \beta\sin\frac{2\pi x}{l} + \gamma\sin\frac{3\pi x}{l} + \text{etc.}$$

His arguments are not mathematical and he nowhere attempts to find the values of the coefficients α, β, γ, etc. A proof of the same nature as Bernoulli's in the mode of approaching the question but much more efficiently developed may be found in Lord Rayleigh's *Theory of Sound*, vol I., cap. VI. Bernoulli observes in § XIII. that Euler had given the same equation as he does (in the memoir of 1748 referred to above), but he holds against Euler that this gives a perfectly general solution.

Euler combats Bernoulli's position in the memoir of 1753 already noticed in connection with D'Alembert. The earlier part of it deals with Bernoulli's solution. Euler admits that if it be general it is much better than his own; but he does not admit its generality, for that would be equivalent to admitting that every curve could be represented by a trigonometric series and this proposition he considers to be certainly false, seeing that a curve given by a trigonometric series is periodic—a property not possessed by all curves. In seeking to establish his position he remarks (p. 200) that it might be argued that since there is an infinite number of disposable constants, α, β, γ, etc., at disposal, it must be possible to make the proposed curve coincide with any given curve, but he states explicitly that Bernoulli himself has not used this argument. Bernoulli indeed does not seem in his memoir of 1753 to have quite grasped the mathematical consequences of his solution; his results seemed so satisfactory in their explanation of the facts of observation that he was prepared to maintain the generality of his solution on that ground alone. In a letter addressed to Clairaut and published in the *Journal des Scavans* for March 1759, pp. 59-80, he states very clearly the substance of his memoirs of 1753 and the line of reasoning that had led him to his treatment of the problem. In criticising Euler's views of his memoirs he (p. 77) explicitly accepts the argument from the infinite number of disposable constants, though in so doing he really detracts from the merit of his work. On p. 78 he indicates a proceeding that would appear to be that subsequently developed by Lagrange. He takes seven points on a curve and says he succeeded in determining α, β, γ, etc., so as to make the trigonometric curve pass through these points, and he adds that the process might be continued. He gives, however, no proof of his statements.

§ 5. When the controversy was at this stage a memoir by Lagrange on the *Nature and Propagation of Sound* appeared in the first volume (1759) of the *Miscellanea Taurinensia* (Lagrange;

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Oeuv., vol. I.). In the introduction he gives a lucid statement of the methods of the three writers we have named, accepts Euler's solution as the most general, but objects to his mode of demonstration, and proposes to obtain a satisfactory solution by first considering the case of a finite number of vibrating particles and then seeking the limit for an infinite number—that is for a chord. The theory deduced in his fourth chapter for a finite number of particles is the same as that of Bernoulli on whose synthetical solution he bestows high praise (§ 32); but for our purposes the thirty-seventh article is the most important, in which he seeks the limit for an infinite number of particles. The length of the string being a and the initial co-ordinates of a point on it being (X, Y) the first part of the equation for the ordinate of the point (x, y) at time t is given by

$$y = \frac{2}{a}\int dx.\mathrm{Y}\Big(\sin\frac{\pi\mathrm{X}}{a}\sin\frac{\pi x}{a}\cos\frac{\pi\mathrm{H}}{\mathrm{T}}t + \sin\frac{2\pi\mathrm{X}}{a}\sin\frac{2\pi x}{a}\cos\frac{2\pi\mathrm{H}}{\mathrm{T}}t$$
$$+ \sin\frac{3\pi\mathrm{X}}{a}\sin\frac{3\pi x}{a}\cos\frac{3\pi\mathrm{H}}{\mathrm{T}}t + \text{etc.}\Big)$$

"where the integral sign $\int$ is used to express the sum of all these series and the integrations are to be made on the supposition that X, Y are the variables and t, x constants." This seems undoubtedly to be a Fourier series in the proper sense of the term; yet it appears to me doubtful if Lagrange actually supposed it to be such. It could hardly have escaped his notice that for a definite value of t this is simply Bernoulli's solution. It was doubtless no part of Lagrange's purpose, as Reiff remarks (p. 134), to determine the co-efficients in Bernoulli's series, but rather to obtain the functional solution given by D'Alembert as he actually does by summing the series by trigonometric methods. At the same time if Lagrange had really meant the summation to be what we now call an integration his subsequent evaluation of the series would not have possessed that generality he contended for, as it starts from a result that implies the continuity of Y. Exactly the same objections he urges (§ 15) against Euler could have been brought against himself. Many parts of the investigation of § 38, where he sums the series, are according to modern notions very loose; yet leaving this aside the investigation shows great analytical skill, and in some respects anticipates the procedure of Fourier as will be pointed out later. All the same I do not think that Lagrange himself nor any of his contemporaries can have understood the above series as anything else than a *finite* series, and I believe that the m used by Lagrange is not made really infinite until he has summed the series and passes to the functional solution. Further Lagrange was quite alive to the merits of Bernoulli's solution and even proposes a proof (*Oeuvres*, I., pp. 514–516) of the proposition that the initial figure of the chord, *when it has one*, is contained in the equation

$$y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \gamma \sin \frac{3\pi x}{a} + \text{etc.}$$

With this result before him it is almost beyond belief that Lagrange would fail to see its identity with his own formula quoted above, had he supposed m to be really infinite. With m infinite his solution would have been complete and the subsequent investigations mere transformations of it without adding anything to it.

Another investigation by Lagrange belonging to the same series of memoirs on Sound and printed on pages 552–554 of the first volume of his collected works is that repeatedly quoted by Poisson and others as the first investigation of the representation of a function by a trigonometric series. I think, however, that this investigation stands on the same footing as that just discussed and I hold that Riemann's view of it is correct. It is no doubt hard for us to understand how near Lagrange came to the conception of expanding an arbitrary function in an infinite series without ever actually attaining to it, especially when we see him in this memoir adopting the method of passing a trigonometric curve through a finite number of points on a given curve and succeeding in solving the necessary equations in the manner used later by Dirichlet (Dove's Repertorium). That he did not really solve the problem of expansion in trigonometric series is I think best understood from the circumstance that neither he nor any of his contemporaries (unless perhaps Bernoulli) believed such expansion to be possible. It would be interesting to have documentary evidence of the truth of Riemann's statement (*Werke*, p. 219) that Lagrange strongly objected to Fourier's conclusions in regard to such expansions.

§ 6. For the next forty years there seems to have been almost no progress made towards a solution of the difficulties raised in these discussions; but before passing to Fourier mention must be made

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of certain results given by Euler. In his memoir *Subsidium Calculi Sinuum* (*Novi. Comm. Petrop.*, tom V., ad annos 1754–55) he obtains (p. 204) the following equations :—

$$\sin\phi - \tfrac{1}{2}\sin 2\phi + \tfrac{1}{3}\sin 3\phi - \text{etc.} = \tfrac{1}{2}\phi$$

$$\cos\phi - \tfrac{1}{4}\cos 2\phi + \tfrac{1}{9}\cos 3\phi - \text{etc.} = \frac{\pi^2}{12} - \frac{\phi^2}{4}.$$

Reiff remarks (p. 128) that these series are the first in which rational functions are expressed by series of sines and cosines of multiples of the variable. It is somewhat remarkable that Euler should have accepted these results, but his views on the validity of results derived from the use of the series were extremely loose.

A more important result for the general theory is contained in a memoir presented by him to the St Petersburg Academy in 1777 but not published till 1798, long after his death. In the *Nova Acta*, vol. XI., p. 114, this memoir appears, and in it he says that if Φ can be expanded in a series of the form $A + B\cos\phi + C\cos 2\phi + \text{etc.}$ then

$A = \frac{1}{\pi}\int \Phi d\phi$, $B = \frac{2}{\pi}\int \Phi d\phi \cos\phi$, etc., where the limits of the integrals are 0 and π. Fourier's method of determining the coefficients is thus explicitly given by Euler as Jacobi remarked (Crelle's *Journal*, vol. II., p. 2); but the use that Euler makes of the series and the words in which he introduces his memoir seem to me to render it doubtful if Euler, as Sachse appears to think (p. 47), drew the hint that led to his method from Lagrange's memoirs. Except for the mode of determining the coefficients the memoir goes but a very little way towards settling the possibility of representing an arbitrary function by a trigonometric series.

§ 7. Glancing for a moment over the work of Euler, Bernoulli, and Lagrange, it is easy for us to see where the difficulties of the subject lay; they lay in the inadequacy of the notion of a function. Both Euler and Lagrange seem at times as if they had in part transcended the limits of their original conception, Euler in giving his geometrical constructions for the solution of the equation for the vibrating chord and Lagrange in his method of constructing the equation to a curve by first finding the equation of a curve passing through the vertices of an inscribed polygon. Yet I do not think either of them got beyond the old notion of continuity and its con-

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sequences in any of their writings on the subject of trigonometric series. But great part of their work could be and was of immense service to Fourier, as he himself indicates (*Théorie Anal. de la Chaleur*, § 428), when he approached the consideration of the subject with his conception of a function as given graphically.

§ 8. Fourier's first investigations on the Theory of Heat were communicated to the Academy of Sciences on the 21st December 1807. The memoir of 1807 has never been printed though it has now been recovered, and it is to the memoir sent to the Academy in 1811 and crowned on the 6th January of the following year that we must look for Fourier's exposition of the representation of arbitrary functions by trigonometric series. In all essential points the treatise *Théorie Analytique de la Chaleur*, published in 1822, is a reproduction of the memoir of 1811, and I shall therefore refer always to the treatise, in the recent edition of it by Darboux.

The third and the ninth chapters of the treatise are those in which the trigonometric expansions are most fully considered, and even a casual reading of these is sufficient to show how thoroughly Fourier cleared away the difficulties which had puzzled his predecessors. Even before the publication of Dirichlet's proof in 1829, which has generally been considered to be the first satisfactory exposition from the mathematical standpoint, Fourier's results had been universally accepted; no doubt some of Fourier's series were criticised but in many cases the errors were those of the printer and not of Fourier himself. At the same time there can be no question of the general acceptance of his main theorem on the subject of the expansion of an arbitrary function.

The treatise is so well known that I need not spend much time in analysing it; but I may call attention to one or two points. In article 428, 12° and 13° Fourier sums up his views on the nature of a function which admits of expansion; it is not necessarily *continuous* in the old sense of that word but may be composed of *separate functions* or *parts of functions*. By these phrases he means a function $f(x)$ which has values while x lies between given limits but is zero for all other values of x. The function may even become infinite between the limits (§ 417) and in general the function need only be given graphically. Again, Fourier has accurate conceptions of the convergency of series (§§ 177, 185, 228, 235, etc.) though he occasionally makes slips (for example in § 218 where he puts

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$1-1+1-1+\text{etc.}=\frac{1}{2}$, see also § 420, p. 506); in this respect both Euler and Lagrange leave much to be desired. A more important question remains, namely, how far did Fourier succeed in his mathematical demonstration that the series which represents the function actually converges to the value of the function? In special cases which he gives the convergency of the series and its equivalence with the function are, as he says, easily demonstrated; but it is usually maintained (*e.g.*, by Riemann, *Werke*, p. 220) that he gave no mathematical proof of the general theorem. I think, however, that in this respect Fourier has received less than justice. No doubt, the investigations of chapter III. can hardly be accepted as doing more than suggesting the truth of the general theorem, but it is different with those of chapter IX. In these as in nearly all the special series of chapter III. he adopts the method, followed afterwards by Dirichlet, of taking n terms of the series and seeking the limit for n infinite. This method indeed seems to me to be that of Lagrange in § 38 of his first memoir referred to above, and it is unquestionably the most satisfactory. Fourier's treatise being in everybody's hands I need hardly do more than refer to § 423 and suggest that it should be compared with Dirichlet's proof. At bottom Fourier's reasoning is, I believe, quite sound and it seems to me to contain the kernel of Dirichlet's proof. No doubt Fourier did not develope his proof with the extreme precision that the importance of the theorem demanded and that Dirichlet afterwards gave to it; still the substantial accuracy of his reasoning is beyond dispute. Darboux in a note pp. 511, 512 of his edition of the treatise calls attention to the matter, and his contention on behalf of Fourier seems to me quite justified by the facts. Before seeing this note I had formed the opinion I have expressed and I was glad to find it confirmed by so able an authority.

I content myself with this meagre reference to Fourier, because his treatise is so universally read even yet by all beginners in the study of mathematical physics that it would be waste of time to delay over it. I cannot pass from it however without remarking that it seems to me peculiarly unfortunate that instead of studying Fourier's own mode of presenting the proof of his series-theorem, or, what would have been even better, taking Dirichlet's memoirs on the same subject as guide, English writers have usually drawn their exposition from Poisson who studiously denied to Fourier his just claims in this field. As a matter of fact Poisson's proof is

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invalid and seems to have been recognised as such almost from the first by the great continental writers. At any rate, Dirichlet does not, I think, allude to it and Cauchy lays his finger on the weak point, as I shall indicate shortly. Nor does Riemann in his historical notice refer to Poisson except to call in question his estimate of Lagrange's position. No doubt the integral that Poisson makes use of is of great importance, and has played a fundamental part in many modern developments; but its value appears *after* the Fourier series has been established and not in the proof of the series itself.

§ 9. Poisson has treated the trigonometric series now dealt with in several places and always in practically the same way. I may refer to the *Journal de l'École Polytechnique* vol. XI. (1820), vol. XII. (1823), and to the treatise *Théorie de la Chaleur* (1835). His process is as follows :—

When $p<1$, $\dfrac{1-p^2}{1-2p\cos(x-a)+p^2}=1+2\sum_{n=1}^{n=\infty} p^n\cos n(x-a)$

Multiplying by $f(a)$ and integrating between $-\pi$ and π, he gets

$$\int_{-\pi}^{\pi}\frac{(1-p^2)f(a)da}{1-2p\cos(x-a)+p^2}=\int_{-\pi}^{\pi}f(a)\{1+2\Sigma p^n\cos n(x-a)\}da$$

When $p=1$, the integral on the left has all its elements zero except when $a=x$. Putting then $p=1-g$, where g is small, and $x-a=z$

he gets for the value of the integral $2f(x)\displaystyle\int_{-\epsilon}^{\epsilon'}\frac{gdz}{g^2+z^2}$ where ϵ, ϵ' are

small; but no error will be introduced by making the limits infinite, so that when $p=1$, the integral is equal to $2\pi f(x)$. Making $p=1$ on the right side he deduces

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(a)\{1+2\Sigma\cos n(x-a)\}da$$

The proof is usually extended so as to include the cases in which $f(x)$ presents discontinuities.

On this proof there are two remarks to be made. In the first

place, if the series $\displaystyle\int_{-\pi}^{\pi}f(a)\{1+2\Sigma p^n\cos n(x-a)\}da$ be denoted by

$\Sigma A_n p^n$, and if we write $F(p)=\Sigma A_n p^n$, then we are only justified

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in assuming $F(1) = \Sigma A_n$ when the series ΣA_n is convergent. This theorem is generally quoted as Abel's Theorem (see Chrystal's *Algebra* Pt. II. p. 133). But in the present case this procedure amounts to assuming that the trigonometric series is convergent, and the convergency of the series is not proved by Poisson. In other words one of the greatest difficulties of the subject is tacitly passed over. It may be added that unless the function $f(x)$ be very greatly restricted it does not seem possible to prove the convergence of the series from a consideration of the integrals which give the coefficients. In the second place, the quantity p has no natural connection with the series and is a source of ambiguity that is not inherent in the series itself. This is seen when the integral $\int_{-\pi}^{\pi} \frac{(1-p^2)f(a)da}{1-2p\cos(x-a)+p^2}$ is more carefully studied, as in the writings of Schwarz (see his two memoirs on the *Integration of the Equation* $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ in his Collected Works, vol. II). If we describe a circle with radius unity and take a point in it having (p, x) for its polar co-ordinates, then the limit of the integral for $p = 1$ depends, except when $f(x)$ is continuous and periodic, on the path by which the point approaches the circumference. Thus if $f(\pi - 0) \neq f(-\pi + 0)$, the limit when $x = \pi$ is $\pi\{f(\pi - 0) + f(-\pi + 0)\} + \theta\{f(\pi - 0) - f(-\pi + 0)\}$ where θ may have any value between π and $-\pi$. But the limit of the *series* when $x = \pi$ is perfectly definite, namely the value of the above expression when $\theta = 0$. However valuable, then, Poisson's integral may be in other respects it does not seem to furnish a satisfactory starting point for the investigation of the series in question.

§ 10. After Poisson, Cauchy attacked the problem in his *Mémoire sur les dévelopements des fonctions en séries périodiques* (*Mem. de l'Inst.* vol. VI.; read 27th Feb. 1826). He starts with the series

$$\int_0^a f(\mu)d\mu + 2\int_0^a \sum_{n=1}^{n=\infty} \cos\frac{2n\pi}{a}(x-\mu)f(\mu)d\mu .$$

To prove that this has for sum $af(x)$ he replaces it by another series

$$\int_0^a f(\mu)d\mu + \sum_{n=1}^{n=\infty}\theta^{n-1}\int_0^a e^{\frac{2n\pi i}{a}(x-\mu)}f(\mu)d\mu + \sum_{n=1}^{n=\infty}\theta^{n-1}\int_0^a e^{-\frac{2n\pi i}{a}(x-\mu)}f(\mu)d\mu$$

where $\theta = 1 - \epsilon$ and ϵ is a small quantity. The series, when summed, gives

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$$\int_0^a \left\{1 + \frac{1}{e^{\frac{-2\pi i}{a}(x-\mu)} - \theta} + \frac{1}{e^{\frac{2\pi i}{a}(x-\mu)} - \theta}\right\} f(\mu)d\mu$$

and this integral being evaluated in Poisson's manner is equal to $af(x)$. But Cauchy recognises one of the faults of Poisson's proof and tries to prove the convergence of the series when $\theta = 1$. To do this he throws it into the form

$$f(x) = \frac{1}{a}\int_0^a f(\mu)d\mu + \frac{1}{ai}\int_0^\infty \left\{\frac{f(a+\nu i) - f(\nu i)}{e^{\frac{2\pi x i}{a}} e^{\frac{2\pi\nu}{a}} - 1} - \frac{f(a-\nu i) - f(-\nu i)}{e^{\frac{-2\pi x i}{a}} e^{\frac{2\pi\nu}{a}} - 1}\right\} d\nu$$

This equation, as Cauchy remarks later, may be deduced by integration of the functions $f(z)/\left\{e^{\pm\frac{2\pi}{a}(z-x)i} - 1\right\}$ round a properly selected boundary. As to the function $f(z)$ it must remain finite for all real or imaginary values of z. He now, instead of examining the integral in its closed form, throws it again into a series of which the general term is, if $z = 2n\pi\nu/a$

$$\frac{1}{2n\pi i} e^{\frac{-2n\pi x i}{a}} \int_0^\infty e^{-z} \left\{f\left(a + \frac{ai}{2n\pi}z\right) - f\left(\frac{ai}{2n\pi}z\right)\right\} dz$$

$$-\frac{1}{2n\pi i} e^{\frac{2n\pi x i}{a}} \int_0^\infty e^{-z} \left\{f\left(a - \frac{ai}{2n\pi}z\right) - f\left(-\frac{ai}{2n\pi}z\right)\right\} dz$$

so that when n is very large the general term approximates to

$$\left(f(0) - f(a)\right) \frac{1}{n\pi} \sin\frac{2n\pi x}{a}$$

The series of which this is the general term is convergent, and he therefore concludes the trigonometric series to be convergent.

Now in regard to this proof two points in particular require notice. First, as Dirichlet noticed, there may be two series whose terms differ infinitely little from each other when $n = \infty$, and yet the one series diverges while the other converges; for example $\Sigma\frac{(-1)^n}{\sqrt{n}}$ converges while $\Sigma\frac{(-1)^n}{\sqrt{n}}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$ diverges. Cauchy's proof of the convergence thus fails. But, as Reiff remarks (p. 189), it is easy to see that the integral in closed form is finite if $0 < x < a$, so that this objection might be overcome. But, secondly, the conditions imposed on $f(z)$ reduce that function to a constant. Riemann, who pointed this out, states that Cauchy's conditions are not really

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necessary for his proof; it is sufficient that the function $f(x+iy)$ be determinable such that for all values of y it remains finite and for $y=0$ becomes $f(x)$. That such a function is determinable Riemann holds to be established, and therefore apparently that Cauchy's proof is valid. This remark of Riemann's is pretty fully considered by Sachse, pp. 48–52, and I content myself with referring to him, only adding that Riemann's proof of the possibility of determining a function by means of its values along a boundary is not now accepted, and that the necessity of using other methods of establishing the proposition in question carries with it the invalidity of Cauchy's proof.

For another and more general investigation by Cauchy I would refer to his *Oeuvres complétes*, vol. VII. (2nd ser.), p. 393.

§ 11. I come now to the classical investigations of Dirichlet. Of his two memoirs dealing with the subject of the Fourier Series the first appeared in Crelle's *Journal*, 1829, vol. IV., pp. 157–169, the second in Dove's *Repertorium der Physik*, vol. I., pp. 152–174. This second memoir is so clear and simple that it has become a model of nearly every discussion on the series in question contained in continental text-books, and probably there is no memoir in the whole range of mathematical journalism that has been so completely and so literally transferred to the text-books. Dirichlet saw that the convergence of the series does not depend solely on the decrease of the terms, but is due also to the presence of negative terms. (See the introduction to his paper on expansion in Spherical Harmonics in Crelle's *Journal*, vol. XVII.). Hence he adopts the method, which Fourier had employed, of summing to n terms and finding the limit for $n=\infty$. It will be convenient to follow the second rather than the first memoir.

The first $2n+1$ terms of the series for $\phi(x)$ may be written

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\alpha . \phi(\alpha)\frac{\sin(2n+1)\frac{\alpha-x}{2}}{2\sin\frac{\alpha-x}{2}}$$

and this integral may be divided into

$$\frac{1}{\pi}\int_0^{\frac{\pi+x}{2}} d\beta . \phi(x-2\beta)\frac{\sin(2n+1)\beta}{\sin\beta}+\frac{1}{\pi}\int_0^{\frac{\pi-x}{2}} d\beta . \phi(x+2\beta)\frac{\sin(2n+1)\beta}{\sin\beta}$$

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and the limit for $n=\infty$ has to be found. The investigation hinges upon the limit for $k=\infty$ of $\int_0^h \frac{\sin k\beta}{\sin\beta} f(\beta)d\beta$ where $k=2n+1$ and $0 < h <$ or $=\pi/2$. The function $f(\beta)$ is supposed in the first place to be continuous, positive, and not increasing, while β goes from 0 to h. The integral is decomposed into a series of partial integrals with limits $0, \pi/k$; $\pi/k, 2\pi/k$; etc.; $r\pi/k, h$ where $r\pi/k$ is the greatest multiple of π/k contained in h. Each of these integrals is less in absolute value than its predecessor and the signs of them are alternately positive and negative. The integral is thus found to lie between limits which for $n=\infty$ coincide in the value $\frac{1}{2}\pi f(0)$. The restrictions on $f(\beta)$ are then partly removed; it may either be constant or negative or a not decreasing function as β goes from 0 to h. It follows immediately that $\underset{k=\infty}{\mathrm{L}} \int_g^h \frac{\sin k\beta}{\sin\beta} f(\beta)d\beta = 0$ if $0 < g < h <$ or $= \frac{\pi}{2}$. By this last result it is possible to extend the first theorem to all continuous functions which have a finite number of maxima and minima, while if $f(\beta)$ be discontinuous for $\beta=0$ the limit is $\frac{1}{2}\pi f(+0)$ if h be positive but $-\frac{1}{2}\pi f(-0)$ if h be negative. The limit for $n=\infty$ of the sum of the first $2n+1$ terms of the trigonometric series is thus $\frac{1}{2}\{\phi(x+0)+\phi(x-0)\}$ if $x \neq \pm\pi$ but $\frac{1}{2}\{\phi(\pi-0)+\phi(-\pi+0)\}$ if $x = \pm\pi$.

The results may therefore be summed up as follows:—The limit for $n=\infty$ of the series $\frac{1}{2}a_0 + \sum_{m=1}^{m=n}(a_m\cos mx + b_m\sin mx)$ where

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(a)\cos ma\,da, \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(a)\sin ma\,da$$

is $\qquad \frac{1}{2}\{\phi(x-0)+\phi(x+0)\}$ if $x \neq \pm\pi$

but $\qquad \frac{1}{2}\{\phi(\pi-0)+\phi(-\pi+0)\}$ if $x = \pm\pi$

provided that while $-\pi =$ or $< x =$ or $< \pi$, $\phi(x)$ has a finite number of maxima and minima, a finite number of discontinuities, and does not become infinite. Of course if $\phi(x)$ is continuous near x, the value is simply $\phi(x)$. These conditions (with another regarding infinite values of $\phi(x)$ to be given presently) are usually called *Dirichlet's conditions.*

It is perhaps worth observing that the mode of conducting the investigation prescribes the order in which the terms are to be taken, and the order is of course essential when the series is semi-convergent.

§ 12. The definite form which Dirichlet gives to the sum of the trigonometric series suggests that the phrases "the function $\phi(x)$ can be expanded in a series" or "the series represents the function $\phi(x)$" should be precisely defined, for where there is a breach of continuity in the function the series has a definite value while the function has not. The natural definition seems to be that adopted by Sachse (p. 55), namely, a series represents a function in a given interval if its values coincide with those of the function for all points in the interval with the exception of a limited number of known points. A Fourier series therefore represents a function which satisfies Dirichlet's conditions.

There is one point in Dirichlet's demonstration which has been subjected to criticism in some quarters. According to Dirichlet the value of the series at a point of discontinuity in the function is the arithmetic mean of the values of the function at that point. It has been contended on the other hand by Schläfli and Du Bois-Reymond that the value is really indeterminate (compare also Thomson and Tait, *Nat. Phil.*, vol. I., pt. I., p. 59) and that the sum may have all values between the two values of the function at the point. Sachse (pp. 56–58) discusses the point and as I have not had access to Schläfli's pamphlet (*Einige Zweifel an der allg. Darst. . . . durch trig. Reihe*, Berne, 1874) nor to Du Bois-Reymond's memoir (*Sprungweise Werthveränderungen*, *Math. Ann.*, vol. VII.) I must simply refer to Sachse for a fuller notice and also to Heine's *Kugelfunctionen*, vol. II., p. 347. At the same time I may say that these objections, so far as I understand them, do not seem to me to be sound as they rest upon the evaluation of a *double limit* while in the case of the series there is but one variable to be considered. I have already referred to the ambiguity of a similar character introduced by Poisson's proof.

§ 13. Had Dirichlet not written his first memoir, the paper which follows his in the same volume of Crelle (vol. IV., p. 170) by Dircksen would have been a notable contribution to the theory of trigonometric series. It proceeds on the same general lines as Dirichlet's though obviously it is quite independent; but neither in elegance nor in generality is it comparable with his, and it has practically fallen into oblivion.

Bessel in Schumacher's *Astronomische Nachrichten*, vol. XVI., p. 229, sought to simplify Dirichlet's proof, but he can hardly be said to have succeeded, and he certainly added nothing to the general theory.

§ 14. The conditions given by Dirichlet in his first memoir, as those which a function must satisfy if it is to be represented by a trigonometric series, are certainly very general, and in an addition to his memoir on the representation of an arbitrary function by a series of Spherical Harmonics (Crelle's *Journal*, 1837, vol. XVII., p. 54) he shows that the function $\phi(\beta)$ may become infinite at a finite number of points provided that $\int\phi(\beta)d\beta$ remain finite and continuous. This condition will be included among *Dirichlet's conditions* when these are referred to. But Dirichlet believed that a function, with fewer restrictions than those implied in his conditions, could be represented by a trigonometric series, and at the end of his first memoir promises a paper on the subject. Nothing, however, except the note in the seventeenth volume of Crelle, just mentioned, has appeared from his pen in the way of carrying out the promise. In particular it should be noticed that Dirichlet's conditions do not include all continuous functions, since they exclude every function with an infinite number of maxima and minima; but if a function have an infinite number of oscillations in the neighbourhood of a point it may be continuous when the amplitude of the oscillations is infinitely small. Thus the function $x\cos(1/x)$ is continuous between $-\pi$ and π on the understanding that it is zero for $x=0$, yet this would be excluded from Dirichlet's conditions. One of the main objects of later investigations has been to extend the limits of the arbitrariness allowable to a function which may still be represented by a trigonometric series, but it is a somewhat striking fact that the conditions do not yet include all continuous functions, and Du Bois-Reymond has even proved that there are continuous functions such that the trigonometric series which represent them become infinite at certain points, that is, cease to represent them at these points. The belief then that every continuous function can be represented by a trigonometric series is unwarranted.

§ 15. The first published attempt to show that a function having an infinite number of maxima and minima may be represented by a trigonometric series is that of Lipschitz in his memoir *De explicatione per series trigonometricas*, etc. (Crelle's *Journal*, 1864, vol. LXIII., p. 296). His proof depends on the evaluation of the two integrals noticed as fundamental in Dirichlet's method, and he shows that these still maintain their validity if in the neighbourhood of those points β for which $f(\beta)$ oscillates $f(\beta+\delta)-f(\beta)$ is less in absolute value than $B\delta^{\alpha}$ where α is positive and B a constant. As an extension of Dirichlet's conditions the result is important, but it is to be observed that there may be continuous functions not satisfying this condition. $f(\beta)$ will be continuous near β if, given an arbitrarily small quantity ϵ, a value h can be found such that for all values of δ less numerically than h, mod. $\{f(\beta+\delta)-f(\beta)\}$ is less than ϵ. Lipschitz's condition implies that $\epsilon = \text{or} < Bh^{\alpha}$ or $h = \text{or} > \sqrt[\alpha]{(\epsilon/B)}$, a relation not necessary for continuity. Again, Lipschitz's results would hold if $\underset{\delta=0}{L} \log\delta\{f(\beta+\delta)-f(\beta)\}=0$ and this is a form which Dini uses in his treatise *Sopra la Serie di Fourier*, and is less restrictive than the other.

§ 16. I now come to Riemann's investigations as contained in his great memoir *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*. Though prepared for his *Habilitationschrift* in 1854, it was not published till after his death, appearing in vol. XIII, of the *Göttingen Abhandlungen*, 1867; it is reprinted in his collected works, pp. 213–253, with notes by Weber.

The memoir is divided into three main sections. The first section, arts. 1–3, is historical and has been several times referred to in the earlier part of this paper. The second, arts. 4–6, contains a thorough investigation of the fundamental principles of definite integrals, and in particular determines in what cases a function has an integral. We see here the great extension of meaning which the word *function* has gained in modern times, chiefly under the guidance of Fourier, Dirichlet, and Riemann himself, and which is essential to the modern function theory. The third section completes the memoir and is devoted to the representation of a function by a trigonometric series without special suppositions as to the nature of the function. The problem proposed for solution is the following:—If a function can be represented by a trigonometric series, what follows respecting the march of the function, respecting the change in its value for a continuous change in the

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argument? The preceding investigations argued from the function to the series; here the series is supposed given and the conclusion is to the nature of the function.

Riemann denotes the series $A_0 + A_1 + A_2 + \text{etc.} + A_n + \text{etc.}$ where $A_0 = \frac{1}{2}b_0$, $A_n = a_n\sin nx + b_n\cos nx$ by Ω, and when it is convergent its value is denoted by $f(x)$, so that $f(x)$ only exists for those values of x for which the series is convergent. He first supposes Ω to be such that for every value of x A_n becomes infinitely small when n becomes infinitely great. If the series Ω be integrated twice and the series thus formed be denoted by $F(x)$ so that

$$F(x) = C + C'x + \tfrac{1}{2}A_0x^2 - A_1 - \text{etc.} - \frac{1}{n^2}A_n - \text{etc.}$$

he shows that $F(x)$ is convergent for every value of x, is continuous, and is integrable. He then proves—

(I.) That when the series Ω converges, the expression

$$\{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)\}/4\alpha\beta$$

converges to the value $f(x)$ when α and β become infinitely small, but such that their ratio remains finite;

(II.) That $\{F(x+2\alpha) + F(x-2\alpha) - 2F(x)\}/2\alpha$ becomes infinitely small with α; and

(III.) That the integral $\mu^2\int_b^c F(x)\cos\mu(x-a)\lambda(x)dx$ becomes infinitely small with $1/\mu$, where b, c denote two arbitrary constants ($c > b$), $\lambda(x)$ a function which with its first derivative is continuous between b and c and vanishes at the limits and whose second derivative has not an infinite number of maxima and minima.

By means of these theorems he proves that if a periodic function $f(x)$, of period 2π, can be represented by a trigonometric series whose terms become ultimately indefinitely small there must exist a continuous function $F(x)$ such that

$$\{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)\}/4\alpha\beta$$

converges to the value $f(x)$ when α, β converge to zero, their ratio remaining finite. Further, the integral of (III), subject to the conditions there given, must become infinitely small with $1/\mu$.

Conversely, when these conditions are satisfied, there exists a trigonometric series whose terms become infinitely small and which is such that, where it converges, it represents the function. For, determining C', A_0 so that $F(x) - C'x - \frac{1}{2}A_0x^2$ has the period 2π, and

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then developing this function by the Fourier method the term A_n where

$$A_n = -\frac{n^2}{\pi}\int_{-\pi}^{\pi}\{F(t) - C't - \tfrac{1}{2}A_0t^2\}\cos n(x-t)dt$$

will become infinitely small with $1/n$ and therefore the series $A_0 + A_1 + A_2 + \text{etc.}$ will, whenever it converges, converge to $f(x)$. In Weber's note (Riemann's *Works*, p. 252) the proof for this assertion about A_n is fully given.

Riemann then shows that the convergence of the series for a definite value of x depends only on the behaviour of the function in the neighbourhood of that value. A proof of this important theorem, independent of Riemann's general theorems and due to Schwarz, is given by Sachse, pp. 89 *et seq.*

It will have been observed that as yet Riemann has given no criterion for determining when the coefficients of the series Ω will in fact become infinitely small. In art. 10 he comes to this point, and he there states that in many cases this question can not be settled by consideration of their expression as definite integrals, but must be determined in other ways. For the very important case in which $f(x)$ is integrable, finite throughout the range of the variable, and (he should have added) has only a finite number of maxima and minima, he proves that the coefficients do become infinitely small and therefore that the series represents $f(x)$ whenever it is convergent.

In art. 11 he takes up the case in which the terms of Ω do not become ultimately indefinitely small for every value of x, and shows that the series can converge only for those values of x which are symmetrically placed with respect to those for which the integral

$$\mu^2\int_b^c F(x)\cos\mu(x-a)\lambda(x)dx$$

does not become infinitely small with $1/\mu$.

In art. 12 he considers the possibility of the function becoming infinite, and gives as necessary and sufficient conditions that when $f(x)$ is infinite for $x=a$, $tf(a+t)$ and $tf(a-t)$ become infinitely small for $t=0$ and $f(a+t)+f(a-t)$ be integrable up to $t=0$, it being understood that $f(x)$ has not an infinite number of maxima and minima.

In the last article, art. 13, he deals with functions having an infinite number of maxima and minima. In this connection he first

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shows by an example that there may be integrable functions having an infinite number of maxima and minima which are yet not capable of representation by a Fourier series. He here takes $f(x)=\frac{d}{dx}\left(x^{\nu}\cos\frac{1}{x}\right)$ where $0<\nu<\frac{1}{2}$. He shows in the second place by examples that there may be functions having a finite number of maxima and minima and not integrable which nevertheless may be represented by a trigonometric series (on these examples, see a paper by Genocchi, *Atti della R. Acc. di Torino*, vol. X., 1875, *Intorno ad alcune serie*).

Riemann has thus given a very general solution of the problem of representation of functions by trigonometric series and his theorems (I), (II), (III), are of fundamental importance in the subsequent investigations of Heine, Cantor, and Du Bois-Reymond. But other methods than those he gives must in many cases be resorted to to determine when the series is convergent, and as a matter of fact, Dirichlet's integrals seem indispensable for this purpose.

§ 17. Hitherto I have said nothing of the contributions of English writers to the theory of expansion in trigonometric series, and I am sorry to add that the main reason for this is that their contributions are so few. It is, I think, very unfortunate that Poisson's treatment of the Fourier series has become the basis of nearly every investigation in our text-books, because, as has been pointed out, that method is radically faulty. Dirichlet's proof seems to have been long unknown, for except in a note to a paper of Stokes's, to be mentioned presently, I do not remember to have seen it even mentioned till the publication of Todhunter's treatise on *Laplace's Functions*. In his *Integral Calculus*, Todhunter makes no mention whatever of it, except in the reference to his treatise on *Laplace's Functions*, and even there it is only given as an alternative to the other. The reference he makes to Abel's theorem on p. 170 of the treatise is curious as it tacitly assumes the whole thing to be proved, for it assumes that $\Sigma(2n+1)u_n$ is convergent.

De Morgan's *Calculus* is often referred to for the demonstration of the Fourier series, but while it is quite true that De Morgan gives many helpful illustrations and examples like other English writers (Donkin, in particular, in his *Acoustics*), it cannot be said that he has advanced beyond Poisson. I do not understand how such a careful writer as De Morgan could have allowed some of the statements he makes to pass. Thus (p, 607) he says $1+\cos\theta+\cos 2\theta+\ldots\ldots=\frac{1}{2}$ in every case unless $\theta=2\pi m$ when it is infinite, and he thinks (p. 640) there is no reason to doubt that the infinite series $1-1+1-$ etc. (namely, the value of that series for $\theta=\pi$) represents half a unit. This example might have been sufficient to show the uncertainty of reasoning from the convergence of $\Sigma a^n u_n$ to that of Σu_n.

§ 18. Hamilton in his great memoir *On Fluctuating Functions* (*Trans. R.I.A.*, vol. XIX., 1842) has much that bears on the subject of periodic series but no set proof of the Fourier series itself. His integrals, however, include the integrals of Dirichlet as particular cases, and the paper deserves more careful study than it usually receives. A good restatement of Hamilton's principal results in regard to these integrals will be found in Neumann's treatise, *Über die nach Kreis–Kugel–und Cylinder–Functionen fortsch. Entwickelungen*, which contains a good statement of the Fourier series and integrals for the case of *vernünftige Functionen.*

Stokes's memoir *On the Critical Values of the Sums of Periodic Series* (*Camb. Phil. Trans.*, 1847, vol. VIII., p. 533, reprinted in *Collected Works*, vol. I., p. 236) is important in the history of series, for he there (section III.) draws attention to what has since been called the uniform convergence of series, though this honour is usually attributed to Seidel, whose paper (evidently quite independent of Stokes's) did not appear till 1848. In the first section Stokes discusses the expansion of a function in a series of sines and also in a series of cosines, and adopts the method of Poisson as that which he employed when he first began the investigations and which best harmonised with the rest of the paper. He points out, however, in a note to page 251 (*Coll. Works*) that had he been aware of Dirichlet's memoir in Crelle and of Hamilton's paper at an earlier stage of his work he would probably have adopted the method of summing the first n terms of the series and then considering the limit of n infinite. The investigation as carried out is a little tedious but it forms a great advance on the way in which Poisson's mode of treating the subject is usually conducted. There are many things in the paper which make it still valuable, but as it is so easily accessible I need do no more than refer to it.

The investigation given by Thomson and Tait in their *Treatise*

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on Natural Philosophy, vol. I., pt. I., pp. 55–60 has much in common with Poisson's proof and also with Cauchy's. It will be noticed that in passing from their equation (11) to equation (12) the continuity of the series up to and including the value $e=1$ is assumed. But as has been repeatedly stated this assumption is only legitimate if the convergence of the series for $e=1$ is otherwise known, so that the same objection applies here as in Poisson's own proof. In general the convergence of the series is only comparable with that whose general term is A/n, and this result does not carry us far in determining the convergence. The remark on p. 59 that "if exactly the critical value is assigned to the independent variable, the series cannot converge to any definite value" is an *ex cathedra* statement which Dirichlet's proof shows to be quite incorrect.

§ 19. The course of the history of the Fourier series now takes a new departure. In the preceding work it has been seen that under certain circumstances the series will converge to the value of the function, but in more recent times it has been recognised that mere convergence is not sufficient for most of the applications for which the series is needed; the convergence must be uniform. Suppose, for instance, that we have for $f(x)$ the series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{n=\infty}(a_n\cos nx + b_n\sin nx)$$

and we wish to evaluate $\int_\alpha^\beta f(x)\phi(x)dx$ by means of the series; then we can only safely assert the equation

$$\int_\alpha^\beta f(x)\phi(x)dx = \tfrac{1}{2}a_0\int_\alpha^\beta \phi(x)dx + \Sigma\int_\alpha^\beta (a_n\cos nx + b_n\sin nx)\phi(x)dx$$

if the series be uniformly convergent. Unless then the series is to be shorn of much of its value its uniform convergence must be established.

Another difficulty that this conception of uniform convergency raises is that the old proof for the uniqueness of the expansion becomes invalid, as resting upon an integration the legitimacy of which is not proved.

§ 20. The first to call attention to the points just mentioned was Heine in a paper contributed to Crelle's *Journal* vol. LXXI. (1870)

p. 353, *Über trigonometrische Reihen.* In § 2 he gives the definition of uniform convergence, and it is interesting to note as illustrative of the immense influence of Weierstrass, in spite of his comparatively few published papers, that Heine's attention seems to have been first directed to the matter of uniform convergence by Weierstrass or one of his pupils rather than by the writings of Seidel or Stokes. (As regards Stokes, Reiff in his *Geschichte* (p. 207) seems to have been the first to give him due credit in this connection.) He shows that the Fourier series can not converge uniformly in the neighbourhood of a point at which the function is discontinuous, and establishes the following theorems :—

(1) The Fourier series for a finite function $f(x)$ with a finite number of maxima and minima converges uniformly if $f(x)$ be continuous for $-\pi = \text{or} < x = \text{or} < \pi$ and $f(-\pi) = f(\pi)$; in all other cases it is only uniformly convergent *in general*, that is, it converges uniformly for every interval which does not include a point of discontinuity, these points being snpposed finite in number. The points $\pm\pi$ are to be considered points of discontinuity if $f(-\pi) \neq f(\pi)$.

(2) If a trigonometric series is in general uniformly convergent, and is in general equal to zero $(-\pi = \text{or} < x = \text{or} < \pi)$ then will every co-efficient be zero. For the proof of this theorem he falls back on Riemann's proposition regarding $\underset{\alpha=0}{\mathrm{L}}\{\mathrm{F}(x+\alpha)+\mathrm{F}(x-\alpha)-2\mathrm{F}(x)\}/\alpha = 0$. The proof of theorem (1) follows the lines of Dirichlet's proof, and is reproduced in greater detail in his *Kugelfunctionen*, vol. I., pp. 53 *et seq.*

§ 21. Heine's second theorem shows that there cannot be two different expansions of a function if these are to be (in general) uniformly convergent. Cantor has proved the more general theorem that even if uniform convergence be not demanded there can be but one convergent expansion in a trigonometric series and it is that of Fourier. Cantor's memoirs appear in Crelle's *Journal*, vol. LXXII., p. 130, *Über einen . . . Lehrsatz* and p. 139 *Beweis dass eine für jeden reelen Werth, etc.*, vol. LXXIII, p. 294, *Notiz zu dem Aufsatze : Beweis, etc.* In the first of these he proves that if two infinite series, a_1, a_2, etc., b_1, b_2, etc., are such that $\underset{n=\infty}{\mathrm{L}}(a_n \sin nx + b_n \cos nx) = 0$ where x is real and lies in a given interval a, b, then $\mathrm{L}a_n = 0$, $\mathrm{L}b_n = 0$ for

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$n = \infty$. In the second memoir he takes the function $F(x)$ of Riemann, the conditions imposed on it being shown, by the proposition just stated, to be satisfied and forms the quotient $\{F(x+a) - 2F(x) + F(x-a)\}/a^2$. This quotient is zero for $a = 0$ and $F(x)$ is continuous; and it now follows by a theorem due to Schwarz that $F(x)$ must be a linear function of x. (Of course, if $F(x)$ be supposed to have continuous first and second derivatives, this theorem is evident). Giving to $F(x)$ a linear value and adopting the notation of Riemann, we have

$$\tfrac{1}{2}A_0x^2 + C_1x + C_2 = A_1 + \tfrac{1}{4}A_2 + \ldots\ldots + \frac{1}{n^2}A_n + \ldots\ldots$$

The right hand member being periodic, it follows that $A_0 = 0 = C_1$ and then by multiplying by $\sin nx$ or $\cos nx$ and integrating between $-\pi$ and π (a process now allowable) it is seen that $a_n = 0 = b_n$ for every value of n. Hence a convergent trigonometric series can represent zero only if every coefficient is zero, from which the uniqueness of the trigonometric expansion at once follows.

In the third memoir quoted above he gives a simplified form of the proof, due to Kronecker, which dispenses with the necessity of the investigation of the first memoir. In an article, *Über die Ausdehnung eines Satzes, etc.*, *Math. Ann.*, vol. V., Cantor extended his theorem to functions having an infinite number of discontinuities, provided these be distributed in a particular way, but, unfortunately, I have not had access to this article.

§ 22. Du Bois-Reymond's name has occurred more than once incidentally in this paper, and one memoir of his has now to be briefly noticed. His contributions to the theory of series in general and of the Fourier series in particular have been both numerous and important, but I can hardly do more than give a brief notice of one memoir and a statement of some interesting results of a second. These two memoirs are very important, and they contain notices of the work of predecessors and full references to his own papers bearing on the subject; but a detailed analysis would carry me far beyond the limits of this paper.

The first of these two memoirs appears in the *Abhandlungen der Bayerischen Academie*, vol. XII. (1875) p. 117, *Beweis dass die Coeff. der trig. Reihe, etc.* He there proves that the coefficients of the series $f(x) = \sum_{p=0}^{p=\infty} (a_p \cos px + b_p \sin px)$ have the values

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$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha),\quad a_p=\frac{1}{\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha)\cos p\alpha,\quad b_p=\frac{1}{\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha)\sin p\alpha,$$

whenever these integrals are finite and determinate. This proposition includes of course the theorem that $f(x)$ can be expanded in only one way in a Fourier series. In the proof Riemann's theorems (I.) and (II.) and Schwarz's theorem, quoted above, play an important part Putting

$$F(x)=\tfrac{1}{2}a_0x^2-\sum_{n=1}^{n=\infty}\frac{1}{p^2}(a_p\cos px+b_p\sin px)\ \ldots\ \ldots\ (1),$$

and supposing, in the first place, $f(x)$ to be continuous, he seeks to express $F(x)$ by $f(x)$. For every value of x between $-\pi$ and π

$$\frac{d^2}{dx^2}\int_{-\pi}^{x}d\alpha\int_{-\pi}^{\alpha}d\beta.f(\beta)=f(x).$$

If $\Phi(x)=F(x)-\int_{-\pi}^{x}d\alpha\int_{-\pi}^{\alpha}d\beta f(\beta)$ it follows that $\underset{\epsilon=0}{L}\,\Delta^2\Phi/\epsilon^2=0$

where $\Delta^2\Phi=\Phi(x+\epsilon)-2\Phi(x)+\Phi(x-\epsilon)$, and therefore $\Phi(x)=c_0+c_1x$

and $F(x)=\int_{-\pi}^{x}d\alpha\int_{-\pi}^{\alpha}d\beta f(\beta)+c_0+c_1x=F_1(x)+c_0+c_1x$, suppose, (2).

Multiplying (1) by $\cos nx$, $\sin nx$ respectively and integrating between $-\pi$ and π we get

$$\int_{-\pi}^{\pi}F(\alpha)\cos n\alpha d\alpha=(-1)^n\frac{2\pi}{n^2}a_0-\frac{\pi}{n^2}a_n\,;\quad \int_{-\pi}^{\pi}F(\alpha)d\alpha=\frac{\pi^3}{3}a_0\,;$$

$$\int_{-\pi}^{\pi}F(\alpha)\sin n\alpha d\alpha=-\frac{\pi}{n^2}b_n.$$

Replacing $F(x)$ by its value given by (2), integrating by parts and noticing that a_n, b_n, $\int_{-\pi}^{\pi}d\alpha f(\alpha)\cos n\alpha$, $\int_{-\pi}^{\pi}d\alpha f(\alpha)\sin n\alpha$ vanish with $1/n$ he finds

$$c_0=\frac{1}{4\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha)\left\{\frac{\pi^2}{3}-(\pi-\alpha)^2\right\},\quad c_1=-\frac{1}{2\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha)(\pi-\alpha),$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha),\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha)\cos n\alpha,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}d\alpha f(\alpha)\sin n\alpha$$

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Du Bois-Reymond now, instead of supposing $f(x)$ to have discontinuities, proceeds to consider the case where $f(x)$ is supposed only to be integrable. In this case $\underset{\epsilon=0}{\mathrm{L}}\, \Delta^2\Phi/\epsilon^2$ is not, or at least is not provable to be, generally zero. He proves, however, that it follows from the integrability of $f(x)$ that $\Phi(x)$ is a linear function of x, but the proof is too long and complicated to be reproduced here. When once this point is established the reasoning is as before. He then considers the possibility of $f(x)$ having infinite values

In the same volume of the *Bayerischen Abhandlungen, Zweite Abtheilung*, pp. 1–102, Du Bois-Reymond has a long article entitled *Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungs-Formeln.* The memoir forms rather laborious reading, but is, nevertheless, a very important contribution to the theory of the Fourier series. It is specially valuable on account of the thorough discussion of the Dirichlet integral $\underset{h=\infty}{\mathrm{L}} \int_0^b d\alpha f(\alpha) \frac{\sin h\alpha}{\alpha}$. By considering special forms of $f(\alpha)$ he succeeds in showing that there do exist continuous functions of x such that for special values of x the Fourier series does not converge. In the last chapter of his essay Sachse gives an example, due to Schwarz, of such a function; the example is included in Du Bois-Reymond's more general ones, but is simpler both in definition and in proof. In the *Comptes Rendus*, vol. XCII, p 915 and p. 962, will be found a short statement by Du Bois-Reymond himself of his investigations on integrals of the form $\underset{h=\infty}{\mathrm{L}} \int_a^b f(x)\phi(x, h)dx$.

§ 23. The memoirs of Du Bois-Reymond may be said in a sense to include all the results of previous writers and to push the inquiry as to the nature of the functions which can be represented by a Fourier series when the co-efficients are determined as definite integrals very near its utmost limits. In what follows I will therefore refer chiefly to certain investigations on the Dirichlet integral, and to some articles which bear on the integrals and which tend to simplify proofs and to clear up one or two doubtful points. But before doing so I would specially recommend to any one who wishes to have in a compact form a thorough and rigorous treatment of the Fourier series in all its bearings the treatise by Ulisse

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Dini, entitled *Serie di Fourier e altre Rappresentazioni analitiche delle Funzioni di una Variabile Reale* (Pisa, 1880). As the title indicates, the book contains much more than the Fourier series proper, and the whole treatment is carried through on a uniform plan and with scrupulous accuracy of statement. A careful reading of it is quite an education in some of the most delicate points of the integral calculus and of the theory of functions.

In the appendix to the second volume of his *Kugelfunctionen* (Berlin, 1881) Heine returns to the discussion of the Fourier series, and shows how, by a certain procedure, great simplification may be introduced into the mode of presenting Dirichlet's proof, which is apt to become rather tedious from the great number of different cases that have to be considered. In particular, the simplification affects the consideration of the uniform convergence of the series, and throws light on certain difficulties raised by Schläfli

In some respects Heine's treatment in this appendix resembles that suggested by Jordan in a paper *Sur la Série de Fourier* (Comptes Rendus, 1881, vol. XCII., p. 228); for the decomposition of the function, as proposed by Heine, into the sum of functions which are either not increasing or not decreasing, secures the same end as Jordan obtains by his conception of *fonctions à oscillation limitée.* In his *Cours d' Analyse*, vol. II. (first edition) Jordan systematically uses the *fonction à oscillation (variation) limitée* in discussing the integrals of Dirichlet and Du Bois-Reymond, and thus simplifies the treatment considerably. In the paper just mentioned he gives a new condition for F(x), such that

$$\underset{p=\infty}{\mathrm{L}} \int_0^b \mathrm{F}(x)\frac{\sin px}{x}dx = \frac{\pi}{2}\mathrm{F}(0)$$

is still true.

Conditions including that of Jordan are developed in an article by O. Hölder, *über eine neue Bedingung, etc.* (Berliner Berichte, 1885, p. 419)

In the *Berliner Berichte* for the same year (p. 641) Kronecker has a memoir *Über die Dirichletsche Integral*, which is particularly noteworthy because of the variety of forms to which he reduces the conditions for the validity of the equation $\underset{n=\infty}{\mathrm{L}} \int_0^h f(x)\frac{\sin nx}{x}dx = \frac{\pi}{2}f(0)$.

These conditions include those of Dirichlet, Lipschitz, Jordan, and Hölder. Kronecker thinks the variety of the results is due to his

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method of putting $f(x) = f_0(x) + f(0)$ so that $f_0(x)$ vanishes with x and using $f_0(x)$ in the integral.

I cannot conclude without calling attention to a remarkable memoir by Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen* (Berliner Ber., 1885, p. 633 and p. 789). He there proves the remarkable theorem that if $f(x)$ be a single-valued, continuous, and periodic function (x real), then, given an arbitrarily small positive magnitude g, a finite Fourier series can be formed in a variety of ways which is such that the difference between it and the function $f(x)$ does not exceed g for any value of x. Further, every such function $f(x)$ (period $= 2c$) may be represented as a sum whose terms are finite Fourier series with the period $2c$. This series converges absolutely for every value of x and uniformly in each finite interval.

In the *Comptes Rendus* for 1891, (vol. CXII., p. 183) Picard has proved the first theorem by using Poisson's integral. *(Sur la représentation approchée des fonctions).*

In the foregoing paper there are some points in connection with the Fourier series which I have not touched upon, and in particular the differentiability of the series. I have also avoided all reference to series other than the Fourier series strictly so called. To have taken up these points would have added considerably to the length of the paper, already perhaps too long. I would fain hope that no important contribution to the theory of the Fourier series has been altogether passed over, and that the paper may prove useful in directing attention to a most interesting side of mathematical theory.