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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF HOMOGENEOUS AND NONHOMOGENEOUS DELAY DIFFERENTIAL EQUATIONS OF EVEN ORDER

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Abstract

In this paper we consider the (non)oscillation properties of two general nonhomogeneous nonlinear delay differential equations of order $2n$

$$(N_{\pm}) \qquad D^n[r(t)D^n y(t)] \pm y_\tau(t) f[t, y_\tau(t)] = Q(t),$$

using as background and motivation the techniques previously applied to the associated homogeneous delay differential equations H_+ and H_-. The equations N_+ and N_- are each reduced to homogeneous form by the introduction of transformations $u(t) = y(t) - R(t)$ and $v(t) = R(t) - y(t)$, where $R(t)$ is a solution of the associated nonhomogeneous differential equation (N). We first extend certain results for the equation H_+ and then develop a classification of the positive solutions of equation H_-. Using this classification and the one developed by Terry (1974) for H_+ we develop a natural classification of the positive solutions of N_+ and N_- according to the sign properties of the derivatives of $u(t)$ and $v(t)$. For each choice of $R(t)$, it is seen that there are $2n + 1$ types of positive solutions of N_+ or N_-. An intermediate Riccati transformation is employed to obtain integral criteria for the nonexistence of some of these solutions. Analysis of the Taylor remainder results in sufficient conditions for the nonexistence of other such solutions.

The purpose of this paper is to discuss the oscillatory and nonoscillatory behavior of solutions of the nonlinear delay differential equations of order $2n$:

$$(N_+) \qquad D^n[r(t)D^n y(t)] + y_\tau(t) f[t, y_\tau(t)] = Q(t)$$

and

$$(N_-) \qquad D^n[r(t)D^n y(t)] - y_\tau(t) f[t, y_\tau(t)] = Q(t),$$

where $Q(t) \not\equiv 0$, $0 < m \leqq r(t) \leqq M$, $y_\tau(t) = y[t - \tau(t)]$ and $0 \leqq \tau(t) < t$. Throughout the paper $f(t, u)$ is assumed to satisfy the following three hypotheses:

(i) $f(t, u)$ is a continuous real valued function on $[0, \infty) \times R$, $R = (-\infty, \infty)$;

(ii) for each fixed $t \in [0, \infty)$, $f(t, u) < f(t, v)$ for $0 < u < v$; and

(iii) for each fixed $t \in [0,\infty)$, $f(t,u) > 0$ and $f(t,u) = f(t,-u)$ for $u \neq 0$.
Before considering the nonhomogeneous equations N_+ and N_- we first review and extend some results concerning the associated equations

$$(H_+) \qquad D^n[r(t)D^n y(t)] + y_\tau(t) f[t, y_\tau(t)] = 0 \text{ and}$$

$$(H_-) \qquad D^n[r(t)D^n y(t)] - y_\tau(t) f[t, y_\tau(t)] = 0.$$

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A solution $y(t)$ of H_+, H_-, N_+ or N_- is said to be oscillatory on $[a,\infty)$ if for each $\alpha > a$, there exists a $\beta > \alpha$ such that $y(\beta) = 0$; it is called nonoscillatory otherwise. Following Terry (1974), we say that a solution of H_+ is of type B_j if for sufficiently large t, $y_k(t) > 0$ for $k = 0,\cdots,2j+1$ and $(-1)^{k+1} y_k(t) > 0$, $k = 2j+2,\cdots,2n-1$ where

$$y_k(t) = \begin{cases} D^k y(t), & k = 0,\cdots,n-1 \text{ and} \\ D^{k-n}[r(t)D^n y(t)], & k = n,\cdots,2n-1. \end{cases}$$

For definiteness, we say that $y(t)$ is of type B_j on $[T_0,\infty)$ if the $y_k(t)$ have the appropriate sign properties for $t \geqq T_0$. It has been shown in Terry (1974) that a positive solution of H_+ is necessarily of type B_j for some $j = 0,\cdots,n-1$. Moreover, under the assumption that $0 \leqq \tau(t) \leqq T < \infty$, the following lemmas were proved:

Lemma 1.1. *Let $y(t)$ be a B_j-solution of H_+ on $[T_0,\infty)$. Then there exist positive constants $N_{k,k-1}$ such that*

$$(t - T_1) y_k(t) \leqq N_{k,k-1} y_{k-1}(t), k = 1,\cdots,2j+1$$

for $t \geqq T_1 = T_0 + T$.

Lemma 1.2. *Let $y(t)$ be a B_j-solution of H_+. Then there exist constants $K_i > 0$ and $t_i > 0$ such that*

$$\frac{y_{i\tau}(t)}{y_i(t)} \geqq K_i, i = 0,\cdots,2j$$

for $t \geqq t_i$.

In an analogous manner we may define a solution of H_- to be of type $\mathscr{B}_j (0 \leqq j \leqq n-1)$ if for t sufficiently large, $y_k(t) > 0$, $k = 0,\cdots,2j$ and $(-1)^k y_k(t) > 0$, $k = 2j+1,\cdots,2n-1$. A solution is of type $\mathscr{B}_n$ if $y_k(t) > 0,\cdots,2n-1$ for large t. We observe that when $n = 2$, $\tau(t) \equiv 0$ and $r(t) \equiv 1$, the solutions of type $\mathscr{B}_0$ reduce to those investigated in Wong (1969), where they were referred to as proper solutions of type I. The solutions of type $\mathcal{B}_1$ and $\mathcal{B}_2$ were collectively referred to as proper solutions of type II. It is easily seen that a positive solution of H_- is necessarily of type $\mathcal{B}_j$ for some $j = 0, \cdots, n$ and that the following analogues of Lemmas 1.1 and 1.2 can be derived when $1 \leqq j \leqq n-1$.

LEMMA 1.1.′ *Let $y(t)$ be a $\mathcal{B}_j$-solution of H_- on $[T_0, \infty)$. Then there exist positive constants $\eta_{k,k-1}$ such that*

$$(t - T_1) y_k(t) \leqq \eta_{k,k-1} y_{k-1}(t), \quad k = 1, \cdots, 2j$$

for $t \geqq T_1 = T_0 + T$.

LEMMA 1.2.′ *Let $y(t)$ be a $\mathcal{B}_j$-solution of H_-. Then there exists constants $\kappa_i > 0$ and $t_i > 0$ such that*

$$\frac{y_{i\tau}(t)}{y_i(t)} \geqq \kappa_i, \quad i = 0, \cdots, 2j-1$$

for $t \geqq t_i$.

The proof of Lemma 1.1′ depends only on the technique of integration by parts and the definition of a $\mathcal{B}_j$-solution and is an imitation of the proof of Lemma 1.1. Moreover, the proof of Lemma 1.2′ follows from Lemma 1.1′ in the same manner that Lemma 1.2 followed from Lemma 1.1 in Terry (1974).

In Section two the basic lemmas given here are extended to the case where $\tau(t)$ is unbounded in a prescribed manner. The resulting lemmas are then used to obtain integral criteria for the nonexistence of solutions of H_+ of type B_j as well as for the oscillation of all solutions of H_+. Section three provides sufficient conditions for the nonexistence of $\mathcal{B}_j$-solutions of H_-. In Sections four and five we let $R(t)$ be a solution of the nonhomogeneous differential equation

$$D^n[r(t) D^n y(t)] = Q(t);$$

for each choice of $R(t)$, the positive solutions of N_+ or N_- may be classified according to $2n+1$ types. The methods of the previous sections are then employed to exclude some of these solutions. If, for a specific choice of $R(t)$, conditions can be given to exclude all $2n+1$ types, we can conclude that N_+ (or N_-) has no positive solutions. We note that even when this is not possible the exclusion of some of the solutions will necessarily give information concerning the asymptotic behavior of the remaining types. With each choice of $R(t)$ additional information is obtained concerning the behavior of the functions $y_k(t)$ for large t. The results are valid for ordinary differential equations as well and are applied to show the nonexistence of bounded positive solutions of certain equations or to show that bounded positive solutions tend to zero ultimately.

The homogeneous equation H_+ has been studied by Terry and Wong (1972) when $n = 2$ and by Terry (1974) for arbitrary n. A slightly different form of H_+ has been considered by Ladas (1971b) when $r(t) \equiv 1$. An initial investigation of the oscillation and separation properties of the nonhomogeneous equation is due to Burton (1952) for the case $\tau(t) \equiv 0$, $n = 1$ and $f(t, u) = p(t)u(t)$. More recent studies include those by Howard (to appear) for the second order differential equation and by Kartsatos (1971) and Kartsatos (1972) in the n-th order case when the forcing term is assumed to be small or periodic.

Since submitting the first draft of this paper, additional papers on nonhomogeneous delay differential equations have been authored by Kusano (1973), Kusano and Onose (1974), Singh (to appear), Singh and Dahiya (to appear) and others.

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In this section we extend the basic lemmas of section one to the case where $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} [t - \tau(t)] = \infty$. Specifically, $\tau(t)$ is assumed to satisfy either

(T1) $$0 \leqq \tau(t) \leqq \mu t, 0 \leqq \mu < 1 \text{ or}$$

(T2) $$0 \leqq \tau(t) \leqq \mu t^{\beta}, 0 \leqq \mu < \infty \text{ and } 0 \leqq \beta < 1.$$

LEMMA. 2.1. *Suppose* $\tau(t)$ *satisfies* (T1) *or* (T2) *and that* $y(t)$ *is a* B_j*-solution of* H_+ *on* $[T_0, \infty)$. *Then there exist constants* $N_{k,k-1} > 0$ *such that* $(t - T_1)y_k(t) \leqq N_{k,k-1}y_{k-1}(t)$ *for* $t \geqq T_1$ *and* $ty_k(t) \leqq 2N_{k,k-1}y_{k-1}(t)$ *for* $t \leqq 2T_1$, *where*

$$T_1 \geqq \min\{t \geqq T_0 \colon t - \tau(t) \geqq T_0 \text{ for } t \geqq T_1\}.$$

PROOF. Suppose that $y(t)$ is a solution of H_+ of type B_j on $[T_o, \infty)$. If $\tau(t)$ satisfies (T1), then $t - \tau(t) \geqq (1 - \mu)t$. On the other hand, if $\tau(t)$ satisfies (T2), then

$$t - \tau(t) \geqq t - \mu t^{\beta} = t^{\beta}[t^{1-\beta} - \mu].$$

Let ε be chosen so that $0 < \varepsilon < 1$. Since

$$\lim_{t\to\infty} \frac{t^{1-\beta} - \mu}{t^{1-\beta}} = 1,$$

$t^{1-\beta} - \mu \geqq (1 - \varepsilon)t^{1-\beta}$ for t sufficiently large. Hence, there is a $T_* \geqq T_0$ such that

$$t - \tau(t) \geqq t^{\beta}[(1 - \varepsilon)t^{1-\beta}] = (1 - \varepsilon)t$$

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for $t \geqq T_*$. It follows that there is a $T_1 \geqq T_*$ such that $t - \tau(t) \geqq T_0$ for $t \geqq T_1$. For the assumption (T2), $T_1 = \max(T_*, (1-\varepsilon)^{-1}T_0)$. For (T1), $T_1 = (1-\mu)^{-1}T_0$. The proof given in Terry (1974) is now valid verbatim with the estimates holding for $t \geqq T_1$. As before, we obtain the intermediate inequalities

$$(t - T_1)y_k(t) \leqq N_{k,k-1}y_{k-1}(t), \; k = 1, \cdots, 2j+1$$

for $t \geqq T_1$.

We note that the estimates on the constants $N_{k,k-1}$ are the same as given in Terry (1974) since the estimates are obtained as part of the proof.

Lemma 2.2. *Let $y(t)$ be a B_j-solution of H_+. Let i be an integer* $(0 \leqq i \leqq 2j)$ *and $\tau(t)$ satisfy* (T2) *or* (T1), *where $\mu < [m^{-1}N_{n,n-1} + 1]^{-1}$, if n is odd, $j \geqq (n-1)/2$ and $i = n-1$; $\mu < [N_{i+1,i} + 1)^{-1})^{-1}$, otherwise. Then there exist positive constants K_i and t_i such that*

$$y_{i_\tau}(t) \geqq K_i y_i(t), t \geqq t_i \qquad (i = 0, \cdots, 2j).$$

Proof. Let $y(t)$ be a B_j-solution of H_+ on $[T_0, \infty)$. Suppose $i \neq n-1$ if n is odd. Then $y(t)$, $y_\tau(t)$, $y_k(t)(k = 1, \cdots, 2j+1)$ are all positive for $t \geqq T_1$ (chosen as in Lemma 2.1). Since $\tau(t) \geqq 0$, $y_{i_\tau}(t) \equiv y_i[t - \tau(t)] \leqq y_i(t)$ for $i = 0, \cdots, 2j$. Moreover, $s - T_1 \geqq t - \tau(t) - T_1$ for any s in the interval $J_\tau(t) = [t - \tau(t), t]$. An application of Lemma 2.1 and a mean-value theorem shows that there is an $s \in J_\tau(t)$ such that

$$\begin{aligned}\left| \frac{y_{i\tau}(t)}{y_i(t)} - 1 \right| &= \frac{|y_{i\tau}(t) - y_i(t)|}{|y_i(t)|} = \frac{y_i(t) - y_{i\tau}(t)}{y_i(t)} \\ &= \tau(t)\frac{y_i'(s)}{y_i(t)} \leqq \frac{\tau(t)}{s - T_1} N_{i+1,i}\frac{y_i(s)}{y_i(t)} \\ &\leqq N_{i+1,i}\frac{\tau(t)}{t - \tau(t) - T_1}.\end{aligned}$$

We first consider the condition (T1). Suppose that $0 \leqq \tau(t) \leqq \mu t$, $0 \leqq \mu < [N_{i+1,i} + 1]^{-1}$. Then we define δ by

$$\mu = [N_{i+1,i} + 1]^{-1} - \delta,$$

where $0 < \delta < [N_{i+1,i} + 1]^{-1} < 1$. Let $\alpha > 1$ and define $\varepsilon > 0$ by

$$\delta = \frac{\alpha\varepsilon N_{i+1,1}}{(N_{i+1,i} + 1)(N_{i+1,i} + 1 - \alpha\varepsilon)}.$$

It follows that

$$0 < \alpha\varepsilon = \frac{(N_{i+1,i} + 1)^2\delta}{N_{i+1,i} + (N_{i+1,i} + 1)\delta}.$$

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This expression is of the form $b^2\delta(a+b\delta)^{-1}$, where $a = N_{i+1,i}$ and $b = a+1$. Since $b\delta < 1$, $ab\delta < a$. Also, since $b = a+1$,

$$b^2\delta = b(a+1)\delta = b\delta + ab\delta < b\delta + a$$

and $\alpha\varepsilon = b^2\delta(a+b\delta)^{-1} < 1$. Since $\alpha > 1$, $\varepsilon < \alpha^{-1} < 1$.

For this choice of ε and for t sufficiently large ($t \geqq t_i$)

$$N_{i+1,i}\frac{\tau(t)}{t-\tau(t)-T_1} \leqq N_{i+1,i}\frac{\mu t}{(1-\mu)t - T_1} = \frac{N_{i+1,i}\,\mu t}{(1-\mu)[t - T_1/(1-\mu)]}$$
$$\leqq N_{i+1,i}\frac{1}{1-\varepsilon}\frac{\mu}{1-\mu}\ .$$

Then

$$\mu = \frac{1}{N_{i+1,i}+1} - \delta = \frac{1-\alpha\varepsilon}{N_{i+1,i}+(1-\alpha\varepsilon)} \text{ and } \frac{y_i(s)}{s-T_1} = \frac{1-\alpha\varepsilon}{N_{i+1,i}}\ .$$

It follows that

$$N_{i+1,i}\frac{\tau(t)}{t-\tau(t)-T_1} \leqq \frac{1-\alpha\varepsilon}{1-\varepsilon} < 1, \quad t \geqq t_i$$

and we may take $K_i = 1-(1-\alpha\varepsilon)/(1-\varepsilon)$.

If n is odd, $j \geqq (n-1)/2$ and $i = n-1$,

$$y_i'(s) = Dy_{n-1}(s) = \frac{y_n(s)}{r(s)} < \frac{y_n(s)}{m} < \frac{y_i(s)}{s-T_1}\frac{N_{n,n-1}}{m}$$

we may repeat the procedure above replacing $N_{i+1,i}$ by $m^{-1}N_{n,n-1}$ and thus obtain the proof of the second assertion.

Now suppose $\tau(t)$ satisfies (T2). Let ε_1 and ε_2 be chosen so that $0 < \varepsilon_i < 1$, $i = 1, 2$. Since $\lim_{t\to\infty}[t-\tau(t)] = \infty$, we have $t - \tau(t) - T_1 \geqq (1-\varepsilon_1)[t-\tau(t)]$ for t sufficiently large. Since $0 \leqq \beta < 1$, $\lim_{t\to\infty} t^{1-\beta} = \infty$ so that $t^{1-\beta} - \mu \geqq (1-\varepsilon_2)\,t^{1-\beta}$ for t sufficiently large. Thus

$$t - \tau(t) \geqq t - \mu t^{\beta} = t^{\beta}[t^{1-\beta} - \mu] \geqq (1-\varepsilon_2)t.$$

Hence,

$$\frac{\tau(t)}{t-\tau(t)-T_1} \leqq \frac{\tau(t)}{(1-\varepsilon_1)[t-\tau(t)]} \leqq \frac{\mu t^{\beta}}{(1-\varepsilon_1)(1-\varepsilon_2)t} = \frac{\mu}{(1-\varepsilon_1)(1-\varepsilon_2)t^{1-\beta}}$$
$$< 1-\varepsilon,$$

if $t > [\mu(1-\varepsilon)^{-1}(1-\varepsilon_1)^{-1}(1-\varepsilon_2)^{-1}]^{1/(1-\beta)}$.

We now state an extended version of Theorem 2.5 of Terry (1974), which was previously proved for $\tau(t) \geqq 0$ and bounded.

THEOREM 2.3. *Let k be an integer* ($k = 0, \cdots, n-1$). *Let $\tau(t)$ satisfy* (T2) *or* (T1) *with*

$$\mu < [m^{-1}N_{n,n-1} + 1]^{-1}, \text{ if } n \text{ is odd and } k = \frac{(n-1)}{2}$$

or

$$\mu < [N_{2k+1,2k} + 1]^{-1}, \text{ otherwise.}$$

Suppose that for all constants $C > 0$

$$\int^{\infty} t^{2k} f(t, Ct^{2k})dt = +\infty.$$

Then H_+ has no solutions of type B_r ($r = k, \cdots, n-1$).

The proof is accomplished by using the intermediate Riccati transformation $z(t) = u_{2n-1}(t)u_{2k}^{-1}(t)$ and is the same as in Terry (1974) except for the use of Lemmas 2.1 and 2.2 instead of Lemmas 1.1 and 1.2. The crucial step of the proof is the consideration of the term $u_{2k,\tau}(t)u_{2k}^{-1}(t)$. In the event that $\tau(t)$ satisfies (T1) with $0 \leqq \mu \leqq [N_{2k+1,2k} + 1]^{-1}$, we may conclude by Lemma 2.2 that this term is bounded away from zero.

In attempting to eliminate B_j-solutions, where $j \geqq k+1$, we are led to consider the term $u_{2j,\tau}(t)u_{2j}^{-1}(t)$. We note, however, that if $y(t)$ is a B_k-solution of H_+ and $k \neq (n-1)/2$, then we may take $N_{2k+1,2k} = 1$: for in this case $y_{2k+1}(t)$ is a positive decreasing function of t for $t \geqq T_1$ and an integration from T_1 to t shows that

$$y_{2k}(t) - y_{2k}(T_1) = \int_{T_1}^{t} y_{2k+1}(s)ds \geqq (t - T_1)y_{2k+1}(t).$$

Since $y_{2k}(T_1) > 0$, $(t - T_1)y_{2k+1}(t) \leqq y_{2k}(t)$.

Similarly, if $y(t)$ is a B_j-solution of H_+, where $j \geqq k+1$, we may take $N_{2j+1,2j} = 1$. If $k = (n-1)/2$, then we may take $N_{2k+1,2k} = N_{n,n-1} = M$ since

$$\begin{aligned}(t - T_1)y_n(t) &= (t - T_1)y_{2k+1}(t) \leqq \int_{T_1}^{t} y_{2k+1}(s)ds = \int_{T_1}^{t} y_n(s)ds \\ &= \int_{T_1}^{t} r(s)Dy_{n-1}(s)ds \leqq M\int_{T_1}^{t} Dy_{n-1}(s)ds \\ &= M[y_{n-1}(t) - y_{n-1}(T_1)] < My_{n-1}(t).\end{aligned}$$

The condition $0 \leqq \mu < [m^{-1}N_{2k+1,2k}+1)^{-1}$ becomes $0 \leqq \mu < m/(m+M)$. If $j \geqq k+1$, the required condition is $0 \leqq \mu < 1/2$, which is already existent since $m/(m+M) < 1/2$. Thus, we may replace the condition of Theorem 2.3 by the slightly stronger condition

$$\text{(T3)} \qquad \begin{gathered} 0 \leqq \tau(t) \leqq \mu t, \text{ where} \\ 0 \leqq \mu < m(m+M)^{-1}, \text{ if } n \text{ is odd;} \\ 0 \leqq \mu < 1/2, \text{ otherwise.} \end{gathered}$$

We will assume in the sequel that $\tau(t)$ satisfies either (T2) or (T3).

THEOREM 2.4. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *Suppose that for some $k = 0, \cdots, n-1$ and for all positive constants C*

$$\text{(2.1)} \qquad \int^{\infty} t^{2n-1} f(t, Ct^{2k})dt = \infty.$$

Then H_+ has no positive B_k-solutions $y(t)$ such that $y_{2k}(t)$ is bounded.

PROOF. Suppose that $y(t)$ is a positive B_k-solution of H_+ for $t \geqq T_0$. Then for $t \geqq T_1$, $y_\tau(t)$ and $y_i(t)$ are positive $(i = 0, \cdots, 2k+1)$ and $(-1)^{i+1}y_i(t) > 0\ (i = 2k+2, \cdots, 2n-1)$. We note that the hypotheses on $\tau(t)$ are not used explicitly below. They are necessary only for the application of Lemma 2.1. Multiplying both sides of H_+ by $t^{2n-2k-1}$ and integrating from T_1 to t yields

$$\text{(2.2)} \quad \int_{T_1}^{t} s^{2n-2k-1} D^n[r(s)D^n y(s)]ds + \int_{T_1}^{t} s^{2n-2k-1} y_\tau(s) f[s, y_\tau(s)]ds = 0.$$

Since $y_{2k+1}(s) > 0$, there is a constant $C > 0$ for which $y_\tau(s) \geqq Cs^{2k}$. Moreover, if $k \geqq n/2$,

$$\int_{T_1}^{t} s^{2n-2k-1} D^n[r(s)D^n y(s)]ds = [P_1(s)]_{T_1}^{t} - [(2n-2k-1)!\, y_{2k}(s)]_{T_1}^{t},$$

where

$$\text{(2.3)} \quad \begin{aligned} P_1(s) &= s^{2n-2k-1} y_{2n-1}(s) \\ &+ \sum_{j=2}^{2n-2k-1} (-1)^{j+1}(2n-2k-1)\cdots(2n-2k+1-j)s^{2n-2k-j} y_{2n-j}(s). \end{aligned}$$

If $k < n/2$, we have

$$\int_{T_1}^{t} s^{2n-2k-1} D^n[r(s)D^n y(s)]ds \geqq [P_2(s)]_{T_1}^{t} - [M(2n-2k-1)!\, y_{2k}(s)]_{T_1}^{t},$$

where

$$(2.4)\quad \begin{aligned} P_2(s) &= s^{2n-2k-1}y_{2n-1}(s) \\ &+\sum_{j=2}^{n}(-1)^{j+1}(2n-2k-1)\cdots(2n-2k+1-j)s^{2n-2k-j}y_{2n-j}(s) \\ &-M\sum_{j=n+1}^{2n-2k-1}(-1)^{j}(2n-2k-1)\cdots(2n-2k+1-j)s^{2n-2k-j}y_{2n-j}(s). \end{aligned}$$

We consider the products $(-1)^{j+1}y_{2n-j}(t)$, $j=2,\cdots,2n-2k-1$. Letting $l=2n-j$, $(-1)^{j+1}=(-1)^{2n-l+1}=(-1)^{-l+1}=(-1)^{l+1}$.
Thus

$$(-1)^{j+1}y_{2n-j}(t)=(-1)^{l+1}y_l(t)>0,\, l=2k+1,\cdots,2n-2$$

since $y(t)$ is a B_k-solution of H_+. So each term of the sum(s) in (2.3) or (2.4) is positive. Substituting the estimates above in (2.2), we have

$$C\int_{T_1}^{t} s^{2n-1}f(s,Cs^{2k})ds \leqq \begin{cases} P_1(T_1)+(2n-2k-1)!\,y_{2k}(t) \\ P_2(T_1)+M(2n-2k-1)!\,y_{2k}(t) \end{cases}$$

which is in contradiction to (2.1) for large t if $y_{2k}(t)$ is bounded.

Letting $k=0$, we obtain a familiar criterion for the nonexistence of bounded positive solutions of H_+, and hence for bounded negative solution of H_+ by condition (iii). We state this as a corollary.

COROLLARY 2.5. *Let $\tau(t)$ satisfy* (T2) *or* (T3). *Suppose that for all $C>0$*

$$\int^{\infty} t^{2n-1}f(t,C)dt = +\infty.$$

Then all bounded solutions of H_+ are oscillatory.

This corresponds to Theorem 4.1 of Ladas (1971a) in the case of the simpler equation

$$D^n y(t)+p(t)f[y(t),y(g(t))]=0,$$

where $p(t)$ is a positive continuous function on $[0,\infty)$, $f\in C[R\times R,R]$, $g(t)\in C[[0,\infty),R]$, $g(t)\leqq t$ for $t\geqq 0$ and $\lim_{t\to\infty} g(t)=+\infty$.

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Criteria for the exclusion of B_j-solutions of H_- are formulated in this section.

THEOREM 3.1. *Let $\tau(t)$ satisfy* (T2) *or* (T3). *Suppose that for some $k = 1, \cdots, n-1$ and for all $C > 0$*

$$\int^{\infty} t^{2n-2} f(t, Ct^{2k-1})dt = +\infty. \tag{3.1}$$

Then H_- has no solutions of type $\mathscr{B}_k$. If (2.1) *holds with $k = 0$ and $y(t)$ is a solution of H_- of type $\mathscr{B}_0$, then $y(t)$ tends to zero as $t \to \infty$.*

PROOF. Let $y(t)$ be a positive solution of H_- of type $\mathscr{B}_k$. Then there is a $T_0 > 0$ such that $y_j(t) > 0$ for $j = 0, \cdots, 2k$ and $(-1)^j y_j(t) > 0$ for $j = 2k+1, \cdots, 2n-1$ provided $t \geqq T_0$. Let $t \geqq T_1$ and $i \leqq n$. Integrating H_- i times over (t, b) results in

$$\begin{aligned} y_{2n-i}(t) &= \sum_{j=0}^{i-1} \frac{y_{2n-i+j}(b)}{j!}(t-b)^j \\ &+ \frac{1}{(i-1)!}\int_b^t (t-s)^{i-1} y_\tau(s) f[s, y_\tau(s)]ds \end{aligned} \tag{3.2}$$

i.e.,

$$\begin{aligned} y_{2n-i}(t) &= \sum_{j=0}^{i-1} \frac{(-1)^j y_{2n-i+j}(b)}{j!}(b-t)^j \\ &+ \frac{(-1)^i}{(i-1)!}\int_t^b (s-t)^{i-1} y_\tau(s) f[s, y_\tau(s)]ds. \end{aligned} \tag{3.3}$$

Since (3.2) is valid for $i = n$ and $y_n(t) \leqq M_0 D y_{n-1}(t)$, where $M_0 = M$ if $y_n(t) > 0$ and $M_0 = m$ if $y_n(t) < 0$, it follows that

$$\begin{aligned} Dy_{n-1}(t) \geqq \frac{1}{M_0}\sum_{j=0}^{n-1} y_{n+j}(b)\frac{(t-b)^j}{j!} \\ + \frac{1}{M_0(n-1)!}\int_b^t (t-s)^{n-1} y_\tau(s) f[s, y_\tau(s)]ds. \end{aligned}$$

In the case $i > n$, an additional $i - n$ integrations of this will result in the analogous inequality

$$\begin{aligned} y_{2n-i}(t) &\geqq \sum_{j=0}^{i-n-1} \frac{(-1)^j y_{2n-i+j}(b)}{j!}(b-t)^j \\ &+ \frac{1}{M_0}\sum_{j=i-n}^{i-1} \frac{(-1)^j y_{2n-i+j}(b)}{j!}(b-t)^j \\ &+ \frac{(-1)^i}{M_0(i-1)!}\int_t^b (s-t)^{i-1} y_\tau(s) f[s, y_\tau(s)]ds. \end{aligned} \tag{3.4}$$

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If $k \geqq n/2$, $2(n-k) \leqq n$ and we may let $i = 2(n-k)$ in (3.3). Similarly, if $k < n/2$, we may let $i = 2(n-k)$ in (3.4). In either case, $2n - i + j = 2k + j$. Each term of each sum is positive since $(-1)^j y_{2k+j}(b) > 0$ for $j = 0, \cdots, 2n-2k-1$. Moreover, $y_{2k+1}(t) < 0$ for $t \geqq T_1$, so that $y_{2k}(T_1) \geqq y_{2k}(t)$. It follows that

$$y_{2k}(T_1) \geqq \omega_k^{-1} \int_t^b (s-t)^{2n-2k-1} y_\tau(s) f[s, y_\tau(s)] ds,$$

where

$$\omega_k = \begin{cases} (2n-2k-1)! & \text{if } k \geqq n/2 \\ M_0(2n-2k-1)! & \text{if } k < n/2. \end{cases}$$

If $k = 1, \cdots, n-1$, then $\lim_{t\to\infty} y_{2k-1}(t) = \infty$; so there is a $C > 0$ such that $y(t) > Ct^{2k-1}$ for $t \geqq T_1$. By Lemma 1.2′ there is a $k_0 > 0$ and a $T_2 \geqq T_1$ such that

$$y_\tau(t) \geqq k_0 y(t) \geqq k_0 C t^{2k-1}, \; t \geqq T_2.$$

Thus, for $k = 1, \cdots, n-1$ and $s \geqq T_2$

$$(s-t)^{2n-2k-1} y_\tau(s) \geqq (s-t)^{2n-2k-1} k_0 C s^{2k-1}.$$

By (ii) $f[s, y_\tau(s)] \geqq f(s, k_0 C s^{2k-1})$ for $s \geqq T_2$. Furthermore, $s - t \geqq s/2$ for $s \geqq 2t$. Now let $t \geqq T_2$ and $b > 2t = T_*$. It follows that

$$y_{2k}(T_1) \geqq k_0 C \omega_k^{-1} \int_t^b (s-t)^{2n-2k-1} s^{2k-1} f(s, k_0 C s^{2k-1}) ds$$

$$\leqq k_0 C \omega_k^{-1} 2^{-2n+2k+1} \int_{T_*}^b s^{2n-2} f(s, k_0 C s^{2k-1}) ds$$

and

$$\int_{T_*}^b s^{2n-2} f(s, k_0 C s^{2k-1}) ds \leqq 2^{2n-2k-1} \omega_k (k_0 C)^{-1} y_{2k}(T_1),$$

which is incompatible with (3.1).

If $k = 0$ and $\lim_{t\to\infty} y(t) = C > 0$, then $y_\tau(t) \geqq y(t) \geqq C$ for $t \geqq T_1$ since $y'(t) < 0$ if $y(t)$ is of type $\mathscr{B}_0$. By (ii) $f[t, y_\tau(t)] \geqq f(t, C)$ for $t \geqq T_1$. For $t \geqq T_1$ and $b > 2t = T_*$ it follows as before that

$$\int_{T_*}^b s^{2n-1} f(s, C) ds \leqq 2^{2n-1} \omega_k C^{-1} y_{2k}(T_1),$$

which is again a contradiction.

In the next corollary, as in the preceding theorem, we assume tacitly that $n \geqq 2$.

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COROLLARY 3.2. *Let $\tau(t)$ satisfy* (T2) *or* (T3). *Suppose that for some $k = 1, \cdots, n-1$ and for all $C > 0$*

$$\int^{\infty} t^{2n-1} f(t, Ct^{2k})dt = +\infty.$$

Then H_- has no $\mathcal{B}_k$-solutions $y(t)$ such that $\lim_{t\to\infty} y_{2k}(t) = \gamma > 0$.

THEOREM 3.3. *Let $\tau(t)$ satisfy* (T2) *or* (T3). *Suppose that for all $B > 0$ and $m = 2n - i$, $i = 1, 2, 3$*

$$\lim_{b\to\infty} t^{-m} \int_t^b s^{2n-1+m} f(s, Bs^m)ds = \infty. \tag{3.5}$$

Then H_- has no solutions of type $\mathcal{B}_{n-1}$ or $\mathcal{B}_n$ which are asymptotic to Ct^m, where $C > 0$.

PROOF. Let $y(t)$ be a $\mathcal{B}_n$-solution of H_-. Then for $T_1 \leqq t \leqq s \leqq b$

$$y(t) \geqq \sum_{k=0}^{n-1} \frac{(-1)^k y_k(b)}{k!}(b-t)^k + \frac{1}{M_0}\sum_{k=n}^{2n-1}\frac{(-1)^k y_k(b)}{k!}(b-t)^k$$
$$+ \frac{1}{M_0(2n-1)!}\int_t^b (s-t)^{2n-1} y_\tau(s) f[s, y_\tau(s)]ds,$$

which is (3.4) with $i = 2n$. Since $(-1)^k y_k(b) \geqq 0$ for k even,

$$t^{-m}y(t) \geqq t^{-m}\sum_{j=0}^{p}\frac{y_{2j+1}(b)}{(2j+1)!}(t-b)^{2j+1}$$
$$+\frac{1}{M_0}t^{-m}\sum_{j=p}^{n-1}\frac{y_{2j+1}(b)}{(2j+1)!}(t-b)^{2j+1}$$
$$+\frac{1}{M_0(2n-1)!t^m}\int_t^b (s-t)^{2n-1}y_\tau(s)f[s, y_\tau(s)]ds$$

where $p = [(n-2)/2]$. Suppose that $y(t) \sim Ct^m$, $C > 0$. Then there is a $T_2 \geqq T_1$ such that

$$1/2 \leqq y(t)/Ct^m \leqq 3/2, \ t \geqq T_2.$$

Using a familiar Tauberian theorem (See Lemma 2.2 of Wong (1969), for example.), there is a $\lambda > 0$ and a $T^* > 0$ such that

$$\left| t^{-m}\sum_{j=0}^{p}\frac{y_{2j+1}(b)}{(2j+1)!}(t-b)^{2j+1} + \frac{1}{M_0}t^{-m}\sum_{j=p}^{n-1}\frac{y_{2j+1}(b)}{(2j+1)!}(t-b)^{2j+1}\right| \leqq \lambda$$

for $b \geqq T^*$. We see that

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$$\frac{3}{2}Ct^m \geqq y(t) \geqq y_\tau(t) \geqq \frac{C}{2}[t-\tau(t)]^m,\ t \geqq T_3,$$

where $T_3 = \min\{t > 0: t - \tau(t) \geqq T_2, t \geqq T_3\}$. If $\tau(t)$ satisfies (T3), then $t - \tau(t) \geqq (1-\mu)t$, on the other hand, if $\tau(t)$ satisfies (T2), there is a $T_4 \geqq T_3$ such that $t - \tau(t) \geqq t/2$ for $t \geqq T_4$ and

$$y_\tau(t) \geqq \nu\frac{C}{2}t^m,\ t \geqq T_4,$$

where

$$\nu = \begin{cases} (1-\mu)^m & \text{if } \tau(t) \text{ satisfies (T3)} \\ 2^{-m} & \text{if } \tau(t) \text{ satisfies (T2)} \end{cases}$$

Let $t \geqq T_4$, $T_* = 2t$ and $b > \max(T_*, T^*)$. Then

$$\frac{3}{2}C \geqq t^{-m}y(t) \geqq -\lambda \times \omega_0^{-1}t^{-m}\int_t^b (s-t)^{2n-1}y_\tau(s)f[s, y_\tau(s)]ds$$

$$\omega_0\left(\frac{3}{2}C + \lambda\right) \geqq \nu\frac{C}{2}t^{-m}\int_t^b (s-t)^{2n-1}f(s, \nu Cs^m/2)ds$$

and

$$t^{-m}\int_{T_*}^b s^{2n-1+m}f(s, C_1s^m)ds \leqq 2^{2n-1}\omega_0C_1^{-1}\left(\frac{3}{2}C + \lambda\right),$$

where $C_1 = \nu C/2$. Letting $b \to \infty$ we obtain a contradiction to (3.5).

When $\tau(t) \equiv 0$, $n = 2$ and $k = 0$, Theorem 3.1 reduces to the necessity of Theorem 2.1 of Wong (1969). When $\tau(t) \equiv 0$ and $n = 2$, Theorem 3.3 reduces to the necessity of Theorem 2.3 of Wong (1969).

4

In this section we consider the nonhomogeneous delay equation N_+ under the assumption that $R(t)$ is a solution of the ordinary differential equation

$$D^n[r(t)D^nR(t)] = Q(t). \tag{4.1}$$

This permits the transformation of N_+ to a homogeneous delay differential equation of order $2n$ for which the methods of the previous selections may be applied. Since the resulting delay equation does not have exactly the same form as H_+ or H_- the arguments have to be duplicated but are entirely analogous. Let us assume that $y(t)$ is a positive solution of N_+ and let $u(t) = y(t) - R(t)$. Then

$$D^n[r(t)D^n u(t)] = D^n[r(t)D^n y(t)] - D^n[r(t)D^n R(t)]$$
$$= -y_\tau(t)f[t, y_\tau(t)]$$
$$= -(u+R)_\tau(t)f[t,(u+R)_\tau(t)],$$

so that $u(t)$ is a solution of the homogeneous equation

$$(H_+^+) \qquad D^n[r(t)D^n u(t)] + (u+R)_\tau(t)f[t,(u+R)_\tau(t)] = 0.$$

Since $y(t) > 0$, $t \geqq T_0$, $(u+R)_\tau(t) > 0$ for $t \geqq T_1$ and $D^n[r(t)D^n u(t)] < 0$ for $t \geqq T_1$ so that $u(t)$ is a nonoscillatory solution of H_+^+. Either $u(t) > 0$ (i.e., $y(t) > R(t)$) or $u(t) < 0$ $(y(t) < R(t))$ for sufficiently large t. If $u(t)$ is a positive solution of H_+^+ of type B_j $(j = 0, \cdots, n-1)$, we will say that $y(t)$ is a solution of N_+ of type B_j^R $(j = 0, \cdots, n-1)$. If $u(t) < 0$, then we further transform the equation by letting $v(t) = -u(t)$. It follows that $v(t)$ is a positive solution of

$$(H_+^-) \qquad D^n[r(t)D^n v(t)] - (R-v)_\tau(t)f[t,(R-v)_\tau(t)] = 0.$$

If $v(t)$ is of type $\mathscr{B}_j$ $(j = 0, \cdots, n)$, we will say that $y(t)$ is a solution of N_+ of type $\mathscr{B}_j^R$. A solution of N_+ is then either oscillatory, negative nonoscillatory, of type B_j^R $(j = 0, \cdots, n-1)$ or of type $\mathscr{B}_j^R$ $(j = 0, \cdots, n)$. We now seek to exclude solutions of N_+ of types B_j^R and $\mathscr{B}_j^R$.

THEOREM 4.1. *Let $\tau(t)$ satisfy* (T2) *or* (T3). *Let $R(t)$ be a bounded solution of* (4.1) *and suppose that for some $k = 1, \cdots, n-1$ and for all $C > 0$*

$$\int^\infty t^{2k} f(t, R_\tau(t) + Ct^{2k})dt = \infty. \tag{4.2}$$

Then N_+ has no B_j^R solutions for $j = k, \cdots, n-1$.

PROOF. Let $y(t)$ be a positive solution of N_+ of type B_k^R. Then $u(t) = y(t) - R(t)$ is a B_k-solution of equation H_+^+. Let $z(t) = u_{2n-1}(t)/u_{2k}(t)$; $z(t)$ is positive for all t sufficiently large, i.e., $t \leqq T_1$. It follows upon differentiating $z(t)$ that

$$z'(t) + \frac{u_{2n-1}(t)Du_{2k}(t)}{[Du_{2k}(t)]^2} - \frac{Du_{2n-1}(t)}{u_{2k}(t)} = 0.$$

Since $u(t)$ is of type B_k, $u_{2n-1}(t)$ and $Du_{2k}(t)$ are both positive. Moreover, $Du_{2n-1}(t) = D^n[r(t)D^n u(t)]$. Thus

$$z'(t) + \frac{(u+R)_\tau(t)}{u_{2k}(t)} f[t,(u+R)_\tau(t)] \leqq 0, \; t \geqq T_1. \tag{4.3}$$

We now consider the term

$$\frac{(u+R)_\tau(t)}{u_{2k}(t)} = \frac{(u+R)_\tau(t)}{u_\tau(t)} \frac{u_\tau(t)}{u_{2k}(t)} .$$

By Lemma 2.1, $t^{2k}u_{2k}(t) \leqq Nu(t)$, where

$$N = \prod_{i=0}^{2k-1} N_{2k-i,\,2k-i-1}.$$

Then, assuming $\tau(t)$ satisfies (T3)

$$(1-\mu)^{2k} t^{2k} u_{2k,\tau}(t) \leqq [t-\tau(t)]^{2k} u_{2k,\tau}(t) \leqq Nu_\tau(t).$$

If $\tau(t)$ satisfies (T2), then $\mu t^{\beta-1} < 1/2$ for t sufficiently large, so that

$$t-\tau(t) \geqq t - \mu t^{\beta} = t(1-\mu t^{\beta-1}) \geqq \frac{t}{2}$$

for $t \geqq T_2 \geqq T_1$. It follows that

$$u_\tau(t) \geqq \mu_k N^{-1} t^{2k} u_{2k,\tau}(t),$$

where

$$\mu_k = \begin{cases} 2^{-2k} & \text{if } \tau(t) \text{ satisfies (T2)} \\ (1-\mu)^{2k} & \text{if } \tau(t) \text{ satisfies (T3)} \end{cases} .$$

By Lemma 2.2, there is a $k_0 > 0$ such that $u_{2k,\tau}(t)\, u_{2k}^{-1}(t) \geqq k_0$. So for $t \geqq T_3 \geqq T_2$

$$\frac{u_\tau(t)}{u_{2k}(t)} \geqq \mu_k N^{-1} k_0 t^{2k}. \tag{4.4}$$

Next we consider the expression

$$\frac{(u+R)_\tau(t)}{u_\tau(t)} = \left[\frac{u+R}{u}\right]_\tau (t) = \left(1+\frac{R}{u}\right)_\tau (t). \tag{4.5}$$

If $R(t)$ is bounded and $y(t)$ is of type B_k^R, $(k = 1, \cdots, n-1)$, then $u(t)$ is unbounded, which implies that $Ru^{-1} \to 0$ as $t \to \infty$. So for $\varepsilon > 0$, $1 + Ru^{-1} \geqq 1 - \varepsilon = C_0$ for t sufficiently large. Thus there is a $T_* \geqq T_3$ such that

$$(u+R)_\tau(t)\, u_\tau^{-1}(t) \geqq C_0,\ t \geqq T_*. \tag{4.6}$$

Moreover, $u_{2k}(t)$ and $u_{2k+1}(t)$ are positive; so there is a $C_1 > 0$ such that $(u+R)_\tau(t) \geqq R_\tau(t) + C_1 t^{2k}$. Substitution of this estimate together with (4.4) and (4.5) via hypothesis (ii) in (4.3) yields

$$z'(t) + C_0 \mu_k k_0 N^{-1} t^{2k} f(t, R_\tau(t) + C_1 t^{2k}) \leqq 0.$$

An integration of this from T_* to t results in

$$\int_{T_*}^{t} t^{2k} f(t, R_\tau(t) + C_1 t^{2k})dt \leqq (C_0\mu_k k_0)^{-1} N[z(T_*) - z(t)] \leqq (C_0\mu_k k_0)^{-1} N z(T_*),$$

which is in contradiction to (4.2).

By hypothesis (ii), the divergence of $t^{2k}f(t, R_\tau(t) + C_1 t^{2k})$ implies that of $t^{2(k+i)}f(t, R_\tau(t) + C_1 t^{2(k+i)})$ for $i = 0, \cdots, n-k-1$. So condition (4.2) is sufficient to exclude B_j^R-solutions ($j = k, \cdots, n-1$) of N_+. If $n = 1$, $j = 0$ is the only possibility and the theorem is vacuously true.

By considering (4.5) more carefully we are able to obtain the following corollaries to the proof of Theorem 4.1.

COROLLARY 4.2. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a solution of* (4.1). *Then N_+ has no B_k^R-solutions $y(t)$ such that $(y - R)_{2k}(t)$ is bounded if for all positive constants C*

$$\int^\infty (R_\tau(t) + Ct^{2k}) f(t, R_\tau(t) + Ct^{2k})dt = \infty. \tag{4.7}$$

COROLLARY 4.3. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a bounded solution of* (4.1). *Then N_+ has no bounded B_k^R-solutions if for all positive constants C*

$$\int^\infty (R_\tau(t) + C) f(t, R_\tau(t) + C)dt = \infty. \tag{4.7'}$$

COROLLARY 4.4. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a bounded solution of* (4.1). *Suppose that for all positive constants C*

$$\int^\infty f(t, R_\tau(t) + C)dt = \infty.$$

Then no bounded B_0^R-solution of N_+ is bounded way from zero as $t \to \infty$.

REMARK 1. In Theorem 4.1 the hypothesis that $R(t)$ is bounded may be replaced by $R(t) = 0(t^{2k-\varepsilon})$, for some $\varepsilon > 0$.

REMARK 2. The conclusion of Theorem 4.1 may be restated as: A positive solution $y(t)$ of N_+ either satisfies $0 < y(t) < R(t)$ for large t or is of type B_j^R, where $j = 0, \cdots, k-1$.

REMARK 3. If $R(t)$ is oscillatory or negative, then $u(t) > 0$ and the conclusion of Theorem 4.1 becomes: A positive solution of N_+ is of type B_j^R ($j = 0, \cdots, k-1$).

It may be easily seen that the integral condition (4.7') is sufficient to guarantee the nonexistence of all positive solutions of N_+ of types B_k^R ($k = 0, \cdots, n-1$). We suppose that $y(t)$ is of type B_k^R on (T_*, ∞). An integration of H_+^+ over (T_*, t) results in

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$$u_{2n-1}(t) - u_{2n-1}(T_*) + \int_{T_*}^{t} (u+R)_\tau(s) f[s, (u+R)_\tau(s)]ds = 0,$$

i.e.,

$$\int_{T_*}^{t} (u+R)_\tau(s) f[s, (u+R)_\tau(s)]ds \leqq u_{2n-1}(T_*).$$

Since $u'(t)$ is positive for $t \geqq T_*$, there is a constant $C > 0$ such that $u_\tau(t) \geqq C$ for $t \geqq T_*$ and

$$\int_{T_*}^{t} (R_\tau(s) + C) f(s, R_\tau(s) + C)ds \leqq u_{2n-1}(T_*).$$

The divergence of the integral above as $t \to \infty$ will result in a contradiction. We have thus proved the following theorem:

THEOREM 4.5. *Let $\tau(t)$ satisfy* (T2) *or* (T3). *Let $R(t)$ be a solution of* (4.1) *and suppose that for all $C > 0$*

$$\int^{\infty} (R_\tau(t) + C) f(t, R_\tau(t) + C)dt = \infty.$$

Then N_+ has no B_j^R-solutions $(j = 0, \cdots, n-1)$.

REMARK 4. If $R(t)$ is oscillatory or negative, the conclusion of Theorem 4.5 may be strengthened to: N_+ has no positive solutions. If, in addition, it is assumed that for all $C > 0$

$$\int^{\infty} (R_\tau(t) - C) f(t, R_\tau(t) - C)dt = -\infty,$$

then N_+ cannot have any negative solutions for large t, i.e., N_+ is oscillatory. The result is essentially part of a theorem due to Kartsatos and Manougian (to appear) provided $f(t, u) = p(t)F(u)$.

Another approach to the question of bounded solutions of N_+ results from applying to equation H_+^* the method used to obtain Theorem 2.4 and Corollary 2.5. We obtain the analogous results:

THEOREM 4.6. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a solution of* (4.1). *Suppose that for some $k = 0, \cdots, n-1$ and for all positive constants C*

$$\int^{\infty} t^{2n-2k-1} (R_\tau(t) + C) f(t, R_\tau(t) + C)dt = \infty. \tag{4.8}$$

Then N_+ has no B_k^R-solutions $y(t)$ for which $[y(t) - R(t)]_{2k}$ is bounded.

COROLLARY 4.7. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a bounded solution of* (4.1). *Suppose that* (4.8) *holds with $k = 0$. Then N_+ has no bounded positive B_k^R-solutions.*

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The corollary follows upon observing that since $R(t)$ is bounded, a B_k^R-solution $y(t)$ is bounded if, and only if, $u(t)$ is bounded. For $k \geqq 1$, $u(t)$ is unbounded by Lemma 2.1. The case $k = 0$ is excluded by Theorem 4.6.

REMARK 5. In view of Remarks 3 and 4 we may assume, without loss of generality, that $R(t) > 0$ in attempting to exclude $\mathscr{B}_j^R$-solutions of N_+.

THEOREM 4.8. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a positive solution of* (4.1). *Suppose that for some $k = 0, \cdots, n-1$ and for all $C > 0$*

$$\int^{\infty} t^{2n-2k-1} f(t, C)\,dt = \infty. \tag{4.9}$$

Then no positive solution of N_+ of type $\mathscr{B}_k^R$ is bounded away from zero as $t \to \infty$.

PROOF. Let $y(t)$ be a positive solution of N_+ of type $\mathscr{B}_k^R$ on $[T_0, \infty)$. Then $v(t) = R(t) - y(t)$ is a positive solution of H_+^- of type $\mathscr{B}_k$ on $[T_0, \infty)$ and $Dv_{2n-1}(t)$ for $t \geqq T_1$. Suppose that $y(t)$ is bounded away from zero as $t \to \infty$. Let $t \geqq T_1$, $b > 2t$ and $i \leqq n$. Integration of H_+^- over (t, b) results in

$$\begin{aligned} v_{2n-i}(t) = {} & \sum_{j=0}^{i-1} (-1)^j v_{2n-i+j}(b) \frac{(b-t)^j}{j!} \\ & + \frac{(-1)^i}{(i-1)!} \int_t^b (s-t)^{i-1} (R-v)_\tau(s) f[s, (R-v)_\tau(s)]\,ds. \end{aligned} \tag{4.10}$$

For the case $i > n$, we obtain

$$\begin{aligned} v_{2n-i}(t) \geqq {} & \sum_{j=0}^{i-n-1} (-1)^j v_{2n-i+j}(b) \frac{(b-t)^j}{j!} \\ & + \frac{1}{M_0} \sum_{j=i-n}^{i-1} (-1)^j v_{2n-i+j}(b) \frac{(b-t)^j}{j!} \\ & + \frac{(-1)^i}{M_0 (i-1)!} \int_t^b (s-t)^{i-1} (R-v)_\tau(s) f[s, (R-v)_\tau(s)\,ds. \end{aligned} \tag{4.10'}$$

Letting $i = 2(n-k)$ in (4.10) if $k \geqq n/2$ or in (4.10′) if $k < n/2$, we obtain

$$v_{2k}(T_1) \geqq v_{2k}(t) \geqq \omega_k^{-1} \int_t^b (s-t)^{2n-2k-1} (R-v)_\tau(s) f[s, (R-v)_\tau(s)]\,ds$$

$$\geqq 2^{2k+1-2n} \omega_k^{-1} \int_T^b s^{2n-2k-1} (R-v)_\tau(s) f[s, (R-v)_\tau(s)]\,ds.$$

Since $y(t) = R(t) - v(t)$ is bounded away from zero as $t \to \infty$, there is a constant $C > 0$ such that $(R - v)_\tau(t) \geqq C$ for $t \geqq T$. Thus

$$\int_T^b s^{2n-2k-1} f(s, C)ds \leqq 2^{2n-2k-1} \omega_k C^{-1},$$

which is in contradiction to (4.9) as $b \to \infty$.

We now apply some of the theorems above to investigate the asymptotic behavior of bounded solutions of the ordinary differential equation

$$(D^4 + 1)y(t) = e^{-t}. \tag{4.11}$$

The general solution of (4.11) is given by

$$y_N(t) = c_1 e^{t/\sqrt{2}} \cos(t/\sqrt{2} + c_2 e^{t/\sqrt{2}} \sin(t/\sqrt{2}) + c_3 e^{-t/\sqrt{2}} \cos(t/\sqrt{2})$$
$$+ c_4 e^{-t/\sqrt{2}} \sin(t/\sqrt{2}) + \frac{1}{2} e^{-t}.$$

Since $e^{-t/\sqrt{2}} > e^{-t}$, a bounded solution $y(t)$ of (4.11) is either oscillatory (if $c_1^2 + c_2^2 + c_3^2 + c_4^2 \neq 0$) or else positive nonoscillatory. In either event, $\lim_{t \to \infty} y(t) = 0$.

We have $Q(t) = e^{-t}$ and $r(t) \equiv 1$, the general solution of $D^4 R(t) = e^{-t}$ is

$$R(t) = e^{-t} + a_1 t^3 + a_2 t^2 + a_3 t + a_4.$$

Let us choose $R(t) = e^{-t}$ and let $y(t)$ be a bounded solution of (4.11). If $y(t)$ is a B_k^R-solution, then $0 < y(t) < R(t)$ and $\lim_{t \to \infty} y(t) = 0$. This fact is confirmed by Theorem 4.8 in the case $k = 0, 1$ since

$$\infty = \int^\infty t^{2n-2k-1} f(t, C)dt = \begin{cases} \displaystyle\int^\infty t^3 dt & \text{if } k = 0 \\ \displaystyle\int^\infty t dt & \text{if } k = 1 \end{cases}.$$

If $y(t)$ is of type B_1^R, then $y(t) - R(t)$ is necessarily unbounded; since $R(t)$ is bounded, this would imply that $y(t)$ is unbounded. Theorem 4.1 shows that there are, in fact, no B_1^R-solutions:

$$\int^\infty t^2 (e^{-(t-\tau(t))} + C)dt = \infty.$$

Corollary 4.4 shows that no B_0^R-solution is bounded away from zero since

$$\int^\infty [e^{-(t-\tau(t))} + C)dt = \infty.$$

5

We now consider the nonhomogeneous delay equation N_- under the assumption that $R(t)$ is a solution of the equation (4.1). As in the previous section, this permits the transformation of N_- to a homogeneous delay equation for which the techniques of sections two and three are applicable.

Let us assume that $y(t)$ is a positive solution of N_- and let $u(t) = y(t) - R(t)$. Then

$$D^n[r(t)D^n u(t)] = D^n[r(t)D^n y(t)] - D^n[r(t)D^n R(t)]$$
$$= y_\tau(t)f[t, y_\tau(t)],$$

so that $u(t)$ is a solution of the homogeneous equation

$$(H_-^-) \qquad D^n[r(t)D^n u(t)] - (u+R)_\tau(t)f[t,(u+R)_\tau(t)] = 0.$$

Since $y(t) > 0$, $(u+R)_\tau(t)$ is positive for sufficiently large t $(t \geqq T_1)$ and $D^n[r(t)D^n u(t)] > 0$ for $t \geqq T_1$. It follows that $u(t)$ is a nonoscillatory solution of H_-^-. Either $u(t) > 0$ or $u(t) < 0$. If $u(t) < 0$, we further transform the equation by letting $v(t) = -u(t) = R(t) - y(t)$. Then $v(t)$ is a positive solution of

$$(H_-^+) \qquad D^n[r(t)D^n v(t)] + (R-v)_\tau(t)f[t,(R-v)_\tau(t)] = 0.$$

We say that a positive solution $y(t)$ of N_- is of type $\mathscr{B}_j^R$ if $u(t)$ is a positive solution of H_-^- of type $\mathscr{B}_j$; it is of type B_j^R if $v(t)$ is a positive solution of H_-^+ of type B_j. A positive solution of N_- is then of type $\mathscr{B}_j^R$ $(j = 0, \cdots, n)$ or of type B_j^R $(j = 0, \cdots, n-1)$. We now seek to exclude $\mathscr{B}_j^R$- and B_j^R-solutions of N_-.

THEOREM 5.1. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and $R(t)$ be a solution of* (4.1). *Suppose that for some $k = 1, \cdots, n-1$ and for all $C > 0$*

$$(5.1) \qquad \int^\infty t^{2n-2k-1}(R_\tau(t) + Ct^{2k-1})f(t, R_\tau(t) + Ct^{2k-1})dt = +\infty.$$

Then N_- has no $\mathscr{B}_k^R$-solutions. Furthermore, if

$$(5.1') \qquad \int^\infty t^{2n-1}(R_\tau(t) + C)f(t, R_\tau(t) + C)dt = +\infty,$$

then any $\mathscr{B}_0^R$-solution $y(t)$ satisfies

$$(5.2) \qquad \lim_{t\to\infty}[y(t) - R(t)] = 0.$$

PROOF. Let $y(t)$ be a $\mathscr{B}_k^R$-solution of N_-. Then $u(t) = y(t) - R(t)$ is a positive solution of H_-^- of type $\mathscr{B}_k^R$. As in the proof of Theorem 4.3, we obtain

$$u_{2n-i}(t) = \sum_{j=0}^{i-1} \frac{u_{2n-i+j}(b)}{j!}(t-b)^j + \frac{(-1)^i}{(i-1)!}\int_t^b (s-t)^{i-1}(u+R)_\tau(s) f[s,(u+R)_\tau(s)]ds, \tag{5.3}$$

where $T_1 \leqq t \leqq s < b$ and $i \leqq n$.

If $i > n$, we may derive the analogous inequality

$$u_{2n-i}(t) \geqq \sum_{j=0}^{i-n-1} \frac{u_{2n-i-j}(b)}{j!}(t-b)^j + \frac{1}{M_0}\sum_{j=i-n}^{i-1} \frac{u_{2n-i+j}(b)}{j!}(t-b)^j + \frac{(-1)^i}{M_0(i-1)!}\int_t^b (s-t)^{i-1}(u+R)_\tau(s) f[s,(u+R)_\tau(s)]ds. \tag{5.4}$$

Letting $i = 2(n-k)$ in the equality (5.3) if $k \geqq n/2$ and in the inequality (5.4) if $k < n/2$ and recalling that $Du_{2k}(t) < 0$ for $t \geqq T_1$, we have

$$\int_t^b (s-t)^{2n-2k-1}(u+R)_\tau(s) f[s,(u+R)_\tau(s)]ds \leqq \omega_k u_{2k}(T_1).$$

For $k = 1, \cdots, n$ there is a constant $C > 0$ such that $u_\tau(s) \geqq Cs^{2k-1}$ for $s \geqq 2T_1 = T_2$. Let $t \geqq T_2$ and $b > 2t = T_*$. Then $s - t \geqq s/2$ and

$$\int_{T_*}^b s^{2n-2k-1}(R_\tau(s) + Cs^{2k-1}) f[s, R_\tau(s) + Cs^{2k-1}]ds \leqq 2^{2n-2k-1}\,\omega_k u_{2k}(T_1),$$

which is in contradiction to (5.1) as $b \to \infty$.

If $k = 0$ and $y(t) - R(t)$ does not tend to zero as $t \to \infty$, there is a constant $C > 0$ such that $u(t) \geqq C$ for t sufficiently large ($t \geqq T_*$). As before we may apply hypothesis (ii) to obtain a contradiction to (5.1′).

Duplicating the arguments which led to Theorems 3.3 and 4.4, we may obtain the following result:

THEOREM 5.2. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and let $R(t)$ be a solution of* (4.1). *Suppose that for some $m = 2n - i$, where $i = 1, 2, 3$, and for all positive constants B*

$$\lim_{b\to\infty} t^{-m}\int_t^b s^{2n-1}(R_\tau(s) + Bs^m) f[s, R_\tau(s) + Bs^m]ds = +\infty.$$

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Then N_- has no solutions of types B_n^R or B_{n-1}^R such that $R(t) - y(t) \sim Ct^m$, where $C > 0$.

If we let $w(t) = v_{2n-1}(t)v_{2k}^{-1}(t)$ in H_-^+ and repeat the procedure which led to Theorems 2.3 and 4.1, we may obtain the following analogue:

THEOREM 5.3. *Let $\tau(t)$ satisfy* (T2) *or* (T3) *and let $R(t)$ be a bounded solution of* (4.1). *Suppose that for some $k = 0, \cdots, n-1$ and for all positive constants C*

$$\int^{\infty} t^{2k} f(t, C)dt = \infty.$$

Then any positive B_k^R-solution $y(t)$ tends to zero as $t \to \infty$.

Summary

We conclude this paper with some observations on the method. Use of the preliminary transformation $Y(t) = -y(t)$ and the techniques of Sections four and five enable us to introduce a natural classification of the negative solutions of N_+ and N_- and to provide sufficient conditions for the nonexistence of such solutions. If the hypothesis $f(t, -u) = f(t, u)$ is omitted, we may still determine conditions for the nonexistence of certain negative solutions. Moreover, the method is applicable even if hypothesis (ii) is replaced by (ii′) there is a ρ $(0 < \rho < 1)$ such that $u^\rho f(t, u)$ is nondecreasing in u. The precise statements and proofs of these analogous results are left to the reader to discover.

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