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On the Smirnov Class Defined by the Maximal Function

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Abstract. H. O. Kim has shown that contrary to the case of H^p-space, the Smirnov class M defined by the radial maximal function is essentially smaller than the classical Smirnov class of the disk. In the paper we show that these two classes have the same corresponding locally convex structure, *i.e.* they have the same dual spaces and the same Fréchet envelopes. We describe a general form of a continuous linear functional on M and multiplier from M into H^p, $0 < p \leq \infty$.

Let $\mathbb{D}$ and $\mathbb{T}$ be the unit disk and the unit circle in the complex plane $\mathbb{C}$. Moreover, let m be the Haar measure on $\mathbb{T}$. The maximal theorem of Hardy and Littlewood states [D, Theorem 1.9]:

For each $0 < p < \infty$ there exists a positive constant C_p such that

$$\text{(Mp)} \qquad \int_{\mathbb{T}} Mf(\omega)^p \, dm(\omega) \leq C_p \sup_{0<r<1} \int_{\mathbb{T}} |f(r\omega)|^p \, dm(\omega)$$

for each holomorphic function f on $\mathbb{D}$, where

$$Mf(\omega) = \sup_{0<r<1} |f(r\omega)| \quad \text{for } \omega \in \mathbb{T}$$

is the *maximal radial function* of f. This implies that the classical Hardy space

$$H^p = H^p(\mathbb{D}) = \Big\{ f \in H(\mathbb{D}) : \|f\|_p^p = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\omega)|^p \, dm(\omega) < \infty \Big\}$$

coincides with the corresponding space

$$M_p = \Big\{ f \in H(\mathbb{D}) : \int_{\mathbb{T}} Mf(\omega)^p \, dm(\omega) < \infty \Big\}$$

defined by the maximal function.

H. O. Kim observed [K] that if we pass to the limit case $p = 0$ then the corresponding statement

$$\int_{\mathbb{T}} \log^+ Mf(\omega) \, dm(\omega) \leq C_0 \sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f(r\omega)| \, dm(\omega),$$

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is not valid. Kim has showed that the class

$$M = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{T}} \log^+ Mf(\omega)\, dm(\omega) < \infty \right\}$$

equipped with the topology induced by the metric

$$d(f,g) = \|f-g\| = \int_{\mathbb{T}} \log\big(1 + M(f-g)(\omega)\big)\, dm(\omega)$$

is a complete topological vector space (an F-space) whose dual separates the points of M. In fact, M is a topological algebra. Kim proved that M is essentially smaller than the Nevanlinna class (a topological vector group [SS])

$$N = \left\{ f \in H(\mathbb{D}) : \sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f(r\omega)|\, dm(\omega) < \infty \right\},$$

and even smaller than the Smirnov class N^+ being the largest linear subspace of N which is a topological vector space in the relative topology. Recall, N^+ is the class of all holomorphic functions f in $\mathbb{D}$ for which the family $\{\log^+ |f_r| : 0 < r < 1\}$ is uniformly integrable, while f belongs to M if and only if this family has an $L^1(\mathbb{T})$-majorant, where $f_r(\omega) = f(r\omega)$ for each $\omega \in \mathbb{T}$.

In this paper we show that the Smirnov class M defined by the maximal function and the classical Smirnov class N^+ have the same corresponding locally convex structure *i.e.* they have the same continuous linear functionals and the same Fréchet envelopes.

Let us recall that if $X = (X, \tau)$ is an F-space whose topological dual X' separates the points of X, then its *Fréchet envelope* $\hat{X}$ is defined to be the completion of the space (X, τ^c), where τ^c is the strongest locally convex topology on X which is weaker than τ. In fact it is known [S] that τ^c is equal to the Mackey topology of the dual pair (X, X'). Since for each metrizable locally convex topology ξ on X, (X, ξ) is a Mackey space, *i.e.*, ξ coincides with the Mackey topology of the dual pair (X, X'_ξ), so the Fréchet envelope $\hat{X}$ of X is up to an isomorphism uniquely defined by the conditions: (a) $\hat{X}$ is a Fréchet space, (b) there exists a continuous embedding j of X onto a dense subspace of $\hat{X}$ such that the mapping $\hat{X}' \ni \Lambda \mapsto \Lambda \circ j \in X'$ is a linear isomorphism of $\hat{X}'$ onto X'.

N. Yanagihara [Y1] proved that the Fréchet envelope $\widehat{N^+}$ of the Smirnov class can be identified with the space

$$F^+ = \left\{ f \in H(\mathbb{D}) : |||f|||_c = \sum_{n=0}^{\infty} |\hat{f}(n)| \exp(-c\sqrt{n}) < \infty \text{ for each } c > 0 \right\},$$

where $\hat{f}(n)$ is the n-th Taylor coefficient of f.

In the sequel we prove

Theorem 1 *F^+ is the Fréchet envelope of both the maximal Smirnov class M and the classical Smirnov class N^+.*

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Indeed, $M \subset N^+ \subset F^+$ with inclusions being continuous. Moreover, polynomials are dense in all this spaces [K, Theorem 3.2]. Thus, M is a dense subspace of F^+. Therefore, for the proof of the Theorem 1 it suffices to show

Theorem 2 *Each continuous liner functional Λ on M is of the form*

$$\text{(C)}\qquad \Lambda(f) = \sum_{n=0}^{\infty} \hat{f}(n) b_n, \quad f \in M,$$

where the sequence $(b_n = \overline{\Lambda(z^n)})$ satisfies

$$\text{(E)}\qquad |b_n| = O\big(\exp(-c\sqrt{n})\big) \quad \textit{for some } c > 0.$$

This implies that each continuous linear functional Λ on M is a restriction of some continuous linear functional on F^+.

For the proof of the theorem we need the following

Lemma 3 *Let*

$$f_{c,r}(z) = \exp\left(c \frac{1 - r^2}{(1 - rz)^2} \right).$$

Then,

(a) $\lim_{c \to 0} \sup_{0<r<1} \|c f_{c,r}\| = 0$,
(b) *for each $c > 0$ there is $d > 0$ such that*

$$\inf_{n \in \mathbb{N}} \sup_{0<r<1} |\widehat{f_{c,r}}(n)| \exp(-d\sqrt{n}) > 0.$$

Proof (a) Define $g_r(z) = (1 - rz)^{-2}$ for each $0 < r < 1$. It is well known that $\|g_r\|_{H^1} = O\big((1 - r^2)^{-1}\big)$ (*cf.* [R, Proposition 1.4.10]). Moreover, by (Mp)

$$\text{(1)}\qquad \|Mg_r\|_{L^1} \le C_1 \|g_r\|_{H^1} = O\big((1 - r^2)^{-1}\big).$$

Let I be an open arc in $\mathbb{T}$ containing 1. Then, $\lim_{c \to 0} \sup_{\omega \in \mathbb{T} \setminus I} \sup_{0<r<1} |cM f_{c,r}(\omega)| = 0$, so

$$\lim_{c \to 0} \sup_{0<r<1} \int_{\omega \in \mathbb{T} \setminus I} \log\big(1 + cM f_{c,r}(\omega)\big)\, dm(\omega) = 0.$$

Using the inequality $\log(1 + cx) \le \log(1 + c) + \log 2 + \log x$, $x \ge 1$, $c \ge 0$, we have

$$\begin{aligned}
&\int_I \log\big(1 + cM f_{c,r}(\omega)\big)\, dm(\omega) \\
&\qquad \le m(I) \log(1 + c) + m(I) \log 2 + c(1 - r^2) \int_I M g_r(\omega)\, dm(\omega) \\
&\qquad \le m(I) \log(1 + c) + m(I) \log 2 + cC
\end{aligned}$$

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for some $C > 0$ and all I (by (1)). This completes the proof since we could choose I as small as we need.

(b) For each $c > 0$ and $0 < r < 1$ we have

$$\begin{aligned} f_{c,r}(z) &= \sum_{j=0}^{\infty} \frac{c^j}{j!}(1-r^2)^j(1-rz)^{-2j} \\ &= 1 + \sum_{j=1}^{\infty} \frac{c^j}{j!}(1-r^2)^j \sum_{n=0}^{\infty} \binom{2j+n-1}{n}(rz)^n \\ &= 1 + \sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \frac{c^j}{j!}(1-r^2)^j \binom{2j+n-1}{n} \right)(rz)^n \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{f_{c,r}}(n) &\geq \frac{c^j}{j!}(1-r^2)^j \binom{2j+n-1}{n} r^n \\ &\geq \frac{c^j}{j!}(1-r^2)^j \frac{n^{2j-1}}{(2j)!} r^n \end{aligned}$$

for each $j \in \mathbb{N}, 0 < r < 1$ and $c > 0$. Let $S(c,n) = \sup_{0<r<1} |\widehat{f_{c,r}}(n)|$. Then, using Stirling's formula we get

$$\begin{aligned} \log S(c,n) \geq\ & j \log c + j \log(1-r^2) + (2j-1)\log n + n \log r \\ & - j \log j + j - O(\log j) - 2j \log 2j + 2j - O(\log 2j) \\ \geq\ & j\left(3 + \log \frac{c(1-r^2)n^2}{4j^3}\right) + n \log r - O(\log n) - O(\log j) \end{aligned}$$

Take $1 - r^2 = \gamma/\sqrt{n}$ and $j=$ the integer part of $\gamma\sqrt{n}$, where γ is being taken so small that $4\gamma^2 < c$. Then,

$$\frac{c(1-r^2)n^2}{4j^3} \geq 1$$

and

$$n \log r = \frac{1}{2} n \log\left(1 - \frac{\gamma}{\sqrt{n}}\right) \geq -n\frac{\gamma}{\sqrt{n}} = -\gamma\sqrt{n}.$$

Finally,

$$\log S(c,n) \geq 3\gamma\sqrt{n} - \gamma\sqrt{n} - O(\log n) = 2\gamma\sqrt{n} - O(\log n).$$

Proof of Theorem 2 It is known that each sequence $b = (b_n)$ satisfying (E) defines by (C) a continuous linear functional on F^+ (*cf.* [Y1]). Since the inclusion mapping $M \subset F^+$ is continuous, so b defines a continuous linear functional on M.

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Suppose now that Λ is a continuous linear functional on M and let $b_n = \overline{\Lambda(z^n)}$ for $n = 0, 1, \dots$. Then, there is an $\varepsilon > 0$ such that

$$\text{(a)} \qquad |\Lambda(f)| \leq 1 \quad \text{for each } f \in M \text{ with } \|f\| < \varepsilon.$$

The uniform topology is stronger than the original topology of M, so for each $\zeta \in \mathbb{D}$

$$\Lambda\big(f(\zeta\cdot)\big) = \sum_{n=0}^{\infty} \hat{f}(n)\zeta^n \Lambda(z^n) = \sum_{n=0}^{\infty} \hat{f}(n)\zeta^n \overline{b_n} = \lambda_f(\zeta).$$

For $\zeta \in \mathbb{D}$ and $f \in M$ we have $\|f(\zeta\cdot)\| \leq \|f\|$. Consequently, λ_f is a holomorphic function on $\mathbb{D}$ bounded by 1 and $\widehat{\lambda_f}(n) = \hat{f}(n)\overline{b_n}$. This implies

$$\text{(b)} \qquad |\hat{f}(n)\overline{b_n}| \leq 1 \quad \text{for each } f \in M,\ \|f\| \leq \varepsilon.$$

Lemma 3(a) implies that we can find $c > 0$ such that

$$\|cf_{c,r}\| < \varepsilon \quad \text{for all } 0 < r < 1.$$

Applying Lemma 3(b) we see that there are $d, \delta > 0$ and $r_n \in (0,1)$, $n = 0, 1, \dots$, such that

$$|\widehat{f_{c,r_n}}(n)| \exp(-d\sqrt{n}) \geq \delta \quad \text{for all } n = 0, 1, \dots.$$

Using this and (b) we obtain

$$|b_n| \leq \delta^{-1} \exp(-d\sqrt{n}) \quad \text{for each } n = 0, 1, \dots,$$

so $b = (b_n)$ satisfies (E). Finally, b defines by (C) a continuous linear functional Λ_b on M which coincides with Λ on all functions whose Taylor series are uniformly convergent. In particular,

$$\Lambda(f) = \lim_{r\to 1} \Lambda\big(f(r\cdot)\big) = \lim_{r\to 1} \Lambda_b\big(f(r\cdot)\big) = \Lambda_b(f) \quad \text{for each } f \in M.$$

The proof is finished.

As a simple consequence of Theorem 1 we obtain

Corollary 4 **[K, Theorem 5.4]** *M is non locally convex.*

Proof The Fréchet envelope $\hat{X}$ of a complete metrizable locally convex space X coincides with X. However, M is a proper subspace of F^+, so M is non locally convex.

Corollary 5 **[K, Theorem 4.5]** *M is non locally bounded.*

Proof It is easily seen that the Fréchet envelope $\hat{X}$ of a locally bounded space X must be locally bounded. For each $c > 0$ the set of monomials $A(c) = \{z^n(c\exp(c\sqrt{n}) : n \in \mathbb{N}\}$ is contained in the ball $|||f|||_c \le c$ but obviously is not bounded in F^+. Consequently, F^+ as well as M can not be locally bounded.

Corollary 6 *If $f \in M$ then*

$$(*) \qquad \hat{f}(n) = O\Big(\exp\big(o(\sqrt{n})\big)\Big).$$

Furthermore, this estimate is best possible: given any sequence (c_n) of positive numbers tending to zero, there exists $f \in M$ such that $\hat{f}(n) \neq O\big(\exp(c_n\sqrt{n})\big)$.

Proof The estimate (*) is well known for the classical Smirnov class N^+ [Y3, Theorem 1] and is the best possible in this case (see also [ST]). The algebra M is smaller than N^+ but the estimate (*) is steel best possible in this case. Indeed, suppose that $\hat{f}(n) = O\big(\exp(c_n\sqrt{n})\big)$ for some sequence (c_n) of positive numbers tending to zero and all $f \in M$. Define the space X consisting of all holomorphic functions such that

$$\|f\| = \sup_n |\hat{f}(n)| \exp(-c_n\sqrt{n}) < \infty.$$

Then, $(X, \|\cdot\|)$ is a Banach space containing M and contained in F^+. This implies that X is the Fréchet envelope of M and $X = F^+$ both as sets and topological vector spaces. This is impossible, since F^+ is non locally bounded (*cf.* Corollary 5).

Let X, Y be two Fréchet spaces of analytic functions on $\mathbb{D}$. Recall, a sequence $\lambda = (\lambda_n)$ is a *multiplier from X into Y* if and only if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in X \quad \text{implies} \quad \sum_{n=0}^{\infty} \lambda_n \hat{f}(n) z^n \in Y.$$

H. O. Kim proved that if λ is a multiplier from M into H^∞ then $\lambda_n = O\big(\exp(-c\sqrt{\frac{n}{\log n}})\big)$ for some $c > 0$ (*cf.* [K, Theorem 5.2]). The next theorem shows that the space of multipliers is smaller. In fact, we give a necessary and sufficient estimate for λ to be a multiplier from M into H^p for an arbitrary $0 < p \le \infty$.

Theorem 7 *If $\lambda = (\lambda_n)$ is a multiplier from M into H^p for some $0 < p \le \infty$ then*

$$\text{(ME)} \qquad \lambda_n = O\big(\exp(-c\sqrt{n})\big) \quad \textit{for some } c > 0.$$

Conversely, each sequence $\lambda = (\lambda_n)$ satisfying (ME) is a multiplier from M into H^∞.

Proof Let $\lambda = (\lambda_n)$ be a multiplier from M into H^p. We can assume $0 < p < 1$. Since $\hat{g}(n) = o(n^{\frac{1}{p}-1})$ for each $f \in H^p$ (*cf.* [D, Theorem 6.4] or [DRS]) so the mapping $g \mapsto \sum \hat{g}(n) n^{-1-\frac{1}{p}}$ is a continuous linear functional on H^p. Consequently,

$$M \ni f \mapsto \sum \lambda_n \hat{f}(n) n^{-1-\frac{1}{p}} \in \mathbb{C}$$

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is a continuous linear functional on M. By Theorem 2

$$b_n = \lambda_n n^{-1-\frac{1}{p}} = O\big(\exp(-2c\sqrt{n})\big) \quad \text{for some } c > 0.$$

Finally, $\lambda_n = O\big(\exp(-c\sqrt{n})\big)$.

On the other side, let λ satisfies (ME) with $c > 0$. Then for each $f \in M$, $\|f\|_{c/2} < \infty$ the series $\sum \lambda_n \hat{f}(n) z^n$ is convergent and defines some bounded function on $\mathbb{D}$.

A similar result for the Smirnov class was obtained by N. Yanagihara [Y2] and the author [N1].

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