ON DIFFERENTIABLE ARCS AND CURVES, VI: SINGULAR OSCULATING SPACES OF CURVES OF ORDER $n+1$ IN PROJECTIVE n-SPACE

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To Donald Coxeter on his sixtieth birthday

A closed curve K^{n+1} of order $n+1$ in real projective n-space R_n has a maximum number of $n+1$ points in common with any $(n-1)$-space. These curves are subjected to certain differentiability assumptions which make it possible to describe their singular points and to provide them with multiplicities in analogy with algebraic geometry. If $N^n{}_p$ denotes the number of $(n-p)$-times singular points, then

$$\text{(1)} \qquad \sum_0^{n-1} (n-p)N^n{}_p \begin{cases} \leqslant n+1, \\ \equiv n+1 \pmod 2; \end{cases}$$

cf. **(4)**. In **(6)**, an interpretation of the difference

$$n+1-\sum_0^{n-1}(n-p)N^n{}_p$$

was given. Necessary and sufficient conditions for equality to hold in (1) can readily be stated **(2; 4)**.

The next step in the study of the K^{n+1} would be the inclusion of certain singular pairs of points. A p-space in R_n was called *special* **(4)** if it met the K^{n+1} $(p+2)$-times and if none of its $(p-1)$-subspaces met the curve $(p+1)$-times. Denote the number of special subspaces through exactly j different points of K^{n+1} which contain the osculating p_1-space of one of them, the osculating p_2-space of another, etc., by

$$N^n{}_{p_1,p_2,\ldots,p_j}.$$

We proved in **(6)** that the numbers $N^n{}_{pq}$ are bounded for a given n $(p+q \leqslant n-2)$; and in **(5)** that

$$\text{(2)} \qquad \sum_0^{n-1}(n-p)N^n{}_p + \sum_0^{n-2}(n-p-1)N^n{}_{p0} < \tfrac{1}{4}n^2 + \tfrac{3}{2}n.$$

In the present paper we wish to improve (1) and (2) and to show that

$$\text{(3)} \qquad \sum_0^{n-1}(n-p)N^n{}_p + \frac{1}{2}\sum_1^{n-2}(n-p-1)N^n{}_{p0} + (n-1)N^n{}_{00} \leqslant n+1.$$

Received May 3, 1967

https://doi.org/10.4153/CJM-1967-095-7 Published online by Cambridge University Press

We conjecture that the factor $\frac{1}{2}$ can be dropped from the middle term; cf. 5.7.

Our discussion will yield the corollary that the numbers $N^n{}_{p00}$ are bounded for given n $(0 \leqslant p \leqslant n-4)$.

1. Arcs and curves.

1.1. A *curve* C is a continuous image of the projective straight line in R_n $(n \geqslant 1)$. The images of distinct points of the parameter line are interpreted as different points of C even if they coincide in R_n. Then the points of C become the continuous 1–1 images of a parameter s ranging through the parameter line. The point of C with the parameter s will also be denoted by s.

An *arc* A is the continuous 1–1 image of a segment in R_n.

A neighbourhood of the parameter s in the parameter space is mapped onto a *neighbourhood* of the point s on C (on A). If a sequence of parameter values converges to the parameter s, the corresponding sequence of points on C (on A) is said to be *convergent* to the point s.

1.2. The *order* of C (of A) is the least upper bound of the number of points that it has in common with an $(n-1)$-space. It obviously is not less than n. The order of a point s is the order of a sufficiently small neighbourhood of s on C (on A).

1.3. We call s a *point of support* (*of intersection*) with respect to an $(n-1)$-space E if some neighbourhood of s has no point $\neq s$ in common with E and if the two arcs into which s decomposes the neighbourhood lie on the same side (on opposite sides) of E. We then call E a supporting (intersecting) $(n-1)$-space at s. Thus E supports if $s \notin E$.

1.4. The point s is called *differentiable* if it has *osculating spaces* $L^n{}_p(s)$ of all dimensions. Define $L^n{}_{-1}(s) = \emptyset$. Suppose we have defined $L^n{}_p(s)$ and postulated its existence. Then we require the $(p+1)$-spaces through $L^n{}_p(s)$ and a point converging on C (on A) to s to converge. Their limit is then the osculating $(p+1)$-space $L^n{}_{p+1}(s)$. Thus $L^n{}_0(s) = s$, $L^n{}_n(s) = R_n$.

A subspace is said to contain $L^n{}_p(s)$ exactly if it contains $L^n{}_p(s)$ but not $L^n{}_{p+1}(s)$.

A curve or arc is called differentiable if each of its points is.

1.5. Let s be a differentiable point on an arbitrary arc which meets $L^n{}_{n-1}(s)$ only a finite number of times. Then there exists a one-row matrix,

$$(a_0, a_1, \ldots, a_{n-1}),$$

the *characteristic* of s with the following properties:

(i) each of the numbers a_i is equal to 1 or 2;

(ii) if an $(n-1)$-space contains exactly $L^n{}_p(s)$, then it supports (intersects) at s if $a_0 + a_1 + \ldots + a_p$ is even (odd).

If $a_0 = a_1 = \ldots = a_{n-1} = 1$, s is called *regular*.

The projection of s from a point on $L^n{}_{p+1}(s)\backslash L^n{}_p(s)$ has the characteristic

$$(a_1, a_2, \ldots, a_{n-1}) \qquad \text{if } p = -1,$$
$$(a_0, a_1, \ldots, a_{n-2}) \qquad \text{if } p = n-1,$$
$$(a_0, a_1, \ldots, a_{p-1}, a'_p, a_{p+2}, \ldots, a_{n-1}) \qquad \text{if } -1 < p < n-1;$$

here $a'_p \equiv a_p + a_{p+1} \pmod 2$.

If a subspace contains $L^n{}_p(s)$ exactly, we count s with the multiplicity $a_0 + a_1 + \ldots + a_p$. In particular, s is counted $(p + 1)$-times if s is regular.

2. Assumptions and lemmas. *The subject of this paper is the differentiable curves K^{n+1} in R_n which are met at most $(n + 1)$-times by any $(n - 1)$-space;* cf. (**3**, §3 and **4**, p. 72).

2.1. The digits of the characteristic $(a_0, a_1, \ldots, a_{n-1})$ of any point $s \in K^{n+1}$ are equal to one with at most one exception. The order of s is equal to n if and only if s is regular. If $a_p = 2$, we call s $(n - p)$-times *singular*. A *cusp* is n-times singular.

A singular point divides a small neighbourhood into two arcs of order n.

At a p-fold point, p distinct points of a curve coincide; cf. 1.1. A K^{n+1} has no more-than-twofold points. The two points which coincide in a *double point* are regular. If $n > 1$, the total number $N^n{}_{00} + N^n{}_0$ of double points and cusps of a K^{n+1}, i.e. the number of special 0-spaces, is 0 or 1.

2.2. We denote by C^n (B^n) a differentiable curve (arc) of order n in R_n. All of its points are regular and simple. It is met by any p-space at most $(p + 1)$-times. The sum of the multiplicities of the points that a C^n has in common with an $(n - 1)$-space is $\equiv n \pmod 2$. The projection of a C^n (B^n) from one of its points is a C^{n-1} (B^{n-1}).

2.3. A p-space spanned by osculating spaces of K^{n+1} meets the curve either $(p + 1)$- or $(p + 2)$-times. In the first case we call it *regular*; in the other case, *abundant*. Any p-space which meets K^{n+1} $(p + 2)$-times is abundant. If it has no proper abundant subspaces, it is called *special*.

2.4. The projection of a K^{n+1} from a regular (abundant) p-space is a K^{n-p} (a C^{n-p-1}), $0 \leqslant p < n - 1$. In particular, its projection from $L^n{}_p(s)$ is a K^{n-p} if and only if s is at most $(n - p - 1)$-times singular and $L^n{}_p(s)$ does not meet K^{n+1} outside s. Thus the projection of K^{n+1} from a double point or cusp is a C^{n-1}; from any other point of the curve, it is a K^n.

2.5. If each of a sequence of subspaces meets a K^{n+1} or C^n at least p-times (in points which converge to s), then any limit subspace will meet the curve at least p-times (at s). In particular, the osculating spaces of a K^{n+1} or C^n are continuous; cf. (**3**).

3. The mapping $t^n{}_{n-3,0}$.

https://doi.org/10.4153/CJM-1967-095-7 Published online by Cambridge University Press

3.1. The mapping $t^n{}_{n-1}$ associated with each point s the point $t^n{}_{n-1}(s)$ at which $L^n{}_{n-1}(s)$ meets K^{n+1} again; cf. **(4)**.

In **(6)**, we studied mappings which associated with each point $s \in K^{n+1}$ the set $t^n{}_m(s)$ of those points which are projected from $L^n{}_m(s)$ into singular points ($m = 0, 1, \ldots, n-2$). It consists of at most $n - m$ points and may be void. We distinguish different points of this set by prefixed indices: ${}_it^n{}_m(s)$.

If the projection of $t \neq s$ from s is q-times singular, we call t a q-fold image point. If t is $(q + m - 1)$-times singular, it is a q-fold *improper* image point of any $s \neq t$. Any other image point is called *proper*.

The original points s can also be provided with multiplicities and s becomes a p-fold original point of the q-fold proper image point $t \neq s$ if and only if $L^n{}_{m+1-p}(s)$ and $L^n{}_{n-m-q}(t)$ span a special subspace ($0 \leqslant m \leqslant n-1$, $0 < p \leqslant m+1$, $0 < q \leqslant n - m$).

3.2. The simply singular points of a K^3 are its inflection points while a twice singular point is a cusp. There are three types of curves K^3:

(i) three inflection points;

(ii) one inflection point and one cusp;

(iii) one inflection point and a double point;

cf. **(4**, 5.4 and 3.6**)**. Thus if we count a cusp twice, there always exists a triple $t^2{}_{-1}$ of points of K^3 which coincide with either a singular or a double point.

In the first two cases, the tangential mapping $t^2{}_1$ is monotonically negative ("retrograde") with the degrees -2 and -1, respectively. In the third case, the degree is zero. The double point then decomposes K^3 into two arcs B^+ and B^- such that $t^2{}_1$ is positive in B^+ and negative in B^-; cf. **(4**, §3**)**.

An improper image point of $t^2{}_0$ is a cusp. As far as this mapping is proper, it is the inverse of $t^2{}_1$; cf. **(6**, 1.2**)**. Thus in the first two cases it is defined on the entire K^3 and its proper image points are monotonically negative. In the third case, it is defined exactly in B^-. If s moves through B^-, then one of the two points $t^2{}_0(s)$ moves positively through B^+ while the other one runs negatively through B^-, the inflection point being a fixed point. This yields: *If the tangential mapping is negative at s, then one of the mappings ${}_it^2{}_0$ is always negative. The other one is negative in case* (i), *improper in case* (ii), *and positive in case* (iii). *Thus if s is any point of K^3, $s \notin t^2{}_{-1}$, then K^3 has a double point if and only if one of the three mappings $t^2{}_1$, ${}_it^2{}_0$ is positive at s;* cf. **(6**, 5.1 and 5.2**)**.

In case (i), the three inflection points and the triple $s, t^2{}_0(s)$ alternate. The pairs $s, t^2{}_1(s)$ and $t^2{}_0(s)$ separate one another.

In case (ii), the inflection point ${}_1t^2{}_{-1}$ and the cusp separate s from its proper image ${}_1t^2{}_0(s)$. The latter lies between $t^2{}_1(s)$ and ${}_1t^2{}_{-1}$.

In case (iii), if s lies in B^-, the point ${}_1t^2{}_0(s)$, say, lies in B^+ while ${}_2t^2{}_0(s)$ and $t^2{}_1(s)$ lie in B^-. The inflection point separates s from ${}_2t^2{}_0(s)$ and $t^2{}_1(s)$, and ${}_2t^2{}_0(s)$ separates s from $t^2{}_1(s)$. If s lies in B^+, $t^2{}_1(s)$ is in B^-.

3.3. If the projection of K^{n+1} from $L^n{}_{n-3}(s)$ is a curve K^3, the mapping $t^n{}_{n-3,0}$ associates with s those points which are projected into a double point or cusp of

K^3. Thus the set $t^n_{n-3,0}(s)$ is either void or a pair ${}_it^n_{n-3,0}(s)$ of distinct or equal points ($i = 1, 2$). If K^{n+1} has a double point or cusp t, then $t^n_{n-3,0}(s) = t$ for all $s \notin t$. We then also define $t^n_{n-3,0}(s) = t$ if $s \in t$ and call the mapping $t^n_{n-3,0}$ *improper. From now on we assume that K^{n+1} has no special* 0*-space.*

3.31. Suppose the points s, ${}_1t$, ${}_2t$ are mutually distinct. Then $\{{}_1t, {}_2t\} = t^n_{n-3,0}(s)$ if and only if ${}_1t$ and ${}_2t$ do not lie on $L^n_{n-3}(s)$ and if there exists an $(n-2)$-space $E(s)$ through $L^n_{n-3}(s)$, ${}_1t$, and ${}_2t$. We call s a *p-fold original point* of the mapping if $L^n_{n-2-p}(s)$, ${}_1t$, and ${}_2t$ span a special $(n-1-p)$-space. The point s is at most simply singular, and ${}_1t$ and ${}_2t$ are at most $(n-p-2)$-times singular ($1 \leqslant p \leqslant n-2$).

$E(s)$ is projected from $L^n_{n-3}(s)$ into a double point and from ${}_1t$ onto $L^{n-1}{}_{n-3}(s)$; this space contains ${}_2t$.

3.32. Let $s \neq t$. Then $\{t, t\} = t^n_{n-3,0}(s)$ if and only if $t \notin L^n_{n-3}(s)$ and if $L^n_{n-3}(s)$ and $L^n_1(t)$ span an $(n-2)$-space $E(s)$. The point t is then called a *double image point.* We assign the multiplicity p to s if $L^n_{n-2-p}(s)$ and $L^n_1(t)$ span a special $(n-1-p)$-flat. The points s and t are then at most simply and $(n-p-2)$-times singular, respectively ($1 \leqslant p \leqslant n-2$). $E(s)$ is projected from $L^n_{n-3}(s)$ into a cusp.

3.33. The pair $s = {}_1t \neq {}_2t$ is projected from $L^n_{n-3}(s)$ into a double point if and only if ${}_2t$ lies exactly on $L^n_{n-2}(s)$. Extending our definition, we call s a p-fold original point of the pair $\{s, {}_2t\}$ if ${}_2t$ lies exactly on $L^n_{n-1-p}(s)$ ($1 \leqslant p \leqslant n-2$).

The *fixed point* s is then regular; ${}_2t$ is at most $(n-p-1)$-times singular.

In this case, we define $E(s) = L^n_{n-2}(s)$. Thus $E(s)$ is projected from $L^n_{n-3}(s)$ onto a double point if $p = 1$. From s, $E(s)$ is projected onto $L^{n-1}{}_{n-3}(s)$; this projection contains ${}_2t$. The projection of s from ${}_2t$ is $(p+1)$-times singular for $p \leqslant n-2$; $E(s)$ is projected onto $L^{n-1}{}_{n-3}(s)$.

3.4. Given s, we can now discuss the existence of the points $t^n_{n-3,0}(s)$.

3.41. Let $L^n_{n-2}(s)$ be regular; cf. 2.3. Thus K^{n+1} is projected from $L^n_{n-3}(s)$ into a curve K^3. By 3.2, K^3 has a double point if and only if one of its mappings t^2_1 and t^2_0 is positive at s. By **(6**, 4.6**)**, the mappings t^2_m and the corresponding mappings t^n_{n-2+m} of K^{n+1} have the same direction at s. Thus *there exists a pair of distinct points* ${}_1t^n_{n-3,0}(s)$, ${}_2t^n_{n-3,0}(s)$ *if and only if one of the mappings*

$$(1) \qquad t^n_{n-1}, \qquad {}_1t^n_{n-2}, \qquad {}_2t^n_{n-2}$$

is positive at s.

3.42. By **(6**, 5.4**)**, the mappings $t^n_m(s)$ have only a bounded number of multiple original points. The point s is a p-fold original point of $t \neq s$ at this mapping if and only if t lies exactly on $L^n_{n-p}(s)$; cf. **(4**, 3.7**)**. It is then a $(p-1)$-fold original point of the pair (s, t) at the mapping $t^n_{n-3,0}$ ($1 < p \leqslant n-1$).

By 3.1, t is a double image point of the $(p-1)$-fold original point s at the mapping $t^n{}_{n-2}$. Hence by **(6**, 4.3**)**, exactly one of the mappings (1) will be positive near s.

3.43. Let s be a p-fold original point of the simple image point t at the mapping $t^n{}_{n-2}$ $(1 < p < n)$. By 3.1 and 3.32, t is then a double image point of the $(p-1)$-fold original point s at the mapping $t^n{}_{n-3,0}$. In a small neighbourhood of s, the mapping $t^n{}_{n-2}$ exists and $t^n{}_{n-1}$ is therefore negative. If p is even, one of the two mappings ${}_it^n{}_{n-2}$, viz. the one that maps s onto t, changes its direction at s. The other one will be negative at s. Thus there are two one-sided neighbourhoods of s such that $t^n{}_{n-3,0}$ is defined everywhere in one of them and nowhere in the other.

If p is odd, then the mappings $t^n{}_{n-2}$ are monotonic at s and $t^n{}_{n-3,0}$ is defined near s if and only if one of them is positive at s; cf. **(6**, 4.2**)**.

3.44. *The points at which the mappings $t^n{}_{n-2}$ change their direction*, i.e. the original points with even multiplicities of simple image points, *decompose K^{n+1} into a bounded number of arcs A*; cf. **(6**, 5.4**)**. Thus they are monotonic at any interior point of A where they exist. By 3.42 and **(6**, 5.2**)**, the number of points of the set $t^n{}_{n-3}(s)$ is constant on A. It is equal to three or one depending on whether all the mappings (1) are negative on A or not. This implies: *The mapping $t^n{}_{n-3}$ is either everywhere single-valued or everywhere triple-valued on an arc A. In the first case, the points of A*, including its end points, *are at most simply singular* **(**cf. **6**, 4.4**)**, *and the mapping $t^n{}_{n-3,0}$ is defined on A*. By 3.42, *every fixed point of $t^n{}_{n-3,0}$ lies in the interior of such an arc A.*

In the second case, A may contain original points s of odd multiplicity > 1 of simple image points t at the mapping $t^n{}_{n-2}$. Then $t^n{}_{n-3,0}(s) = \{t, t\}$. But $t^n{}_{n-3,0}$ is not defined elsewhere in A.

3.5. We continue the discussion of the first case of 3.44 and begin the proof that *$E(s)$ and $t^n{}_{n-3,0}$ are continuous on A;* cf. 3.3 and 3.8.

Suppose the sequence of points $s_\lambda \in A$ converges to s_0. For all but a finite number of indices, the points

$$s_\lambda, \qquad {}_1t_\lambda = {}_1t^n{}_{n-3,0}(s_\lambda), \quad \text{and} \quad {}_2t = {}_2t^n{}_{n-3,0}(s_\lambda)$$

are mutually distinct. Thus $E(s_\lambda)$ is the $(n-2)$-space through $L^n{}_{n-3}(s_\lambda)$, ${}_1s_\lambda$, and ${}_2s_\lambda$. We may assume that the points ${}_it_\lambda$ are convergent, say to ${}_it$, and that $E(s_\lambda)$ converges, say to E. By 2.5, E is abundant and contains $L^n{}_{n-3}(s_0)$, ${}_1t$, and ${}_2t$; cf. 2.3. Since s_0 lies in the closure of A, it is at most simply singular and the points s_0, ${}_1t$, and ${}_2t$ are not all equal to each other. If $L^n{}_{n-3}(s_0)$ does not meet K^{n+1} elsewhere, then E is projected from $L^n{}_{n-3}(s_0)$ into a special 0-space. Thus

$$E = E(s_0) \quad \text{and} \quad \{{}_1t, {}_2t\} = t^n{}_{n-3,0}(s_0).$$

If $L^n{}_{n-3}(s_0)$ meets K^{n+1} at a second point t_0, then one of the points ${}_it$ is equal to

https://doi.org/10.4153/CJM-1967-095-7 Published online by Cambridge University Press

t_0. We shall show in 3.8 that the other point is equal to s_0 and shall thus complete our continuity proof.

3.6. In this section we prepare the discussion of the double image points and the fixed points of $t^n{}_{n-3,0}$.

Suppose $L^n{}_{n-3}(s)$ and s' span the regular $(n-2)$-space P; cf. 2.3. Then K^{n+1} is projected from $L^n{}_{n-3}(s)$ and s' into a K^3 and K^n, respectively. P is projected into the regular subspaces $L^2{}_0(s')$ and $L^{n-1}{}_{n-3}(s)$, respectively. Let $t^{n-1}{}_m$ $(t^2{}_m)$ denote the mappings of K^n (K^3). Then

$$t^2{}_0(s') = t^{n-1}{}_{n-3}(s). \tag{2}$$

3.61. If $t^{n-1}{}_{n-2}$ is positive at s, then (2) is void; cf. **(6,** 5.2**)**. Hence $t^2{}_1$ is positive at s' and K^3 has a double point $t^n{}_{n-3,0}(s)$. Its two points separate s' from the inflection point $t^n{}_{n-3}(s)$ of K^3 and from the point $t^n{}_{n-1}(s) = t^2{}_1(s)$; cf. 3.2.

3.62. Suppose $t^{n-1}{}_{n-2}$ is negative at s and the points

$$_it^n{}_{n-3,0}(s) \qquad (i = 1, 2) \tag{3}$$

exist and are distinct. Then the two points (2) exist. By 3.2, $t^2{}_1$ is negative at s' and the pairs (2) and (3) alternate. The points (3) separate s from s' if and only if $t^2{}_1$ is positive at s. By **(4,** 3.42**)**, this is equivalent to $t^n{}_{n-1}$ being positive at s.

3.7. The double image points. *Let*

$$t^n{}_{n-3,0}(s_0) = \{t_0, t_0\}.$$

Let B be a sufficiently small one-sided neighbourhood of s_0. Suppose $t^n{}_{n-3,0}$ is defined in B. Then the two points $t^n{}_{n-3,0}(s)$ converge to t_0 from opposite sides as s tends to s_0.

Proof. By 3.32, t_0 is a double image point of s_0 at the mapping $t^n{}_{n-3}$. Projection from $L^n{}_{n-3}(s_0)$ shows that s_0 has a third image point $_1t^n{}_{n-3}(s_0) \neq t_0$ at this mapping. By 3.4, $t^n{}_{n-3}$ is single valued on B. It then follows from **(6,** 4.3**)** that the mapping $t^{n-1}{}_{n-2}$ of the projection of K^{n+1} from t_0 is positive on B, and the point $t^n{}_{n-3}(s)$ converges to $_1t^n{}_{n-3}(s_0)$ as s tends to s_0 on B. Applying 3.61 to s and $s' = t_0$, we obtain that the points $t^n{}_{n-3,0}(s)$ separate t_0 from $t^n{}_{n-3}(s)$. By 3.5 they converge to t_0 as s approaches s_0. This yields our statement.

3.8. The fixed points of $t^n{}_{n-3,0}$. *Let $t_0 \in L^n{}_{n-2}(s_0)$, $t_0 \neq s_0$.* Thus

$$t^n{}_{n-3,0}(s_0) = \{s_0, t_0\}; \qquad \text{cf. 3.33.}$$

By 3.44, *the mapping $t^n{}_{n-3,0}$ is defined in a small neighbourhood B of s_0.* On account of 3.5, *we may assume that, for example, $_2t^n{}_{n-3,0}(s)$ converges to t_0 as s tends to s_0.*

On the projection K^n of K^{n+1} from t_0, the point s_0 is at least twice singular. Hence the three mappings

$$t^{n-1}{}_{n-2} \quad \text{and} \quad t^{n-1}{}_{n-3}$$

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of K^n exist on B and have a fixed point at s_0. They are either negative on B or improper.

Let $s \in B$, $s \neq s_0$. By 3.62, the pairs $t^n{}_{n-3,0}(s)$ and $t^{n-1}{}_{n-3}(s)$ separate one another, i.e. ${}_1t^n{}_{n-3,0}(s)$ lies between the two points $t^{n-1}{}_{n-3}(s)$. If s converges monotonically to s_0, the points $t^{n-1}{}_{n-3}(s)$ converge to s_0 from the opposite direction. Hence the same applies to ${}_1t^n{}_{n-3,0}(s)$. Thus ${}_1t^n{}_{n-3,0}$ *is continuous and negative at its fixed point* s_0. This completes, in particular, the proof of the continuity of $t^n{}_{n-3,0}$.

The points ${}_2t^n{}_{n-3,0}(s)$ and $t^n{}_{n-1}(s)$ lie near t_0. By 3.62, s and t_0 are separated by the points $t^n{}_{n-3,0}(s)$ if and only if $t^n{}_{n-1}$ is positive at s. This is the case if and only if it is positive between s_0 and s, i.e. if the pairs $\{s_0, t_0\}$ and $\{s, t^n{}_{n-1}(s)\}$ separate one another. Since s and ${}_1t^n{}_{n-3,0}(s)$ lie on opposite sides of s_0, we obtain first that the points s_0 and t_0 are separated by the pair $t^n{}_{n-3,0}(s)$ if and only if they are separated by s and $t^n{}_{n-1}(s)$, and then that the points

(4) $${}_2t^n{}_{n-3,0}(s) \quad \text{and} \quad t^n{}_{n-1}(s)$$

lie on opposite sides of t_0. Hence *the two points* (4) *converge to* t_0 *from opposite directions as* s *tends to* s_0. We can now deduce from **(4**, 3.7**)** that ${}_2t^n{}_{n-3,0}$ *changes its direction at* s_0 *if and only if* s_0 *is an original point of odd multiplicity;* cf. 3.33.

3.9. Since the numbers $N^n{}_{pq}$ $(p + q < n - 1)$ are finite, the mapping $t^n{}_{n-3,0}$ has only a finite number of fixed points and double image points; cf. the introduction and 3.3. We wish to show that *the numbers*

$$N^n{}_{p00} \qquad (0 \leqslant p < n - 3),$$

are finite. Thus this mapping has altogether only a finite number of multiple original points; cf. 5.1.

Suppose our assertion was false. Then there exists a convergent sequence of multiple original points $s_\lambda \to s_0$. We may assume that the points s_λ, $t^n{}_{n-3,0}(s_\lambda)$ are mutually distinct. Since $t^n{}_{n-3,0}$ is continuous, the pairs $t^n{}_{n-3,0}(s_\lambda)$ converge to

$$\{{}_1t_0, {}_2t_0\} = t^n{}_{n-3,0}(s_0).$$

At least one of the ${}_it_0$'s, say ${}_1t_0$, is distinct from s_0. Project K^{n+1} from ${}_1t_0$ into a K^n. Since the numbers $N^{n-1}{}_{p0}$ are finite for $p \leqslant n - 3$, we can have

(5) $${}_1t^n{}_{n-3,0}(s_\lambda) \in L^{n-1}{}_{n-3}(s_\lambda)$$

only a finite number of times. Thus we may assume that (5) does not occur. Hence

(6) $${}_1t_0 \notin t^n{}_{n-3,0}(s_\lambda) \quad \text{and} \quad t^{n-1}{}_{n-4,0}(s_\lambda) = t^n{}_{n-3,0}(s_\lambda)$$

for all λ's. In particular, the points s_λ, $t^{n-1}{}_{n-4,0}(s_\lambda)$ are mutually distinct. Since $t^{n-1}{}_{n-4,0}$ is continuous, we obtain from (6)

$$t^{n-1}{}_{n-4,0}(s_0) = t^n{}_{n-3,0}(s_0).$$

By 2.4, s_0 is a multiple original point of $t^n{}_{n-3,0}$.

If ${}_2t_0 = s_0$, then ${}_1t_0 \in L^n{}_{n-3}(s_0)$ and s_0 would be a multiple singular point of K^n. Since the mapping $t^{n-1}{}_{n-4,0}$ is proper by (6), this is impossible; cf. 3.3.

Suppose then that ${}_2t_0 \neq s_0$. If ${}_1t_0 \neq {}_2t_0$, then $L^n{}_{n-4}(s_0)$, ${}_1t_0$, and ${}_2t_0$ lie in an $(n - 3)$-space and $\{{}_1t_0, {}_2t_0\}$ becomes a double point of the projection K^4 of K^{n+1} from $L^n{}_{n-4,0}(s_0)$. The projection of K^n from $L^{n-1}{}_{n-4}(s_0)$ is identical with that of K^4 from ${}_1t_0$. Thus it is a C^3; cf. 2.4. But, $t^{n-1}{}_{n-4,0}$ being proper, this is not possible for ${}_2t_0 \neq s_0$.

The case ${}_1t_0 = {}_2t_0$ is similar.

4. The direction of $t^n{}_{n-3,0}$. The discussion of the directions of the mappings ${}_it^n{}_{n-3,0}$ will be based on the decomposition of K^{n+1} into the arcs A introduced in 3.44.

4.1. We start out with the case $n = 3$ and assume that K^4 has neither a double point nor a cusp. Thus the three points

$$(1) \qquad s, \quad t^3{}_{00}(s) = \{{}_1t^3{}_{00}(s), {}_2t^3{}_{00}(s)\}$$

lie on a special straight line. Obviously, the relation between them is symmetric, and no point of K^4 lies on more than one such line.

If the points (1) are mutually distinct, the mappings ${}_it^3{}_{00}$ are locally one-to-one and hence monotonic. Their fixed points are those points whose tangents meet K^4 again. Thus they are identical with the points where the mapping $t^3{}_2$ changes its direction. The end points of the arcs A, i.e. the points where one of the mappings $t^3{}_1$ changes its direction, are those points which lie on the tangents of other points.

If $t^3{}_2$ is negative on the entire K^4, then it has fixed points. They are the singular points of K^4 and the fixed points of the mappings $t^3{}_1$. The latter are defined and monotonic everywhere. In fact, being negative at their fixed points, they are monotonically negative, and the two points ${}_it^3{}_1(s)$ are distinct outside a double singular point. By **(6**, 5.2**)**, the mapping $t^3{}_0$ is triple-valued on K^4; it is the inverse of $t^3{}_2$. By 3.4, $t^n{}_{n-3,0}$ is nowhere defined. For such a K^4, we have

$$\sum_0^3 (3 - m)N^n{}_m = 4.$$

If $t^3{}_2$ is monotonically positive, then $t^3{}_{00}$ is defined on the whole curve. Since no tangent meets K^4 three times, the mappings ${}_it^3{}_{00}$ are monotonic. Having no fixed points, they are monotonically positive.

Suppose the mappings $t^3{}_2$ are not monotonic. Then there are points s_0 whose tangents meet the curve again, say at t_0. If s passes monotonically through s_0, then one of the points $t^3{}_{00}(s)$ moves through s_0 in the opposite sense while the other one changes its direction at t_0; more accurately, it is separated from $t^3{}_2(s)$ by t_0. If s moves from t_0 into an arc A, then the two points $t^3{}_{00}(s)$ move from s_0 in opposite directions; cf. 3.7 and 3.8. Since the number of the points s_0, t_0 is finite and since the mappings ${}_it^3{}_{00}$ are monotonic elsewhere, we obtain: If $t^3{}_2$

is negative and one of the mappings $t^3{}_1$ is positive, then the two mappings $t^3{}_{00}$ have opposite directions.

4.2. Suppose the mapping $t^n{}_{n-3,0}$ is defined at s_0 and proper, and the three points s_0, $t^n{}_{n-3,0}(s_0)$ are mutually distinct. Then there exists a closed arc B which contains s_0 in its interior and a neighbourhood C of s_0 with the following property: If $s \in C$, $s \neq s_0$, then the mappings $t^3{}_{00}$ of the projection of K^{n+1} from $L^n{}_{n-4}(s)$ are defined in B and proper. In B, they are monotonic and without fixed points.

Proof. By our assumptions, $L^n{}_{n-2}(s_0)$ is regular. Hence $L^n{}_{n-4}(s_0)$ is so too and the projection of K^{n+1} from $L^n{}_{n-4}(s_0)$ is a curve K^4. The abundant $(n - 2)$-spaces through $L^n{}_{n-4}(s_0)$ are projected into abundant straight lines; cf. 2.3.

By our assumptions, the points

$$s_0, \qquad t^3{}_{00}(s_0) = t^n{}_{n-3,0}(s_0)$$

are mutually distinct. Thus $L^3{}_1(s_0)$ is regular and any abundant straight line meets K^4 outside s_0 at least twice. Since K^4 has only a finite number of abundant tangents, we obtain: All of the abundant $(n - 2)$-spaces through $L^n{}_{n-4}(s_0)$ meet K^{n+1} outside s_0 at least twice; only a finite number of them meet K^{n+1} at only two points $\neq s_0$. We choose the closed neighbourhood B such that it contains none of these points.

Let C be a sufficiently small neighbourhood of s_0. By 3.44 and 3.8, the mapping $t^n{}_{n-3,0}$ is defined and continuous in C and proper. Thus we may assume that the points

$$s, \qquad t^n{}_{n-3,0}(s)$$

are mutually distinct for all $s \in C$. On account of 3.9, we may assume that each s is a simple original point of $t^n{}_{n-3,0}$. If we project K^{n+1} from the regular subspace $L^n{}_{n-4}(s)$, then

$$t^n{}_{n-3,0}(s) = t^3{}_{00}(s)$$

will be a proper pair of distinct image points of s. Thus the mapping $t^3{}_{00}$ is defined near s and proper and the three points s', $t^3{}_{00}(s')$ are mutually distinct if s' is near s.

Suppose our assertion were false for B and for every choice of C. Then there would exist a sequence of points $s_\lambda \to s_0$, $s_\lambda \neq s_0$, such that the projection of K^{n+1} from each $L^n{}_{n-4}(s_\lambda)$ would possess abundant tangents which would meet this projection in B. Hence each $L^n{}_{n-4}(s_\lambda)$ would lie in some abundant $(n - 2)$-space which would meet K^{n+1} in not more than two points outside s_λ such that at least one of them would lie in B. A limit space of these $(n - 2)$-spaces would be an abundant $(n - 2)$-space through $L^n{}_{n-4}(s_0)$ and not more than two points distinct from s_0, at least one of which would lie in B. Such subspaces have been excluded by our construction of B.

4.3. *Suppose $t^n{}_{n-3,0}$ is defined at s_0 and proper, and the three points*

$$s_0, \qquad t^n{}_{n-3,0}(s_0)$$

are mutually distinct and non-collinear. Let $t^{n-1}{}_{n-4,0}$ *denote the mapping of the projection* K^n *of* K^{n+1} *from* s_0. Thus

$$t^{n-1}{}_{n-4,0}(s_0) = t^n{}_{n-3,0}(s_0),$$

$t^{n-1}{}_{n-4,0}$ is proper, and the mappings

$$(2) \qquad t^n{}_{n-3,0} \quad \text{and} \quad t^{n-1}{}_{n-4,0}$$

are defined and continuous near s_0. We label them such that

$$_it = {}_it^n{}_{n-3,0}(s_0) = {}_it^{n-1}{}_{n-4,0}(s_0) \qquad (i = 1, 2).$$

Then the point ${}_it^{n-1}{}_{n-4,0}(s)$ *lies between* ${}_it$ *and* ${}_it^n{}_{n-3,0}(s)$ *for every* s *sufficiently close to* s_0 $(s \neq s_0)$. *In particular,*

$$_it^n{}_{n-3,0} \quad \textit{and} \quad {}_it^{n-1}{}_{n-4,0}$$

have the same direction at s_0.

Proof. Choose the neighbourhoods B and C of s_0 according to 4.2. We may assume that ${}_1t$ and ${}_2t$ do not lie in B, that $C \subset B$, and that the mappings (2) exist in C. Construct small neighbourhoods C_i about ${}_it$ such that B, C_1, C_2 are mutually disjoint and make C so small that

$$\{{}_it^n{}_{n-3,0}(s),\ {}_it^{n-1}{}_{n-4,0}(s)\} \subset C_i \qquad \text{for all } s \in C;\ i = 1, 2.$$

If we project K^{n+1} from ${}_1t$, then ${}_2t \in L^{n-1}{}_{n-3}(s_0)$. Thus the mapping $\bar{t}^{n-1}{}_{n-4,0}$ of this projection is proper and

$$\bar{t}^{n-1}{}_{n-4,0}(s_0) = \{s_0, {}_2t\}.$$

The mapping $\bar{t}^{n-1}{}_{n-4,0}$ is defined at any point s_1 sufficiently close to s_0. One of the points $\bar{t}^{n-1}{}_{n-4,0}(s_1)$, say ${}_1\bar{t}$, lies in B and is separated from s_1 by s_0 while the other one, ${}_2\bar{t}$, lies in C_2; cf. 3.8. We finally make C so small that this is the case for every $s_1 \in C$, $s_1 \neq s_0$, and that $L^n{}_{n-4}(s_1)$ and s_0, as well as $L^n{}_{n-4}(s_1)$ and ${}_1t$, span regular $(n - 3)$-spaces for all these s_1.

Let $s_1 \in C$ now be fixed $(s_1 \neq s_0)$. Let $t^3{}_{00}$ denote the mapping of the projection of K^{n+1} from $L^n{}_{n-4}(s_1)$. Thus

$$t^3{}_{00}(s_1) = t^n{}_{n-3,0}(s_1).$$

Choose the notation such that

$$_it^3{}_{00}(s_1) = {}_it^n{}_{n-3,0}(s_1) \qquad (i = 1, 2).$$

By our construction

$$(3) \qquad t^3{}_{00}({}_1\bar{t}) = \{{}_1t, {}_2\bar{t}\} \quad \text{and} \quad t^3{}_{00}(s_0) = t^{n-1}{}_{n-4,0}(s_1).$$

Let s move on B from s_1 to ${}_1\bar{t}$. Then the two points $t^3{}_{00}(s)$ depend continuously and monotonically on s, and the three points s, $t^3{}_{00}(s)$ remain mutually distinct;

cf. 4.2. Hence their order on the oriented curve remains unchanged and (3) implies that

$$ {}_1t_{00}({}_1\bar{t}) = {}_1t, \qquad {}_2t_{00}({}_1\bar{t}) = {}_2\bar{t}. $$

Since s_0 lies between s_1 and ${}_1\bar{t}$, the point ${}_it^3{}_{00}(s_0)$ lies between ${}_it^3{}_{00}(s_1)$ and ${}_it^3{}_{00}({}_1\bar{t})$ In particular, it lies in C_i. Hence, (3) yields

$$ {}_it^3{}_{00}(s_0) = {}_it^{n-1}{}_{n-4,0}(s_1) \qquad (i = 1, 2); $$

and the point ${}_1t^{n-1}{}_{n-4,0}(s_1)$ lies between the points

$$ {}_1t^3{}_{00}(s_1) = {}_1t^n{}_{n-3,0}(s_1) \quad \text{and} \quad {}_1t^3{}_{00}({}_1\bar{t}) = {}_1t. $$

Since ${}_2t^{n-1}{}_{n-4,0}(s_1)$ lies between

$$ {}_2t^3{}_{00}(s_1) = {}_2t^n{}_{n-3,0}(s_1) \quad \text{and} \quad {}_2t^3{}_{00}({}_1\bar{t}) = {}_2\bar{t}^{n-1}{}_{n-4,0}(s_1), $$

we note that ${}_2t^n{}_{n-3,0}(s_1)$ *and* ${}_2\bar{t}^{n-1}{}_{n-4,0}(s_1)$ *lie on the same side of* ${}_2t$.

4.4. Combining the last remark with 3.8, we readily obtain conditions for $t^n{}_{n-3,0}$ to change its directions. But the following discussion will yield more detailed information.

Suppose $t^n{}_{n-3,0}$ *is defined at* s_0 *and proper, and the three points*

$$ s_0, \qquad t^n{}_{n-3,0}(s_0) \tag{4} $$

are mutually distinct. Using 4.1 and 4.3, we readily verify by induction that the mappings ${}_it^n{}_{n-3,0}$ are monotonic at s_0 if s_0 is a simple original point. Thus they can change their directions only at multiple original points or at those points s_0 where two of the points (4) coincide; cf. 3.7 and 3.8. By 3.9, the number of these points is finite. We prove:

Let s *lie sufficiently close to* s_0. *Then the two pairs of points*

$$ t^n{}_{n-3,0}(s) \quad \textit{and} \quad t^n{}_{n-3,0}(s_0) $$

alternate if and only if the mapping $t^n{}_{n-1}$ *is positive at* s_0. This implies: *If the mappings* ${}_it^n{}_{n-3,0}$ *are monotonic at* s_0, *then they have the same or opposite directions depending on whether* $t^n{}_{n-1}$ *is positive or negative at* s_0. *If one of the two changes its direction at* s_0, *then so does the other.*

By 4.1, our assertion is true for $n = 3$. Suppose it has been proved up to $n - 1$. Choose a small neighbourhood B of s_0 with the following properties: The mappings $t^n{}_{n-3,0}$ are defined in B, and the points

$$ s, \quad t^n{}_{n-3,0}(s) \qquad (s \in B) $$

are mutually distinct; with the possible exception of s_0, B contains no multiple original points. Thus these mappings are monotonic on the two subarcs into which B is decomposed by s_0.

Let $s_1 \in B$, $s_1 \neq s_0$; and let $t^{n-1}{}_*$ denote mappings of the projection of K^{n+1} from s_1. Choose s between s_0 and s_1 sufficiently close to s_1. By our induction assumption, the pairs of points

$$ t^{n-1}{}_{n-4,0}(s) \quad \text{and} \quad t^{n-1}{}_{n-4,0}(s_1) = t^n{}_{n-3,0}(s_1) $$

alternate if and only if $t^{n-1}{}_{n-2}$ is positive at s_1. By **(4**; 3.42**)**, this mapping and $t^n{}_{n-1}$ have the same direction at s_1. Since B contains no multiple original points of the latter, it is monotonic in B; cf. **(4**; 3.7**)**. Thus $t^{n-1}{}_{n-2}$ is positive at s_1 if and only if $t^n{}_{n-1}$ is positive at s_0. Since s_1 is a simple original point of $t^n{}_{n-3,0}$, 4.3 implies that the two points

$$ {}_it^{n-1}{}_{n-4,0}(s) \quad \text{and} \quad {}_it^n{}_{n-3,0}(s) $$

lie on the same side of ${}_it^n{}_{n-3,0}(s_1)$ $(i = 1, 2)$. Altogether, the pairs

$$ t^n{}_{n-3,0}(s) \quad \text{and} \quad t^n{}_{n-3,0}(s_1) $$

alternate if and only if $t^n{}_{n-1}$ is positive at s_0. The mappings ${}_it^n{}_{n-3,0}$ being monotonic between s_0 and s_1, we can now drop the restriction that s be close to s_1. Letting s tend to s_0, we obtain our statement.

4.5. Suppose the points

$$ (5) \qquad s_0, \qquad {}_1t = {}_1t^n{}_{n-3,0}(s_0), \qquad {}_2t = {}_2t^n{}_{n-3,0}(s_0) $$

are mutually distinct but collinear. The following remark is a substitute for 4.3: Let B be a closed neighbourhood of s_0 which does not contain ${}_1t$ and ${}_2t$. Suppose the neighbourhood C of s_0 is sufficiently small $(s_1 \in C,\ s_1 \neq s_0)$. Then the mappings $t^{n-1}{}_{n-4,0}$ of the projection of K^{n+1} from s_1 are defined on B and proper. They are monotonic outside s_0 and the three points

$$ (6) \qquad s, \qquad t^{n-1}{}_{n-4,0}(s) $$

are mutually distinct.

Proof. By 3.4 and 3.8, the mapping $t^n{}_{n-3,0}$ is defined and continuous in a neighbourhood C of s_0. By 3.9, we may choose C so small that it contains no multiple original points $\neq s_0$. Thus $t^{n-1}{}_{n-4,0}$ will be defined at s_1 and proper.

Suppose there is a sequence of points $s_1 \to s_0$, $s_1 \neq s_0$ and to each s_1 a point $s \in B$ such that two of the points (6) are identical. Thus to each s_1 of this sequence there exists an abundant $(n - 2)$-space through s_1 and through not more than two other points of K^{n+1}, not more than one of them lying outside B. Letting s_1 tend to s_0, we obtain an abundant $(n - 2)$-space through s_0 with the same property. It is projected from s_0 into an abundant $(n - 3)$-space F which meets the projection K^n of K^{n+1} from s_0 at most once outside B. Since the points ${}_1t$, ${}_2t$ are projected into a double point of K^n and lie outside B, F cannot contain the double point. The $(n - 2)$-flat through F and that point would meet K^n not less than $[(n - 1) + 2]$-times.

We can now choose $C \subset B$ so small that for every point $s_1 \in C$, $s_1 \neq s_0$, and for every $s \in B$, the points (6) are mutually distinct if they exist. But 3.4 implies now that $t^{n-1}{}_{n-4,0}$ is defined not only at s_1 but in the whole of B.

It remains to be shown that the mappings ${}_it^{n-1}{}_{n-4,0}$ are monotonic outside s_0. Let $s \in B$, $s \neq s_0$. If one of our mappings would change its direction at s, then s would be a multiple original point. Thus $L^n{}_{n-5}(s)$, s_1, and the two points

${t^{n-1}}_{n-4,0}(s)$ would lie in an abundant $(n - 3)$-space. It is projected from ${L^n}_{n-5}(s)$ onto an abundant straight line through s_1 and the points ${}_i{t^{n-1}}_{n-4,0}(s)$. This line and the straight line through the projections of the points (5) are distinct. They span a subspace of dimension $\leqslant 3$. It would meet the projection K^5 of K^{n+1} at least six times (at least five times) if its dimension were three (were two).

4.6. *Suppose the mappings ${t^n}_{n-3,0}$ are defined at s_0 and proper and the three points*

(7) $$s_0, \qquad {}_it = {}_i{t^n}_{n-3,0}(s_0) \quad (i = 1, 2)$$

are mutually distinct. Then these mappings change their directions at s_0 if and only if the multiplicity of s_0 is even.

Let s_0 be a p-fold original point. Thus ${L^n}_{n-p-2}(s_0)$ and the ${}_it$ span a special subspace. Projecting from ${L^n}_{n-p-3}(s_0)$ and making use of 4.3, we reduce our assertion to the case $p = n - 2$. Thus we may assume that the points (7) are collinear, and we have to show that the mappings ${t^n}_{n-3,0}$ change their directions if and only if n is even.

We choose a closed neighbourhood B and a neighbourhood $C \subset B$ of s_0 such that ${t^n}_{n-3,0}$ is defined in C and monotonic outside s_0 and that we can apply 4.2 and 4.5. It is sufficient to prove: Let C be sufficiently small. Then if s' and s'' lie in C and are separated by s_0, the points

$${}_it' = {}_i{t^n}_{n-3,0}(s') \quad \text{and} \quad {}_it'' = {}_i{t^n}_{n-3,0}(s'')$$

lie on the same side of ${}_it$ if and only if n is even $(i = 1, 2)$.

Let ${t^3}_{00}$ and ${t^{n-1}}_{n-4,0}$ denote the mappings of the projections of K^{n+1} from ${L^n}_{n-4}(s')$ and s'', respectively. We number them such that

(8) $${}_i{t^3}_{00}(s_0) = {}_i{t^{n-1}}_{n-4,0}(s_0) = {}_it \qquad (i = 1, 2).$$

Obviously,

$${t^3}_{00}(s') = {t^n}_{n-3,0}(s') \quad \text{and} \quad {t^{n-1}}_{n-4,0}(s'') = {t^n}_{n-3,0}(s'').$$

By 4.2, ${L^n}_{n-4}(s')$ and s'' span a regular $(n - 3)$-space. Hence

$${t^{n-1}}_{n-4,0}(s') = {t^3}_{00}(s'').$$

If s moves on C from s_0 to s' or to s'', then the points ${t^3}_{00}(s)$ and ${t^{n-1}}_{n-4,0}(s)$ move continuously, and s and its image points remain mutually distinct. Hence, their order on the oriented curve remains unchanged. Therefore

(9) $${}_i{t^3}_{00}(s) = {}_it', \qquad {}_i{t^{n-1}}_{n-4,0}(s'') = {}_it'',$$

and

(10) $${}_i{t^{n-1}}_{n-4,0}(s') = {}_i{t^3}_{00}(s'') \qquad (i = 1, 2).$$

Choose small neighbourhoods of the points ${}_it$ and make C so small that the points (9) lie in these neighbourhoods $(i = 1, 2)$. If s' and s'' converge to s_0,

https://doi.org/10.4153/CJM-1967-095-7 Published online by Cambridge University Press

then the abundant $(n-2)$-space through $L^n_{n-4}(s')$, s'', and the two points (10) will have an abundant limit space through $L^n_{n-3}(s_0)$, i.e. it will converge to the $(n-2)$-space through $L^n_{n-3}(s_0)$, ${}_1t$, and ${}_2t$. Hence the pair $t^3{}_{00}(s'')$ converges to the pair $\{{}_1t, {}_2t\}$. Since the triples s'', $t^3{}_{00}(s'')$ and s_0, $t^3{}_{00}(s_0)$ have the same order on the oriented curve, the points (10) must converge to ${}_it$. Hence we may choose C so small that ${}_it^3{}_{00}(s'')$ lies in the neighbourhood of ${}_it$ $(i = 1, 2)$.

For $n = 3$, our assertion follows from 4.1. Suppose it is proved up to $n - 1$. Thus the points ${}_it^{n-1}{}_{n-4,0}(s')$ and ${}_it^{n-1}{}_{n-4,0}(s'')$, i.e. the points ${}_it^3{}_{00}(s'')$ and ${}_it''$, lie on the same side of ${}_it$ if and only if $n - 1$ is even. The mappings $t^3{}_{00}$ being monotonic in C, the points ${}_it^3{}_{00}(s') = {}_it'$ and ${}_it^3{}_{00}(s'')$ are separated by ${}_it^3{}_{00}(s_0) = {}_it$; cf. (8). Combining these two observations, we obtain our assertion.

5. Some global properties of the mapping $t^n{}_{n-3,0}$.

5.1. *The numbers $N^n{}_{p00}$ are bounded for given n* $(0 \leqslant p \leqslant n-4)$.

Trivially, $N^4{}_{000} \leqslant 1$. Suppose our statement has been proved up to $n - 1$. By 3.9, the numbers $N^n{}_{p00}$ were finite. Project K^{n+1} from a point which is neither a fixed point nor a multiple original point of $t^n{}_{n-3,0}$. If s is such a multiple original point, then

$$(1) \qquad t^{n-1}{}_{n-4,0}(s) = t^n{}_{n-3,0}(s).$$

Our projection has decreased the multiplicity of s by one.

These points s decompose K^{n+1} into a finite number of arcs B. We divide the set of these arcs into two classes. B shall belong to the first class if and only if all the mappings

$$(2) \qquad {}_it^n{}_{n-3,0} \quad \text{and} \quad {}_it^{n-1}{}_{n-4,0}$$

are defined and monotonic on B and have no fixed points in B. Thus any arc B of the second class either contains multiple original points of ${}_it^{n-1}{}_{n-4,0}$ or fixed points of one of the mappings (2). By our induction assumption and the introduction, the number of these points and hence that of the arcs B of the second class is bounded.

Let B be an arc of the first class. We wish to show that the mappings

$$(3) \qquad {}_1t^n{}_{n-3,0} \quad \text{and} \quad {}_1t^{n-1}{}_{n-4,0}$$

have the same direction on B. Since exactly one of them changes its direction at an end point of B, an arc adjacent to B cannot belong to the first class; cf. 4.6. Thus the number of the arcs of the first class is not greater than that of the second and it is bounded too.

If s moves on B, the three mutually distinct points s and $t^n{}_{n-3,0}(s)$ (s and $t^{n-1}{}_{n-4,0}(s)$) move continuously on the curve. Hence their order on the oriented curve remains unchanged. Furthermore, (1) holds true if s is equal to one of the end points s' and s'' of B. Hence the mappings (3) can be labelled such that

$${}_it^n{}_{n-3,0}(s) = {}_it^{n-1}{}_{n-4,0}(s) \qquad \text{for } s = s' \text{ and } s = s''; i = 1, 2.$$

If s moves from s' through B to s'', the points

$$ {}_1t^n{}_{n-3,0}(s) \quad \text{and} \quad {}_1t^{n-1}{}_{n-4,0}(s) $$

move continuously and monotonically from a common initial point to a common end point. Since neither mapping has a fixed point, they must be monotonic in the same direction. This completes our proof.

5.2. The mapping $t^n{}_{n-3}$ was extended in **(6**; 5.3**)** to a mapping $\bar{t}^n{}_{n-3}$ which was triple-valued and continuous on the whole curve. The additional image points were the positive image points of the mappings $t^n{}_{n-1}$ and $t^n{}_{n-2}$, each of them counted twice.

On account of 3.4, *we can complete* $t^n{}_{n-3}$ *in another fashion to a mapping* $\tilde{t}^n{}_{n-3}$ *which is triple-valued and continuous everywhere.* Define $\tilde{t}^n{}_{n-3}(s) = t^n{}_{n-3}(s)$ if $t^n{}_{n-3}(s)$ consists of three points. If $t^n{}_{n-3}$ is single-valued at s, define

$$ \tilde{t}^n{}_{n-3}(s) = \{t^n{}_{n-3}(s),\, t^n{}_{n-3,0}(s)\}. $$

This mapping could be discontinuous only when the number of image points of $t^n{}_{n-3}$ changes. That is the case at s_0 if and only if the multiplicity of s_0 is odd and that of its image point t_0 is two. But if s converges monotonically to s_0, then either two points ${}_it^n{}_{n-3}(s)$ or two points ${}_it^n{}_{n-3,0}(s)$ converge to t_0 from opposite directions; cf. 3.8 and **(6**; 4.2 and 4.3**)**. Thus these points s_0 are exactly those points where pairs $t^n{}_{n-3,0}(s)$ change into pairs $t^n{}_{n-3}(s)$ and vice versa. Thus $\tilde{t}^n{}_{n-3}$ remains continuous at s_0. If s passes through s_0, two of the points $\tilde{t}^n{}_{n-3}(s)$ move through t_0 monotonically in opposite directions.

The improper image points of $\tilde{t}^n{}_{n-3}$ are the cusps (counted twice), the $(n-1)$-times singular points, and the double points. By the introduction, each point of K^{n+1} is the proper image point of a bounded number of points.

The proper fixed points of $\tilde{t}^n{}_{n-3}$ are the multiple original points of $t^n{}_{n-1}$, each of them counted once, and the singular points; cf. 3.8. Simple singular points and cusps have to be counted once; the twice or $(n-1)$-times singular points are counted twice; any other singular point is counted three times; cf. **(6**; 3.1**)**. By **(6**; 4.4**)**, a q-fold fixed point of $\tilde{t}^n{}_{n-3}$ is the fixed point of q different mappings ${}_it^n{}_{n-3}$, $q > 1$. Each mapping ${}_i\tilde{t}^n{}_{n-3}$ is negative at a fixed point.

5.3. *Suppose t is not a fixed point of the mapping* $\tilde{t}^n{}_{n-3}$. *Then the number of the negative original points of t at this mapping minus that of its positive ones is equal to*

$$ (4) \qquad N^n{}_{n-1} + 2N^n{}_{n-2} + 3\sum_{0}^{n-3} N^n{}_m + \sum_{1}^{n-2} N^n{}_{m0} + 2N^n{}_{00} - 3 \qquad (n \geqslant 3). $$

Proof. Let h denote the sum of the multiplicities of any improper images of $\tilde{t}^n{}_{n-3}$; thus

$$ h = N^n{}_1 + 2N^n{}_0 + 2N^n{}_{00}. $$

The proper part of $\tilde{t}^n{}_{n-3}$ being $(3-h)$-valued, we can uniformize it to a single-valued continuous mapping of the $(3-h)$-times covered K^{n+1} into itself.

Its fixed points decompose the covering curve into a finite number of arcs B which have no fixed points in their interiors. The new mapping still being negative at its fixed points, the number of negative original points of the point t exceeds that of its positive ones by one in each arc B which does not contain t. For each of the $3 - h$ arcs B which contain t, these two numbers are equal. Thus our difference is equal to the number of arcs B minus $(3 - h)$. But the number of these arcs is equal to the number of the fixed points of $\tilde{t}^n_{n-3}$, each of them counted with its multiplicity, i.e. it is equal to

$$N^n_{n-1} + 2N^n_{n-2} + 3\sum_{2}^{n-3} N^n_m + 2N^n_1 + N^n_0 + \sum_{1}^{n-2} N^n_{m0};$$

cf. 5.2. Subtracting $3 - h$, we obtain (4).

5.4. *Suppose the point t has no multiple original points at the mappings*

$$t^n_{n-1} \quad \text{and} \quad t^n_{n-2}. \tag{5}$$

Then the number of its negative original points at the mapping $t^n_{n-3,0}$ minus that of its positive ones is equal to

$$\sum_{1}^{n-2} N^n_{m0} + 2N^n_{00} \tag{6}$$

minus twice the number of its positive original points at the mappings (5).

Proof. Suppose first that t is in addition regular and that the original points of t at the mapping t^n_{n-3} are simple. By (**6**; 5.1), the number of negative original points of t at that mapping minus that of its positive ones is equal to

$$N^n_{n-1} + 2N^n_{n-2} + 3\sum_{0}^{n-3} N^n_m - 3$$

plus twice the number of positive original points of t at the mappings (5). Comparing this relation with 5.3, we obtain our assertion for these points, i.e. for all the points $t \in K^{n+1}$ with a finite number of exceptions.

If we assume only that the original points of t at the mappings (5) are simple, then both the difference before (6) and the expression following it are the same for t as they are for points near t. Thus our remark remains valid under these weaker assumptions.

5.5. Let $n \geqslant 3$. Suppose t has no multiple original points at the mappings (5) or at $t^n_{n-3,0}$. Project K^{n+1} from t into a K^n.

A regular point $s \neq t$ of K^n is at most simply singular on K^{n+1}. The osculating space $L^{n-1}_m(s)$ is special if and only if the subspace through t and $L^n_m(s)$ is abundant while that through t and $L^n_{m-1}(s)$ is not. Hence $L^{n-1}_m(s)$ is special if and only if either $L^n_m(s)$ or the subspace through $L^n_m(s)$ and t is special. This yields

$$\begin{cases} N^{n-1}_{n-3,0} = N^n_{n-3,0} + \text{no. of orig. pts. of } t \text{ at the mapping } t^n_{n-3,0} \text{ if } n > 3, \\ N^{n-1}_{m0} = N^n_{m0} \qquad (0 \leqslant m < n-3). \\ 2N^2_{00} = 2N^3_{00} + \text{no. of orig. pts. of } t \text{ at } t^3_{00}. \end{cases} \tag{7}$$

Hence, by 5.4,

$$(8)\quad \sum_{1}^{n-2} (n-m-1)N^n{}_{m0} + 2(n-1)N^n{}_{00}$$

$$= \left[\sum_{1}^{n-3} (n-m-2)N^n{}_{m,0} + 2(n-2)N^n{}_{00}\right] + \left[\sum_{1}^{n-2} N^n{}_{m0} + 2N^n{}_{00}\right]$$

$$= \sum_{1}^{n-3} (n-m-2)N^{n-1}{}_{m,0} + 2(n-2)N^{n-1}{}_{00}$$

$-\ 2 \times$ number of pos. orig. pts. of t at the mapping $t^n{}_{n-3,0}$

$+\ 2 \times$ number of pos. orig. pts. of t at the mappings (5).

By induction, we obtain from (8) that

$$\sum_{1}^{n-2} (n-m-1)N^n{}_{m0}$$

is even.

By (4; 3.9), we have

$$(9)\quad \sum_{0}^{n-1} (n-m)N^n{}_m = \sum_{0}^{n-2} (n-m+1)N^{n-1}{}_m + 1$$

$-\ 2 \times$ number of pos. orig. pts. of t at the mapping $t^n{}_{n-1}$.

Define

$$\sum\nolimits_n = \sum_{0}^{n-1} (n-m)N^n{}_m + \frac{1}{2}\sum_{1}^{n-2} (n-m-1)N^n{}_{m0} + (n-1)N^n{}_{00} \qquad (n = 2, 3, \ldots).$$

Then (8) and (9) yield

$$(10)\quad \sum\nolimits_n = \sum\nolimits_{n-1} + 1$$

$-$ number of pos. orig. pts. of t at the mapping $t^n{}_{n-1}$

$+$ number of pos. orig. pts. of t at the mapping $t_n{}^n{}_{-2}$

$-$ number of pos. orig. pts. of t at the mapping $t^n{}_{n-3,0}$.

If we drop the assumption that t has no multiple original points at the mapping $t^n{}_{n-3,0}$, then (7) has to be replaced by the relations:

$N^{n-1}{}_{m0} = N^n{}_{m0} +$ number of $(n-m-2)$-fold original points of t at the mappings $t^n{}_{n-3,0}$, $\quad 0 < m \leqslant n-3$;

$2N^{n-1}{}_{00} = 2N^n{}_{00} +$ number of $(n-2)$-fold original points of t at this mapping.

We then have to replace the equality signs in (8) and (10) by "$\leqslant$."

5.6. *Suppose the point t is regular, that it is a simple proper image point at all the mappings $t^n{}_m$, and that all of its original points at these mappings are simple $(m = 0, 1, \ldots, n-1)$. Thus $L^n{}_{n-2}(t)$ is regular.*

Let $3 \leqslant p < n$. Project K^{n+1} from $L^n{}_{n-p-1}(t)$ into a K^{p+1}. A multiple original point of t at the mapping $t^p{}_m$ would also be a multiple original point of t at the mapping $t^n{}_m$. Hence the original points s of t at each $t^p{}_m$ are simple. If

$$ {}_it^p{}_m(s) = {}_it^n{}_m(s) = t, $$

then ${}_it^p{}_m$ and ${}_it^n{}_m$ have the same direction at s; cf. **(6**; 4.2**)**.

We apply 5.5 to each K^{p+1} and add over $p = 3, 4, \ldots, n$, obtaining

$$
\begin{aligned}
\textstyle\sum_n \leqslant \sum_2 + (n-2) & \\
& - \text{number of pos. orig. pts. of } t \text{ at the mappings } t^n{}_2, \ldots, t^n{}_{n-1} \\
& + \text{number of pos. orig. pts. of } t \text{ at the mappings } t^n{}_1, \ldots, t^n{}_{n-2} \\
& - \text{number of pos. orig. pts. of } t \text{ at all the } t^p{}_{p-3,0} \quad (p = 3, 4, \ldots, n) \\
& = (\textstyle\sum_2 + \text{number of pos. orig. pts. of } t \text{ at the mapping } t^2{}_1) \\
& + (n-2) - \text{number of pos. orig. pts. of } t \text{ at the mapping } t^n{}_{n-1} \\
& - \text{number of pos. orig. pts. of } t \text{ at all the } t^p{}_{p-3,0} \quad (p = 3, 4, \ldots, n).
\end{aligned}
$$

By 3.2, the parenthesis is equal to 3 − number of positive image points of t at the same mapping and hence also at the mapping $t^n{}_{n-1}$. This finally yields that

(11) $\sum_n \leqslant n + 1$ − *number of positive original and image points of t at the mapping* $t^n{}_{n-1}$
− *number of pos. orig. pts. of t at all the* $t^p{}_{p-3,0}$ $(p = 3, 4, \ldots, n)$.

In particular,

$$ \boxed{\textstyle\sum_n \leqslant n + 1.} $$

Equality holds in (11) if and only if the original points of t at all the mappings $t^p{}_{p-3,0}$ are simple.

5.7. Suppose the point t satisfies the assumptions of 5.5. Let

$$ S_n = \sum_0^{n-1} (n-m)N^n{}_m + \sum_1^{n-2} (n-m-1)N^n{}_{m0} \qquad (n \geqslant 2). $$

By (8) and (9)

(12) $S_n = S_{n-1} + 1 + 2 \times$ number of pos. orig. pts. of t at the mapping $t^n{}_{n-2}$
$- 2 \times$ number of pos. orig. pts. of t at $t^n{}_{n-3,0}$
$- 2(n-1)N^n{}_{00} + 2(n-2)N^{n-1}{}_{00}$.

If t satisfies the assumptions of 5.6, we deduce, for example, from (11) that

$$ S_n \leqslant 2n + 1 - \sum_0^{n-2} (n-1-m)N^{n-1}{}_m \tag{13} $$

$- 2 \times$ number of positive *image* points of t at $t^n{}_{n-3,0}$
$- 2 \times$ number of pos. orig. pts. of t at all the $t^p{}_{p-3,0}$ $(p = 3, 4, \ldots, n)$.

Again, equality will hold if and only if the original points of t at all the mappings $t^p{}_{p-3,0}$ are simple.

We conjecture that

$$S_n \leqslant n + 1. \tag{14}$$

Trivially, $S_2 + 2N^2{}_{00} = 3$. It is not hard to prove that

$$S_3 = \begin{cases} 4 - 2N^3{}_{00} & \text{if } K^4 \text{ is homotopic to zero,} \\ 0 & \text{otherwise.} \end{cases}$$

(14) is trivial if K^{n+1} has a cusp or double point.

Let $n > 3$. With some effort, the K^{n+1} have been determined with

$$N^n{}_{10} + N^n{}_{000} > 0.$$

If $N^n{}_{10} > 0$, then

$$S_n = N^n{}_{n-1} + 2N^n{}_{n-2} + N^n{}_{n-2,0} + (n-2)N^n{}_{10} = n + 1.$$

Similarly if $N^n{}_{000} > 0$, then

$$S_n = N^n{}_{n-1} + 2N^n{}_{n-2} + N^n{}_{n-2,0} = \begin{cases} 3 & \text{if } n \text{ is even,} \\ 0 \text{ or } 4 & \text{if } n \text{ is odd.} \end{cases}$$

These results imply the formula for K^5

$$S_4 + 2N^4{}_{000} + 4N^4{}_{00} = 5.$$

An approach to (14) via (12) faces the difficulty that $S_n > S_{n-1} + 1$ can occur. In order to utilize (13), it seems that certain more general and rather difficult mappings $t^n{}_{m0}$ would have to be studied. Using these mappings the author could at least prove the finiteness of the numbers

$$N^n{}_{pqr}, \qquad p + q + r \leqslant n - 4.$$

REFERENCSS

1. O. Haupt, *Ein Satz über die reellen Raumkurven vierter Ordnung und seine Verallgemeinerung*, Math. Ann., *108* (1933), 126–142.
2. Wm. F. Pohl, *On a theorem related to the four-vertex theorem*, Ann. of Math., *84* (1966), 356–367.
3. P. Scherk, *Ueber differenzierbare Kurven und Bögen III. Ueber Punkte $(n + 1)$-ter Ordnung auf Bögen im R_n*, Annali di Mat. (4), *17* (1938), 291–305.
4. ——— *On differentiable arcs and curves IV. On the singular points of curves of order $n + 1$ in projective n-space*, Ann. of Math., *46* (1945), 68–82.
5. ——— same title *IVa. On certain singularities of curves of order $n + 1$ in projective n-space*, Ann. of Math., *46* (1945), 175–181.
6. ——— same title *V. On a class of mappings of the curves of order $n + 1$ in projective n-space into themselves*, Ann. of Math., *47* (1946), 786–805.

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