Locally finite varieties of groups arising from Cross varieties

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Let \( V \) be a Cross variety and let \( n \) be the least integer such that \( V^{(n)} \) is locally finite; then \( n \leq 2d + 3 \) where \( d \) is an upper bound for the number of generators of certain critical groups in \( V \).

1. Introduction

If \( V \) is a Cross variety then, by the Oates-Powell Theorem, \( V = V^{(n)} \) for some \( n \) and hence \( V^{(n)} \) is locally finite, but, of course, \( V^{(n)} \) can be locally finite even though \( V \neq V^{(n)} \); for instance if \( V \) is the variety generated by the dihedral group of order \( 2^{p+1} \) then \( V^{(2)} \) is locally finite, although \( V \neq V^{n-1} \) \([4]\). Can anything be said in general about the local finiteness of \( V^{(n)} \)? Certainly \( V^{(1)} \) is not always locally finite \([7]\), but I conjecture that \( V^{(2)} \) is. Certain evidence to support this exists: R.M. Bryant \([1]\) has shown that the two variable laws of \( \text{PSL}(2, q) \) imply local finiteness, thus extending the results of \([2]\).

Both proofs of the Oates-Powell Theorem \([8], [3]\) give values of \( n \) for which \( V^{(n)} \) is locally finite, though these tend to be somewhat large. In this paper I extend the results of \([8], \S 3\), to prove:-

THEOREM A. Let \( V \) be a Cross variety with a chain of subvarieties...
\[ E = V_n \subset V_{n-1} \subset \ldots \subset V_1 = V, \]
each maximal in the succeeding one, and let \( V_i = \text{var}(V_{i-1}, D_i) \) where \( D_i \) is critical and can be generated by \( d \) (or fewer) elements; then \( V_i^{(2d+3)} \) is locally finite.

2. Notation

Notation and terminology follow that of Hanna Neumann [6].

3. Outline of proof

It is clearly sufficient to prove the following theorem:

**THEOREM B.** Let \( V \) be a Cross variety and \( U \) a maximal subvariety of \( V \) such that \( V = \text{var}(U, D) \) where \( D \) is a critical \( d \)-generator group. If \( U^{(2d+3)} \) is locally finite so is \( V^{(2d+3)} \).

Theorem A follows by induction on \( r \) from Theorem B, since it is trivially true for \( E \).

The proof of Theorem B divides into two parts according as \( OD \) is abelian or non-abelian.

4. \( OD \) abelian

**DEFINITION 4.1.** Let \( \{\tilde{w}_1 = 1, \ldots, \tilde{w}_k = 1\} \) be a basis for the \( (2d+3) \)-variable laws of \( U \) (by a result of B.H. Neumann [5] such a finite basis exists) and let \( W(G) \) be the corresponding word subgroup, so that

\[ G/W(G) \in U^{(2d+3)} \]

and \( G/N \not\in U^{(2d+3)} \) if \( N < W(G) \). Similarly let \( \{w_1 = 1, \ldots, w_L = 1\} \) be a basis for the \( d \)-variable laws of \( U \), and \( w(G) \) the corresponding word subgroup. Note that, since \( U^{(d)} \supset U^{(2d+3)} \), \( w(G) \leq W(G) \).

**LEMMA 4.2.** If \( G \in U \), \( w(G) = W(G) \).

Proof. Suppose there is \( G \in U \) such that \( w(G) < W(G) \), then,
(\star)
\[
\begin{aligned}
G/\omega(G) \in \mathbb{V} \cap \mathbb{U}^{(d)} \\
G/\omega(G) \notin \mathbb{V} \cap \mathbb{U}^{(2d+3)}.
\end{aligned}
\]

However
\[
\mathbb{V} \supset \mathbb{V} \cap \mathbb{U}^{(d)} \supset \mathbb{U},
\]
\[
\mathbb{V} \supset \mathbb{V} \cap \mathbb{U}^{(2d+3)} \supset \mathbb{U}
\]
and \( \mathbb{U} \) is maximal in \( \mathbb{V} \).

Moreover \( D \notin \mathbb{U}^{(d)} \) (since it is a \( d \)-generator group not in \( \mathbb{U} \)). It follows that
\[
\mathbb{V} \cap \mathbb{U}^{(d)} = \mathbb{U} = \mathbb{V} \cap \mathbb{U}^{(2d+3)},
\]
contradicting (\star). Hence \( \omega(G) \notin \mathbb{W}(G) \).

**Lemma 4.3.** If \( G \in \mathbb{V} \) then \( \mathbb{W}(G) \) is elementary abelian of exponent \( p \), where \( \omega D \) is a \( p \)-group.

**Proof.** \( D/\omega D \in \mathbb{U} \) (since \( D \) is critical and \( \mathbb{U} \) is maximal in \( \mathbb{V} \)) so \( \mathbb{W}(G) \leq \omega D \). But \( D \notin \mathbb{U}^{(2d+3)} \) so \( \mathbb{W}(D) \neq 1 \). It follows that \( \mathbb{W}(D) = \omega D \) is elementary abelian, and \( D \) satisfies the laws:
\[
[w_i, w_j] = 1, \quad w_i^p = 1 \ (i, j = 1, \ldots, k)
\]
where the sets of variables in \( w_i \) and \( w_j \) are disjoint.

Since \( \mathbb{U} \) also satisfies these laws, \( \mathbb{V} \) must satisfy them and hence \( \mathbb{W}(G) \) is elementary abelian for every \( G \in \mathbb{V} \).

**Corollary 4.4.** \( \mathbb{V} \) satisfies the \((2d)\)-variable laws
\[
[w_i, w_j] = 1, \quad w_i^p = 1 \ (i, j = 1, \ldots, l).
\]

**Proof of Theorem B for abelian \( \omega D \).** Let \( G \) be a finitely generated group in \( \mathbb{V} \). \( G/\mathbb{W}(G) \in \mathbb{U}^{(2d+3)} \) and so is finite. It follows that \( \mathbb{W}(G) \) is finitely generated and so is generated by a finite number of words of the form \( w_i^{g_1}, \ldots, g_{2d+3} \).

Let \( H = gp\{g_1, \ldots, g_{2d+3}\} \); then \( H \in \mathbb{V} \) and so \( \mathbb{W}(H) = \omega(H) \). Thus
Hence $W(G) = \omega(G)$ and $\omega(G)$ is also finitely generated. But the laws $[\omega_i, \omega_j] = 1$ and $\omega_i^p = 1$ hold in $\mathcal{V}(2d+3)$ (being $(2d)$-variable laws of $\mathcal{V}$). It follows that $\omega(G)$ is a finitely generated elementary abelian $p$-group and so is finite. Hence $G$ is finite as required.

5. $\sigma(D)$ non-abelian

Consideration of Section 3 [8] shows that, for the purposes of proving local finiteness $a$ (the number of variables in a basis for $\mathcal{V}$) can be replaced by $2d + 3$ (the number of variables needed to ensure local finiteness) and $b$ (the size of a generating set for $D$ which includes one for $\sigma(D)$) can be replaced by at worst $d + 1$, since it is sufficient to work with a generating set for $D$ which contains one element from $\sigma(D)$. Thus, by the results of §3.4 of [8] we have that $\mathcal{V}_n(\mathcal{U})$ is locally finite, for

$$\max(2d+3, 2(d+1)+1) = 2d + 3.$$

References


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