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A Note on the Weierstrass Preparation Theorem in Quasianalytic Local Rings

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Abstract. Consider quasianalytic local rings of germs of smooth functions closed under composition, implicit equation, and monomial division. We show that if the Weierstrass Preparation Theorem holds in such a ring, then all elements of it are germs of analytic functions.

1 Introduction and Main Results

Since the original work of Borel [5, 6], the notion of quasianalytic rings of infinitely differentiable functions has been studied intensively (see for example the expositary article on quasianalytic local rings by V. Thilliez [21]). Recall that a ring $\mathcal{C}_n$ of smooth germs at the origin of $\mathbb{R}^n$ is called *quasianalytic* if the only element of $\mathcal{C}_n$ that admits a zero Taylor expansion is the zero germ.

The early works of Denjoy [9] and Carleman [7] show a deep connection between the growth of partial derivatives of $\mathcal{C}^\infty$ germs at the origin and the quasianalyticity property, leading to the notion of *quasianalytic Denjoy–Carleman classes of functions*. The algebraic properties of such rings, namely their stability under several classical operations, such as composition, differentiation, implicit function, is well understood (see for example [16] and [18]).

These stability properties have allowed a study of quasianalytic classes from the point of view of *real analytic geometry*, that is, the investigation of the properties of subsets of the real spaces locally defined by equalities and inequalities satisfied by elements of these rings. For example, it is shown in [3] how the resolution of singularities extends to the quasianalytic framework.

However, two classical properties, namely Weierstrass division and Weierstrass preparation, seem to cause trouble in the quasianalytic setting. For example it has been proved by Childress [8] that quasianalytic Denjoy–Carleman classes might not satisfy Weierstrass division. Since Weierstrass preparation is usually introduced as a consequence of Weierstrass division, it is classically considered that Weierstrass preparation should fail in a quasianalytic framework. So far, no explicit counterexample has been given. Moreover, we do not know any example of a ring of smooth functions for which the Weierstrass preparation theorem holds, but the Weierstrass division fails.

We are interested here in what we call in the next section a *quasianalytic system*, that is, a collection of quasianalytic rings of germs of smooth functions that contains

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the analytic germs and is closed under composition, partial differentiation and implicit function. Such systems have been investigated in several works from the point of view of real analytic geometry or o-minimality [3, 12, 19, 20]. It is worth noticing that, in these papers, the possible failure of Weierstrass preparation leads to a study mostly based on resolution of singularities.

In such a context, a nice result has been obtained by Elkhadiri and Sfouli in [15]. They prove, in a remarkably simple way, that if a quasianalytic system satisfies Weierstrass division, then it coincides with the analytic system: all its germs are analytic. The proof is based on the following idea. In order to prove that a given real germ f is analytic at the origin of $\mathbb{R}^n$, they prove that f extends to a holomorphic germ at the origin of $\mathbb{C}^n$. This extension is built by considering the *complex formal extension* $\hat{f}(x + iy) \in \mathbb{C}[[x, y]]$, where $\hat{f}$ is the Taylor expansion of f at the origin. The real and imaginary parts of this series statisfy the Cauchy–Riemann equations. Moreover, the Weierstrass division of $f(x + t)$ by the polynomial $t^2 + y$ shows that these real and imaginary parts are the Taylor expansions of two germs that belong to the initial quasianalytic system. By quasianalyticity, these two germs also satisfy the Cauchy–Riemann equations. They consequently provide the real and imaginary parts of a holomorphic extension of f.

Our goal is to use Elkhadiri and Sfouli's methods to prove that a quasianalytic system in which Weierstrass preparation holds coincides with the analytic system. This result apply in particular to the examples of quasianalytic systems mentioned above. We still don't know if any ring of a quasianalytic system strictly bigger than the analytic one, besides the ring of one variable germs, is noetherian or not.

A similar property has been announced in [1] for quasianalytic Denjoy–Carleman classes. More precisely, the statement made in [1] claims that if a Denjoy–Carleman class contains strictly the analytic system, then Weierstrass preparation does not hold, even if we allow the unit and the distinguished polynomial to be in any wider quasianalytic Denjoy–Carleman class. The approach there, pretty different from ours, leads to a precise investigation of the following *extension problem*: does a function belonging to a quasianalytic Denjoy–Carleman class defined on the positive real axis extend to a function belonging to a wider Denjoy–Carleman class defined on the real axis? The authors actually produce an explicit example of non-extendable function with additional properties that permit contradicting Weierstrass preparation.

2 Notations and Main Result

Notation 2.1 For $n \in \mathbb{N}$, we denote by $\mathcal{E}_n$ the ring of smooth germs at the origin of $\mathbb{R}^n$ and by $\mathcal{A}_n \subset \mathcal{E}_n$ the subring of analytic germs.

For every $f \in \mathcal{E}_n$, we denote by $\hat{f} \in \mathbb{R}[[x_1, \dots, x_n]]$ its (infinite) Taylor expansion at the origin.

Finally, we denote $(x_1, \dots, x_n)$ by x and $(x_1, \dots, x_{n-1})$ by x'.

Definition 2.2 Consider a collection $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ of rings of germs of smooth functions at the origin of $\mathbb{R}^n$. We say that $\mathcal{C}$ is a *quasianalytic system* if the following properties hold for all $n \in \mathbb{N}$:

(i) The algebra $\mathcal{A}_n$ is contained in $\mathcal{C}_n$.
(ii) (Stability by composition) If $f \in \mathcal{C}_n$ and $g_1, \dots, g_n \in \mathcal{C}_m$ with $g_1(0) = \cdots = g_n(0) = 0$, then $f(g_1, \dots, g_n) \in \mathcal{C}_m$.
(iii) (Stability by implicit equation, assuming $n > 0$) If $f \in \mathcal{C}_n$ satisfies $f(0) = 0$ and $(\partial f/\partial x_n)(0) \neq 0$, then there exists $\varphi \in \mathcal{C}_{n-1}$ such that $\varphi(0) = 0$ and $f\big(x', \varphi(x')\big) = 0$.
(iv) (Stability under monomial division) If $f \in \mathcal{C}_n$ satisfies $f(x', 0) = 0$, then there exists $g \in \mathcal{C}_n$ such that $f(x) = x_n g(x)$.
(v) (Quasianalyticity) For every $n \in \mathbb{N}$, the Taylor expansion at the origin map $f \mapsto \hat{f}$ is injective on $\mathcal{C}_n$.

Remark 2.3 It can easily be seen that the above properties imply that the algebras $\mathcal{C}_n$ are closed under partial differentiation (see [19, p. 423] for example).

Definition 2.4 A germ $f \in \mathcal{E}_n$ is *of order d in the variable x_n* if $f(0, x_n) = x_n^d u(x_n)$, where $u(0) \neq 0$ (that is, u is a *unit* of $\mathcal{E}_1$).

Definition 2.5 We say that a quasianalytic system $\mathcal{C}$ *satisfies Weierstrass preparation* if, for all $n \in \mathbb{N}$, the following statement $(\mathcal{W}_n)$ holds: every $f \in \mathcal{C}_n$ of order d in the variable x_n can be written

$$f = U(x)\big(x_n^d + a_1(x')x_n^{d-1} + \cdots + a_d(x')\big),$$

where $U \in \mathcal{C}_n$, $a_1, \dots, a_d \in \mathcal{C}_{n-1}$, $U(0) \neq 0$ and $a_1(0) = \cdots = a_d(0) = 0$.

Our main result is the following:

Theorem *If the quasianalytic system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ satisfies Weierstrass preparation, then it is contained in the analytic system: for all $n \in \mathbb{N}$, $\mathcal{C}_n = \mathcal{A}_n$.*

Remark 2.6 We will actually prove that the conclusion of the theorem is true once $\mathcal{W}_3$ holds.

3 Proof of the Theorem

We consider in this section a quasianalytic system $\mathcal{C}$ that satisfies Weierstrass preparation.

In order to prove the theorem, it is enough to prove that $\mathcal{C}_1 = \mathcal{A}_1$. In fact, it is noticed in [15] that the equality $\mathcal{C}_1 = \mathcal{A}_1$ implies $\mathcal{C}_n = \mathcal{A}_n$ for all $n \in \mathbb{N}$. The argument is the following. If $f \in \mathcal{C}_n$ (and $n > 1$) then, for every $\xi \in \mathbb{S}^{n-1}$, the germ $f_\xi \colon t \mapsto f(t\xi)$ belongs to $\mathcal{C}_1$. Hence, under the assumption $\mathcal{C}_1 = \mathcal{A}_1$, the germ f_ξ is analytic. Thanks to a result of [4], this implies that $f \in \mathcal{A}_n$.

Lemma *Let $f \in \mathcal{C}_n$ such that $f(0, x_n) = x_n^2 + x_n^3 + h(x_n)$, where $h \in \mathcal{C}_1$ has order greater than 3. Then there exists $f_0, f_1 \in \mathcal{C}_n$ such that*

$$f(x) = f_0(x', x_n^2) + x_n f_1(x', x_n^2).$$

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Proof We introduce the germs

$$g_0\colon x \mapsto \big(f(x',x_n)+f(x',-x_n)\big)/2 \quad \text{and} \quad g_1\colon x \mapsto \big(f(x',x_n)-f(x',-x_n)\big)/2,$$

which both belong to $\mathcal{C}_n$ and satisfy $f = g_0 + g_1$. They are respectively even and odd in the variable x_n. Hence the exponents of x_n in their Taylor expansions at the origin are respectively even and odd.

The order of g_0 in the variable x_n is exactly 2, so is the order in x_n of the germ $F\colon (x,t) \mapsto g_0(x) - t$, which belongs to $\mathcal{C}_{n+1}$. Since the system $\mathcal{C}$ satisfies Weierstrass preparation, there exist $\varphi_1, \varphi_2 \in \mathcal{C}_n$ and a unit $U \in \mathcal{C}_{n+1}$ such that $\varphi_1(0) = \varphi_2(0) = 0$ and

$$F(x,t) = \big(x_n^2 + \varphi_1(x',t)x_n + \varphi_0(x',t)\big) \cdot U(x,t).$$

We claim that $\varphi_1 = 0$. In fact, considering the Taylor expansions, we have

$$\widehat{F}(x,t) = \big(x_n^2 + \hat{\varphi}_1(x',t)x_n + \hat{\varphi}_0(x',t)\big) \cdot \widehat{U}(x,t).$$

Now it stems from the classical proof of the Weierstrass Preparation Theorem for formal series that the support of $x_n^2 + \hat{\varphi}_1(x',t)x_n + \hat{\varphi}_0(x',t)$ is contained in the sub-semigroup of $\mathbb{N}^{n+1}$ generated by the support of $\widehat{F}$. Hence this support contains only even powers of the variable x_n, and $\hat{\varphi}_1 = 0$. Since the system $\mathcal{C}$ is quasianalytic (see Definition 2.2 (v)), $\varphi_1 = 0$.

Notice that the order of the germ $(x',z,t) \mapsto z + \varphi_0(x',t)$ in the variable t is 1. Since the system $\mathcal{C}$ is closed under implicit equation (see Definition 2.2 (iii)), there exists a germ $f_0 \in \mathcal{C}_n$ such that

$$z + \varphi_0(x',t) = 0 \Longleftrightarrow t = f_0(x',z).$$

We deduce that

$$t = g_0(x) \Longleftrightarrow F(x,t) = 0 \Longleftrightarrow x_n^2 + \varphi_0(x',t) = 0 \Longleftrightarrow t = f_0(x',x_n^2),$$

that is, $g_0(x) = f_0(x',x_n^2)$.

In the same way, we notice that the order of g_1 in the variable x_n is exactly 3. Moreover, $g_1(x',0) = 0$. By stability under monomial division (see Definition 2.2 (iv)) there exists $\bar{g}_1 \in \mathcal{C}_n$ such that $g_1(x) = x_n\bar{g}_1(x)$. The germ $\bar{g}_1$ is even in the variable x_n and its order in this variable is exactly 2.

Therefore there exists a germ $f_1 \in \mathcal{C}_n$ such that $\bar{g}_1(x) = f_1(x',x_n^2)$, and the lemma is proved. ∎

Remark It is well known that, for every $f \in \mathcal{E}_n$, there exist f_0 and f_1 in $\mathcal{E}_n$ such that $f(x) = f_0(x^2) + xf_1(x^2)$ (see, for example, [17, p. 12]). But the classical proof, whose first step consists in transforming f into a flat germ, cannot work in a quasianalytic system.

Proof of the Theorem Consider a germ $h \in \mathcal{C}_1$. Up to adding a polynomial, we may suppose that $h(x_1) = x_1^2 + x_1^3 + \ell(x_1)$, where the order of ℓ in the variable x_1 is greater than 3. We define the germ $f \in \mathcal{C}_2$ by $f\colon (x_1, x_2) \mapsto h(x_1 + x_2)$. According to the lemma, there exist two germs, f_0 and f_1, in $\mathcal{C}_2$ such that

$$f\colon (x_1, x_2) \longmapsto f_0(x_1, x_2^2) + x_2 f_1(x_1, x_2^2).$$

We introduce the complex germ H defined by

$$H\colon z = x_1 + \mathrm{i}x_2 \in \mathbb{C} \longmapsto f_0(x_1, -x_2^2) + \mathrm{i}x_2 f_1(x_1, -x_2^2).$$

We see that $H(x_1, 0) = f(x_1, 0) = h(x_1)$. Hence the theorem is proved once we have proved that the germ H is holomorphic, that is, that its real and imaginary parts satisfy the Cauchy–Riemann equations.

Consider the Taylor expansion $\hat{h}(x_1) = \sum_{n\geq 0} h_n x_1^n \in \mathbb{R}[[x_1]]$ of the germ h. The real and imaginary parts of the formal series $\widehat{H}(x_1 + \mathrm{i}x_2) \in \mathbb{C}[[x_1, x_2]]$ defined by

$$\widehat{H}(x_1 + \mathrm{i}x_2) = \sum_{n\geq 0} h_n (x_1 + \mathrm{i}x_2)^n$$

are the series $\hat{f}_0(x_1, -x_2^2)$ and $x_2 \hat{f}_1(x_1, -x_2^2)$. These series satisfy the Cauchy–Riemann equations. By quasianalyticity, the germs $f_0(x_1, -x_2^2)$ and $x_2 f_1(x_1, x_2^2)$ satisfy the same equations.

We deduce that the complex germ H is holomorphic, and thus the germ h is analytic. ∎

Remark 3.1 In the proof of the theorem, the lemma is applied to the germ f, which belongs to $\mathcal{C}_2$. Hence the single hypothesis $\mathcal{W}_3$ is actually required.

4 Corollaries

Well-behaved Quasianalytic Systems. We say that a quasianalytic system $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is *noetherian* if all the rings $\mathcal{C}_n$ are noetherian. To our knowledge the problem of noetherianity of quasianalytic systems is still open. At the end of [8], Childress conjectures that *a quasianalytic system is noetherian if and only if it satisfies Weierstrass division*. By Elkhadiri and Sfouli [15], Weierstrass division does not hold in quasianalytic systems that extend strictly the analytic system. Hence the resolution of Childress' conjecture would ensure that such quasianalytic systems cannot be noetherian.

Elkhadiri proves in [13] that Childress' conjecture is true for *well-behaved* quasianalytic systems. This result could lead to the research of well-behaved, hence non-noetherian, quasianalytic systems. However, it follows easily from the proof of our main result that *the only well-behaved quasianalytic system is the analytic system.*

Definition A quasianalytic system $\mathcal{C}$ is *well-behaved* if, for all $n, d \in \mathbb{N} \setminus \{0\}$, every formal power series $\hat{f} \in \mathbb{R}[[x_1, \ldots, x_n]]$ such that one of $\hat{f}(x_1x_2, x_2, \ldots, x_n)$ and $\hat{f}(x_1^d, x_2, \ldots, x_n)$ is the Taylor expansion of a germ in $\mathcal{C}_n$ is itself the Taylor expansion a germ in $\mathcal{C}_n$.

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The lemma in the previous section holds in a well-behaved quasianalytic system $\mathcal{C}$. Indeed, let $f \in \mathcal{C}_n$. We may write f as the sum $f = g_0 + g_1$ of two elements of $\mathcal{C}_n$, even and odd respectively in the variable x_n. The Taylor series of g_0 is $\hat{f}_0(x', x_n^2)$ for some power series $\hat{f}_0 \in \mathbb{R}[[x', x_n]]$. If $\mathcal{C}$ is well behaved, the power series $\hat{f}_0$ is the Taylor expansion of a $f_0 \in \mathcal{C}_n$. By quasianalyticity, $g_0(x) = f_0(x', x_n^2)$. Similarly $g_1(x) = x_n f_1(x', x_n^2)$ for a $f_1 \in \mathcal{C}_n$. We complete the proof as in the previous section.

Weierstrass Preparation for Definable Analytic Germs. Note that the proofs of the previous section hold as well for quasianalytic systems that do not necessarily contain the analytic system. (We replace Definition 2.2 (i) by asking that $\mathcal{C}_n$ contains the polynomial germs.) If $\mathcal{C}$ is a subsystem of an analytic system that satisfies Weierstrass preparation, our method shows that every germ $f \in \mathcal{C}_1$ admits a holomorphic extension $F(x_1 + ix_2) = f_0(x_1, x_2) + i f_1(x_1, x_2)$, $f(x_1) = F(x_1)$, where f_0 and f_1 belong to $\mathcal{C}_2$.

Such systems appear in the framework of o-minimal structures (see, for example, [11] for the basic definitions). Consider an o-minimal expansion $\mathcal{S}$ of the real field. If f is a function defined on an open neighborhood of $0 \in \mathbb{R}^n$ and definable in $\mathcal{S}$, its germ at the origin is called a *definable germ*. For every $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the ring of the analytic definable germs at $0 \in \mathbb{R}^n$. The collection $\mathcal{C} = \{\mathcal{C}_n, n \in \mathbb{N}\}$ is obviously a quasianalytic system. The possible failure of Weierstrass division in such a system is a question asked by van den Dries in [10].

Elkhadiri and Sfouli address this question in [14]. They prove that the system $\mathcal{C}$ associated to the structure $\mathbb{R}_e = (\mathbb{R}, \exp_{|[0,1]})$ does not satisfy Weierstrass division (here, the *restricted exponential* $\exp_{|[0,1]}$ is the function which coincides with the exponential function on the interval $[0, 1]$ and is extended by 0 on $\mathbb{R} \setminus [0, 1]$).

We claim that *this system does not satisfy Weierstrass preparation either*. Otherwise, our main result would imply that the real and imaginary parts of the germ at $0 \in \mathbb{C}$ of the complex exponential function would be definable in $\mathbb{R}_e$. This would contradict a result of Bianconi [2], which states that the restriction of the sine function to any interval is not definable in $\mathbb{R}_e$.

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