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RESEARCH ARTICLE

The global Gan-Gross-Prasad conjecture for unitary groups. II. From Eisenstein series to Bessel periods

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Abstract
We state and prove an extension of the global Gan-Gross-Prasad conjecture and the Ichino-Ikeda conjecture to the case of some Eisenstein series on unitary groups $U_n \times U_{n+1}$. Our theorems are based on a comparison of the Jacquet-Rallis trace formulas. A new point is the expression of some interesting spectral contributions in these formulas in terms of integrals of relative characters. As an application of our main theorems, we prove the global Gan-Gross-Prasad and the Ichino-Ikeda conjecture for Bessel periods of unitary groups.

Contents

1. Introduction

1.1. Arthur parameters and weak base change

1.1.1.

In some sense, this paper is a sequel of [BPCZ22], where we proved the global Gan-Gross-Prasad (see [GGP12, section 24]) and the Ichino-Ikeda conjectures for a product of unitary groups $U(n) \times U(n+1)$ (see [II10] and [Har14]). The goal of the present paper is two-fold: first, we state and prove an extension of these two conjectures to the case of some Eisenstein series. Second, we show that this extension, when applied to some specific Eisenstein series, implies the global Gan-Gross-Prasad conjecture and its refinement à la Ichino-Ikeda for general Bessel periods of unitary groups. To state our results, we first review the notion of Arthur parameter.

1.1.2. Hermitian Arthur parameter.

Let E/F be a quadratic extension of number fields and c be the nontrivial element of the Galois group $\mathrm{Gal}(E/F)$. Let $\mathbb{A}$ be the ring of adèles of F. Let $n \geqslant 1$ be an integer. Let G_n be the group of automorphims of the E-vector space E^n. We view G_n as an F-group by Weil restriction. For an automorphic representation Π of $G_n(\mathbb{A})$, we denote by Π^* its conjugate-dual. Let us introduce some definitions. A *discrete Hermitian Arthur parameter* of G_n is an irreducible automorphic representation Π of $G_n(\mathbb{A})$ such that

- Π is isomorphic to the full induced representation $\mathrm{Ind}_Q^{G_n}(\Pi_1 \boxtimes \ldots \boxtimes \Pi_r)$, where Q is a parabolic subgroup of G_n with Levi factor $G_{n_1} \times \ldots \times G_{n_r}$ where $n_1 + \ldots + n_r = n$;
- Π_i is a conjugate self-dual cuspidal automorphic representation of $G_{n_i}(\mathbb{A})$, and the Asai L-function $L(s, \Pi_i, \mathrm{As}^{(-1)^{n+1}})$ has a pole at $s = 1$ for $1 \leqslant i \leqslant r$.
- the representations Π_i are mutually non-isomorphic for $1 \leqslant i \leqslant r$;

The integer r and the representations $(\Pi_i)_{1 \leqslant i \leqslant r}$ are unique (up to a permutation). We set $S_\Pi = (\mathbb{Z}/2\mathbb{Z})^r$.

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For our purpose, we need more general Arthur parameters of G_n, which we call *regular Hermitian Arthur parameters* and which are by definition the automorphic representations Π of G_n such that

- Π is isomorphic to the full induced representation $\mathrm{Ind}_Q^{G_n}(\Pi_1 \boxtimes \ldots \boxtimes \Pi_r \boxtimes \Pi_0 \boxtimes \Pi_r^* \boxtimes \ldots \boxtimes \Pi_1^*)$, where Q is a parabolic subgroup of G_n with Levi factor $M_Q = G_{n_1} \times \ldots \times G_{n_r} \times G_{n_0} \times G_{n_r} \times \ldots \times G_{n_1}$, where $n_0 + 2(n_1 + \ldots + n_r) = n$;
- Π_0 is a discrete Hermitian Arthur parameter of G_{n_0};
- Π_i is a cuspidal automorphic representation of $G_{n_i}(\mathbb{A})$ (with character central trivial on $A_{G_{n_i}}^\infty$) for $1 \leqslant i \leqslant r$;
- the representations $\Pi_1, \ldots, \Pi_r, \Pi_1^*, \ldots, \Pi_r^*$ are mutually non-isomorphic .

The representation Π_0 is uniquely determined by Π and is called the discrete component of Π. We set

$$S_\Pi = S_{\Pi_0}.$$

The parabolic subgroup Q depends on the ordering on the representations $\Pi_1, \ldots, \Pi_r, \Pi_1^*, \ldots, \Pi_r^*$: we fix one.

Let $\mathfrak{a}_{Q,\mathbb{C}}^*$ be the complex vector space of unramified characters of $Q(\mathbb{A})$. We have the real subspaces $\mathfrak{a}_Q^*$ and $i\mathfrak{a}_Q^*$, respectively, of real and unitary characters. Let w be the permutation that exchanges the two blocks G_{n_i} corresponding to Π_i and Π_i^* for all $1 \leqslant i \leqslant r$. We set

$$\mathfrak{a}_{\Pi,\mathbb{C}}^* = \{\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^* \mid w\lambda = -\lambda\}.$$

For any $\lambda \in \mathfrak{a}_{\Pi,\mathbb{C}}^*$, we define Π_λ as the full induced representation

$$\mathrm{Ind}_Q^{G_n}\left((\Pi_1 \boxtimes \ldots \boxtimes \Pi_r \boxtimes \Pi_0 \boxtimes \Pi_r^* \boxtimes \ldots \boxtimes \Pi_1^*) \otimes \lambda\right).$$

If $\lambda \in i\mathfrak{a}_\Pi^* = \mathfrak{a}_{\Pi,\mathbb{C}}^* \cap i\mathfrak{a}_Q^*$, then Π_λ is irreducible.

1.1.3. Unitary groups and (weak) base change

For any integer $n \geqslant 1$, let $\mathcal{H}_n$ be the set of isomorphism classes of nondegenerate c-Hermitian spaces h over E of rank n. We identify any $h \in \mathcal{H}_n$ with a representative, and we denote by $U(h)$ its automorphism group. Let $h \in \mathcal{H}$ and $P \subset U(h)$ be a parabolic subgroup with Levi factor M_P. There exist a decomposition $n_0 + 2(n_1 + \ldots + n_r) = n$ and $h_{n_0} \in \mathcal{H}_{n_0}$ such that M_P is identified with $G_{n_1} \times \ldots \times G_{n_r} \times U(h_{n_0})$. Let σ be a cuspidal automorphic subrepresentation of $M_P(\mathbb{A})$ (with central character trivial on the central subgroup A_P^∞ defined in §2.1.6). Accordingly, we have $\sigma = \Pi_1 \boxtimes \ldots \boxtimes \Pi_r \boxtimes \sigma_0$ with Π_i a cuspidal automorphic representation of G_{n_i} (with central character trivial on $A_{G_{n_i}}^\infty$).

We shall say that a regular Hermitian Arthur parameter Π of G_n is a *weak base-change* of (P, σ) if there exist a parabolic subgroup Q of G_n with Levi factor $M_Q = G_{n_1} \times \ldots \times G_{n_r} \times G_{n_0} \times G_{n_r} \times \ldots \times G_{n_1}$ and a discrete Hermitian Arthur parameter Π_0 of G_{n_0} such that

1. Π is isomorphic to the full induced representation $\mathrm{Ind}_Q^{G_n}(\Pi_1 \boxtimes \ldots \boxtimes \Pi_r \boxtimes \Pi_0 \boxtimes \Pi_r^* \boxtimes \ldots \boxtimes \Pi_1^*)$;
2. for almost all places of F that split in E, the local component $\Pi_{0,v}$ is the split local base change of $\sigma_{0,v}$.

Note that this implies that the representations $\Pi_1, \ldots, \Pi_r, \Pi_1^*, \ldots, \Pi_r^*$ are mutually non-isomorphic and that Π_0 is in fact the discrete component of Π. If condition 2 above is satisfied, we shall also say that Π_0 is the weak base change of σ_0.

If Π is a weak base-change of (P, σ), we can naturally identify the space $\mathfrak{a}_{P,\mathbb{C}}^*$ of unramified characters of $P(\mathbb{A})$ with $\mathfrak{a}_{\Pi,\mathbb{C}}^*$, and so we will not distinguish between the two spaces. Thus, for $\lambda \in \mathfrak{a}_{\Pi,\mathbb{C}}^*$, we can consider the full induced representation $\Sigma_\lambda = \mathrm{Ind}_P^{U_h}(\sigma \otimes \lambda)$.

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1.1.4.
We can extend the notions above to the case of a product. Let $n, n' \geqslant 1$ be integers. A *regular* Hermitian Arthur parameter of $G_n \times G_{n'}$ is then an automorphic representation of the form $\Pi = \Pi_n \boxtimes \Pi_{n'}$, where Π_k is a *regular* Hermitian Arthur parameter of G_k for $k = n, n'$. Then we set $S_\Pi = S_{\Pi_n} \times S_{\Pi_{n'}}$ and $\mathfrak{a}^*_{\Pi,\mathbb{C}} = \mathfrak{a}^*_{\Pi_n,\mathbb{C}} \times \mathfrak{a}^*_{\Pi_{n'},\mathbb{C}}$ etc. For $\lambda = (\lambda_n, \lambda_{n'}) \in \mathfrak{a}^*_{\Pi,\mathbb{C}}$, we set $\Pi_\lambda = \Pi_{n,\lambda_n} \boxtimes \Pi_{n',\lambda_{n'}}$. A parameter Π is discrete if both Π_n and $\Pi_{n'}$ are discrete.

Let $h \in \mathcal{H}_n$ and $h' \in \mathcal{H}_{n'}$. Let $P = P_n \times P_{n'}$ be a parabolic subgroup of $U(h) \times U(h')$. We say that a regular Hermitian Arthur parameter $\Pi = \Pi_n \boxtimes \Pi_{n'}$ of $G_n \times G_{n'}$ is a weak base-change of (P, σ) if Π_n and $\Pi_{n'}$ are, respectively, weak base-changes of (P_n, σ_n) and $(P_{n'}, \sigma_{n'})$, where $\sigma = \sigma_n \boxtimes \sigma_{n'}$.

1.2. An extension of the Gan-Gross-Prasad conjecture to some Eisenstein series

1.2.1. Corank 1 and regular Hermitian Arthur parameter

Let $n \geqslant 1$. Consider the 'corank 1' case $G = G_n \times G_{n+1}$. Let $\Pi = \Pi_n \boxtimes \Pi_{n+1}$ be a regular Hermitian Arthur parameter of G. We can write $\Pi_k = \mathrm{Ind}_{Q_k}^{G_k}(\Pi_{1,k} \boxtimes \ldots \boxtimes \Pi_{r_k,k})$ for some parabolic subgroup $Q_k \subset G_k$ for $k = n, n+1$ and some cuspidal automorphic representations of $G_{n_{i,k}}(\mathbb{A})$ with $n_{1,k} + \ldots + n_{r_k,k} = k$. We shall say that the parameter Π is *H-regular* if for all $1 \leqslant i \leqslant r_n$ and $1 \leqslant j \leqslant r_{n+1}$, the representation $\Pi_{i,n}$ is not isomorphic to the contragredient of $\Pi_{j,n+1}$.

Remark 1.2.1.1. In the core of the paper, H will stand for the diagonal subgroup G_n of G, and the term H-regular refers to the fact that H-regular Hermitian Arthur parameter features some particularly nice properties with respect to the (regularized) Rankin-Selberg period over that subgroup (roughly stemming from the fact that the Rankin-Selberg L-function of Π has no poles). A discrete Hermitian Arthur parameter is necessarily H-regular. Otherwise, we would get a self-conjugate cuspidal representation Π_i of G_{n_i} for some $n_i \geqslant 1$ such that both Asai L-functions $L(s, \Pi_i, \mathrm{As}^+)$ and $L(s, \Pi_i, \mathrm{As}^-)$ have a pole at $s = 1$: this is not possible.

On the unitary side, let $h_0 \in \mathcal{H}_1$ be the element of rank 1 given by the norm $N_{E/F}$. Then we attach to any $h \in \mathcal{H}_n$ the following algebraic groups over F:

- the product of unitary groups $U_h = U(h) \times U(h \oplus h_0)$, where $h \oplus h_0$ denotes the orthogonal sum;
- the unitary group U'_h of automorphisms of h viewed as a subgroup of U_h by the obvious diagonal embedding.

1.2.2.
Let $P = M_P N_P \subset U_h$ be a parabolic subgroup with Levi factor M_P and unipotent radical N_P. Let σ be a cuspidal automorphic subrepresentation of $M_P(\mathbb{A})$ with central character trivial on A_P^∞. Let $\mathcal{A}_{P,\sigma}(U_h)$ be the space of automorphic forms on the quotient $A_P^\infty M_P(F) N_P(\mathbb{A}) \backslash U_h(\mathbb{A})$ such that for all $g \in U_h(\mathbb{A})$,

$$m \in M_P(\mathbb{A}) \mapsto \delta_P(m)^{-\frac{1}{2}} \varphi(mg)$$

belongs to the space of σ. Here, N_P is the unipotent radical of P and δ_P is the modular character of $P(\mathbb{A})$. The representation of $U_h(\mathbb{A})$ on $\mathcal{A}_{P,\sigma}(U_h)$ is isomorphic to the induced representation $\Sigma = \mathrm{Ind}_P^{U_h}(\sigma)$. Let $\varphi \in \mathcal{A}_{P,\sigma}(U_h)$. For $\lambda \in \mathfrak{a}^*_{\Pi,\mathbb{C}}$, we introduce the Eisenstein series $E(\varphi, \lambda)$ and the Ichino-Yamana regularized period

$$\mathcal{P}_{U'_h}(\varphi, \lambda) = \int_{[U'_h]} \Lambda_u^T E(x, \varphi, \lambda)\, dx, \qquad (1.2.2.1)$$

where $[U'_h] = U'_h(F) \backslash U'_h(\mathbb{A})$ is equipped with the Tamagawa measure, Λ_u^T is the truncation operator introduced by Ichino-Yamana in [IY19] depending on an auxiliary parameter T whose definition is recalled in §3.3.2. The integral is absolutely convergent. Moreover, if the base change of Σ is a H-regular Arthur parameter (which will be our assumption), then the integral does not depend on T (see

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proposition 3.5.3.1 below). In this case, $\mathcal{P}_{U'_h}(\varphi, \lambda)$ is a meromorphic function, which is regular outside the singularities of the Eisenstein series. In particular, it is holomorphic on $i\mathfrak{a}^*_\Pi$.

1.2.3. The Gan-Gross-Prasad conjecture for some Eisenstein series

Theorem 1.2.3.1. *Let Π be a H-regular Hermitian Arthur parameter of G and let $\lambda \in i\mathfrak{a}^*_\Pi$. The following two statements are equivalent:*

1. *The complete Rankin-Selberg L-function of Π_λ (including Archimedean places) satisfies*

$$L\left(\frac{1}{2}, \Pi_\lambda\right) \neq 0;$$

2. *There exist $h \in \mathcal{H}_n$, a parabolic subgroup $P \subset U_h$ with Levi factor M_P and σ an irreducible cuspidal automorphic subrepresentation of $M_P(\mathbb{A})$ such that Π is a weak base change of (P, σ) and the period integral $\varphi \mapsto \mathcal{P}_{U'_h}(\varphi, \lambda)$ induces a nonzero linear form on $\mathcal{A}_{P,\sigma}(U_h)$.*

Remark 1.2.3.2. The Levi subgroup M_P is determined up to conjugation by the parameter Π. Moreover, we have $P = U_h$ if Π is discrete. In this case, the theorem is proved in [BLZZ21, Theorem 1.8] if Π is cuspidal and in [BPCZ22, Theorem 1.1.5.1] for a general discrete Hermitian parameter. The novelty of the theorem is to consider *non-discrete* Arthur parameters and thus periods of *proper* Eisenstein series on unitary groups.

1.2.4. Factorization of periods of some Eisenstein series à la Ichino-Ikeda

Let $h \in \mathcal{H}_n$. Let P be a parabolic subgroup of U_h with Levi factor M_P and let σ be an irreducible cuspidal automorphic subrepresentation of $M_P(\mathbb{A})$ such that the weak base change of (P, σ) is a regular Hermitian Arthur parameter Π. We have a restricted tensor product decomposition $\sigma = \bigotimes'_{v \in V_F} \sigma_v$ over the set V_F of places of F. We assume that σ is tempered – that is, for every place v, the local representation σ_v is tempered. Let $\lambda \in i\mathfrak{a}^*_\Pi$. We define Π_λ and Σ_λ as above. Let $\Sigma_{\lambda,v} = \mathrm{Ind}_P^{U_h}(\sigma_v \otimes \lambda)$ and $\Pi_{\lambda,v}$ be their local components.

We set

$$\mathcal{L}(s, \Sigma_\lambda) = \left(s - \frac{1}{2}\right)^{-\dim(\mathfrak{a}^*_\Pi)} \prod_{i=1}^{n+1} L(s + i - 1/2, \eta^i) \frac{L(s, \Pi_\lambda)}{L(s + 1/2, \Pi_\lambda, \mathrm{As}')},$$

where η denotes the quadratic idele class character associated to the extension E/F, $L(s, \eta^i)$ is the completed Hecke L-function associated to η^i and $L(s, \Pi_\lambda, \mathrm{As}')$ is the L-function associated to $\mathrm{As}^{(-1)^n} \boxtimes \mathrm{As}^{(-1)^{n+1}}$. Note that with our hypothesis, the function $L(s, \Pi_\lambda, \mathrm{As}')$ has a pole of order $\dim(\mathfrak{a}^*_\Pi)$ at $s = 1$. Thus, the function $(s-1)^{-\dim(\mathfrak{a}^*_\Pi)} L(s, \Pi_\lambda, \mathrm{As}')$ is holomorphic and nonvanishing at $s = 1$. In particular, the function $\mathcal{L}(s, \Sigma_\lambda)$ is holomorphic at $s = \frac{1}{2}$.

We denote by $\mathcal{L}(s, \Sigma_{\lambda,v})$ the corresponding quotient of local L-factors; namely, for s in some half-space, we have

$$\mathcal{L}(s, \Sigma_\lambda) = \left(s - \frac{1}{2}\right)^{-\dim(\mathfrak{a}^*_\Pi)} \prod_{v \in V_F} \mathcal{L}(s, \Sigma_{\lambda,v}).$$

For each place v of F, we define a *local normalized period* $\mathcal{P}^\natural_{h,\sigma_v} : \Sigma_v \times \Sigma_v \to \mathbb{C}$ as follows:

$$\mathcal{P}^\natural_{h,\Sigma_{\lambda,v}}(\varphi_v, \varphi'_v) = \mathcal{L}\left(\frac{1}{2}, \Sigma_{\lambda,v}\right)^{-1} \int_{U'_h(F_v)} (\Sigma_{\lambda,v}(h_v)\varphi_v, \varphi'_v)_v dh_v, \quad \varphi_v, \varphi'_v \in \Sigma_v.$$

It depends on the choice of a Haar measure dh_v on $U'_h(F_v)$ as well as an invariant inner product on σ_v which gives in the usual way an invariant product on Σ_v denoted by $(\cdot,\cdot)_v$. By the temperedness

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assumption, the integral is absolutely convergent [Har14, Proposition 2.1] and the local factor $\mathcal{L}(s,\Sigma_{\lambda,v})$ has neither zero nor pole at $s=\frac{1}{2}$.

We introduce on $\mathcal{A}_{P,\sigma}(U_h)$ the Petersson inner product given by

$$(\varphi,\varphi)_{\mathrm{Pet}} = \int_{A_P^\infty M_P(F)N_P(\mathbb{A})\backslash U_h(\mathbb{A})} |\varphi(g)|^2 dg, \quad \varphi\in\sigma.$$

Recall that we have normalized the period integral $\mathcal{P}_{U'_h}(\lambda)$ by choosing the invariant Tamagawa measures on $[U'_h]$. We also normalize the Petersson product by using the quotient of Tamagawa measures. We assume that the local Haar measures dh_v on $U'_h(F_v)$ are such that the product $\prod_v dh_v$ gives the Tamagawa measure on $U'_h(\mathbb{A})$.

Theorem 1.2.4.1. *Let* Π *and* (P,σ) *as above. For* $\lambda\in i\mathfrak{a}_\Pi^*$ *and every nonzero factorizable vector* $\varphi=\otimes'_v\varphi_v\in\mathcal{A}_{P,\sigma}(U_h)\simeq\otimes'_{v\in V_F}\Sigma_v$*, we have*

$$\frac{|\mathcal{P}_{U'_h}(\varphi,\lambda)|^2}{(\varphi,\varphi)_{\mathrm{Pet}}} = |S_\Pi|^{-1}\mathcal{L}\left(\frac{1}{2},\Sigma_\lambda\right)\prod_v \frac{\mathcal{P}^\natural_{h,\Sigma_{\lambda,v}}(\varphi_v,\varphi_v)}{(\varphi_v,\varphi_v)_v}. \tag{1.2.4.1}$$

Remark 1.2.4.2. By [Har14, Theorem 2.12] and our choice of measures, almost all factors in the right-hand side are equal to 1. As in remark 1.2.3.2, the statement reduces to [BPCZ22, Theorem 1.1.6.1] for a discrete Hermitian Arthur parameter Π and even to [BLZZ21, Theorem 1.10] if Π is moreover simple.

1.3. The case of Bessel periods

1.3.1.

Let $n\geqslant m\geqslant 0$ be two integers of the same parity. We have $n=m+2r$ for some $r\geqslant 0$. Recall that we denote by h_0 the 1-dimensional Hermitian space given by the norm $N_{E/F}$. Let $h_s\in\mathcal{H}_2$ be the orthogonal sum of h_0 and $-h_0$. For any $h\in\mathcal{H}_m$, we define $\tilde{h}\in\mathcal{H}_n$ to be the orthogonal sum of h and r copies of h_s denoted by $h_s^1,\ldots,h_s^r$. For each $1\leqslant i\leqslant r$, let (x_i,y_i) be a hyperbolic basis of h_s^i; that is, we have $h_s^i(x_i,x_i)=h_s^i(y_i,y_i)=0$ and $h_s^i(x_i,y_i)=1$. We consider also the orthogonal sum $h_{n+1}=\tilde{h}\oplus h_0\in\mathcal{H}_{n+1}$. We denote by v_0 the vector of h_0 corresponding to $1\in E$. We have a diagonal embedding

$$U(h)\hookrightarrow\mathcal{G}_h=U(h)\times U(h_{n+1})$$

for which the image of $U(h)$ in $U(h_{n+1})$ is the subgroup which acts by the identity on $h_0\oplus h_s^1\oplus\ldots\oplus h_s^r$.

Let $B\subset U(h_{n+1})$ be the stabilizer of the isotropic flag

$$(0)\subsetneq \mathrm{vect}(x_1)\subsetneq\mathrm{vect}(x_1,x_2)\subsetneq\ldots\subsetneq\mathrm{vect}(x_1,\ldots,x_r). \tag{1.3.1.1}$$

Let N be the unipotent radical of B. Then the group $U(h)$ normalizes N. Let $\mathcal{B}_h=U(h)\ltimes(\{1\}\times N)$: this is the so-called Bessel subgroup of $\mathcal{G}_h$.

1.3.2. Bessel periods

Let $\psi:\mathbb{A}/F\to\mathbb{C}^\times$ be a nontrivial continuous character. We define a character $\psi_N:[N]\to\mathbb{C}^\times$ by

$$\psi_N(u)=\psi\left(\sum_{i=1}^{r-1}h_{n+1}(ux_{i+1},y_i)+h_{n+1}(uv_0,y_r)\right),\quad u\in[N].$$

This character extends uniquely to a character $\psi_{\mathcal{B}_h}:[\mathcal{B}_h]\to\mathbb{C}^\times$ that coincides with ψ_N on $[N]$ and is trivial on $[U(h)]$. Let σ be a cuspidal automorphic subrepresentation of $\mathcal{G}_h(\mathbb{A})$. We define the *global*

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Bessel period for $\varphi \in \sigma$ by the absolute convergent integral

$$\mathcal{P}_{\mathcal{B}_h,\psi}(\varphi) = \int_{[\mathcal{B}_h]} \varphi(g)\psi_{\mathcal{B}_h}(g)dg.$$

1.3.3. The Gan-Gross-Prasad conjecture for Bessel periods

Let $G^\flat = G_m \times G_{2n+1}$. We can now state our first theorem about Bessel periods.

Theorem 1.3.3.1. *Let Π be a discrete Hermitian Arthur parameter of $G^\flat$. The following assertions are equivalent:*

1. *The complete Rankin-Selberg L-function of Π satisfies*

$$L\left(\frac{1}{2}, \Pi\right) \neq 0;$$

2. *There exist a Hermitian form $h \in \mathcal{H}_m$ and an automorphic cuspidal subrepresentation σ of $\mathcal{G}_h(\mathbb{A})$ such that its weak base to $G^\flat$ is Π and the Bessel period*

$$\varphi \mapsto \mathcal{P}_{\mathcal{B}_h,\psi}(\varphi)$$

does not vanish identically on σ.

Remarks 1.3.3.2.

- The case $r = 0$ is just a particular case of Theorem 1.3.3.1.
- Assume $m = 0$. Then the L-function is the constant function of value 1. So the assertion 1 is automatically satisfied. However, the group $\mathcal{G}_h$ is the quasi-split unitary group U_{2r+1} of rank $2r + 1$. Moreover, the Bessel subgroup is a maximal unipotent subgroup of U_{2r+1}. Then the Bessel period is the so-called Fourier-Whittaker coefficient. The theorem is proved in the work of Ginzburg-Rallis-Soudry; see [GRS11].
- The direction $2 \Rightarrow 1$ is also proved by D. Jiang-L. Zhang; see [JZ20, Theorem 5.7].

1.3.4.

In our approach, Theorem 1.3.3.1 is a consequence of Theorem 1.2.3.1. To explain this, we may and shall assume $r > 0$. We start with a discrete Hermitian Arthur parameter Π of $G^\flat$. It can be written $\Pi = \Pi_m \boxtimes \Pi_{n+1}$, where Π_m and Π_{n+1} are respective discrete parameters of G_m and G_{n+1}. Let $\alpha_1, \ldots, \alpha_r$ be r characters of $E^\times\backslash\mathbb{A}_E^1$ such that the characters $\alpha_1, \ldots, \alpha_r, \alpha_1^*, \ldots, \alpha_r^*$ are two by two distinct (we recall that α_i^* denotes the conjugate-dual of α_i). Let $Q_n \subset G_n$ be a parabolic subgroup of Levi factor $G_1^r \times G_m \times G_1^r$. Then

$$\tilde{\Pi} = \mathrm{Ind}_{Q_n}^{G_n}(\alpha_1 \boxtimes \ldots \boxtimes \alpha_r \boxtimes \Pi_m \boxtimes \alpha_1^* \boxtimes \ldots \boxtimes \alpha_r^*) \boxtimes \Pi_{n+1}$$

is a regular Hermitian Arthur parameter of $G = G_n \times G_{n+1}$. Even if $\tilde{\Pi}$ is not discrete, it is at least H-regular in the sense of §1.2.1: this is an obvious consequence of remark 1.2.1.1 and the assumption on the characters α_i. We have an identification $\mathbb{C}^r \simeq \mathfrak{a}^*_{\tilde{\Pi},\mathbb{C}}$ such that if $\lambda_s \in \mathfrak{a}^*_{\tilde{\Pi},\mathbb{C}}$ is the image of $(s, \ldots, s)$ with $s \in \mathbb{C}$, we have

$$\tilde{\Pi}_{\lambda_s} = \mathrm{Ind}_{Q_n}^{G_n}(\alpha_1|\cdot|_E^s \boxtimes \ldots \boxtimes \alpha_r|\cdot|_E^s \boxtimes \Pi_m \boxtimes \alpha_1^*|\cdot|_E^{-s} \boxtimes \ldots \boxtimes \alpha_r^*|\cdot|_E^{-s}) \boxtimes \Pi_{n+1}.$$

For simplicity, we set $\tilde{\Pi}_s = \tilde{\Pi}_{\lambda_s}$. By elementary properties of Rankin-Selberg L-function, it is clear that assertion 1 of 1.3.3.1 is equivalent to $1'$:

$1'$. There exists $s \in i\mathbb{R}$ such that $L(\frac{1}{2}, \tilde{\Pi}_s) \neq 0$.

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Let $h \in \mathcal{H}_m$ and σ be an automorphic cuspidal subrepresentation of $\mathcal{G}_h$ whose weak base to $G^\flat$ is Π. Let $P_n \subset U(\tilde{h})$ be the parabolic subgroup stabilizing the isotropic flag

$$0 \subsetneq \mathrm{vect}(x_r) \subsetneq \mathrm{vect}(x_r, x_{r-1}) \subsetneq \ldots \subsetneq \mathrm{vect}(x_r, \ldots, x_1).$$

(Note that this flag is opposite position to (1.3.1.1).) and set $P = P_n \times U(h_{n+1})$; a parabolic subgroup of $U_{\tilde{h}} = U(\tilde{h}) \times U(h_{n+1})$. Then $G_1^r \times \mathcal{G}_h$ is a Levi factor M_P of P. Set $\tilde{\sigma} = \alpha_1 \boxtimes \ldots \boxtimes \alpha_r \boxtimes \sigma$. This is an automorphic cuspidal representation of $M_P(\mathbb{A})$, and $\tilde{\Pi}$ is the weak base change of (P, σ). Let $\varphi \in \mathcal{A}_{P,\tilde{\sigma}}(U_{\tilde{h}})$. As in subsection 1.2, we denote by $U'_{\tilde{h}}$ the 'diagonal' subgroup of $U_{\tilde{h}}$. In the case at hand, the restriction of the Eisenstein series $E(\varphi, \lambda)$ to $[U'_{\tilde{h}}]$ is rapidly decreasing for any $\lambda \in \mathfrak{a}_{\tilde{\Pi},\mathbb{C}}$, where the Eisenstein series is regular, and for any such λ, we have

$$\mathcal{P}_{U'_{\tilde{h}}}(\varphi, \lambda) = \int_{[U'_{\tilde{h}}]} E(x, \varphi, \lambda)\, dx$$

where the left-hand side is defined according to (1.2.2.1) and the right-hand side is absolutely convergent (see Proposition 3.5.3.1 assertion 3). Moreover, the map $s \mapsto \mathcal{P}_{U'_{\tilde{h}}}(\varphi, \lambda_s)$ is meromorphic and holomorphic on $i\mathbb{R}$. We prove in Proposition 8.8.2.1 that the map $\varphi \in \mathcal{A}_\sigma(\mathcal{G}_h) \mapsto \mathcal{P}_{\mathcal{B}_h,\psi}(\varphi)$ does not vanish identically if and only if there is $s \in \mathbb{C}$ such that the map $\varphi \mapsto \mathcal{P}_{U'_{\tilde{h}}}(\varphi, \lambda_s)$ does not vanish identically on $\mathcal{A}_{P,\tilde{\sigma}}(U_{\tilde{h}})$. This last fact is eventually a consequence of some unfolding identity that roughly takes the following form:

$$\mathcal{P}_{U'_{\tilde{h}}}(\varphi, \lambda_s) = \int_{\mathcal{B}'(\mathbb{A}) \backslash U'_{\tilde{h}}(\mathbb{A})} \mathcal{P}_{\mathcal{B}_h,\psi}(\varphi_s(h))dh$$

for $\varphi \in \mathcal{A}_{P,\tilde{\sigma}}(U_{\tilde{h}})$, where φ_s stands for the corresponding element of $\mathrm{Ind}_P^{U_{\tilde{h}}}(\tilde{\sigma} \otimes \lambda_s)$ (given through the choice of a suitable Iwasawa decomposition $U_{\tilde{h}}(\mathbb{A}) = P(\mathbb{A})K$ that is implicit in the definition of the Eisenstein series $E(\varphi, \lambda_s)$) and $\mathcal{B}' = U(h) \rtimes V$ with V the unipotent radical of the parabolic subgroup of $U(\tilde{h})$ stabilizing the isotropic subspace $\mathrm{vect}(x_1, \ldots, x_r)$. It should be emphasized however that this identity does not make sense per se, as the Eulerian integral on the right-hand side is not absolutely convergent in general. More precisely, it has to be 'interpreted in the sense of L-functions', which requires some nontrivial unramified computations of local integrals involving Bessel functions. We refer the reader to Section 8 and, more specifically, 8.7 and 8.8 for details.

It follows that condition 2 of Theorem 1.3.3.1 holds for $h \in \mathcal{H}_m$ and σ if and only if the following assertion holds:

2'. There exists $s \in i\mathbb{R}$ such that $\varphi \mapsto \mathcal{P}_{U'_{\tilde{h}}}(\varphi, \lambda_s)$ does not vanish identically on $\mathcal{A}_{P,\sigma}(U_{\tilde{h}})$.

It is then straightforward to deduce Theorem 1.3.3.1 from Theorem 1.2.3.1.

1.3.5. Local Bessel periods

From now on, we fix $h \in \mathcal{H}_m$ and a decomposition of the character $\psi = \otimes_{v \in V_F} \psi_v$ from which we get a decomposition $\psi_{\mathcal{B}_h} = \otimes_{v \in V_F} \psi_{\mathcal{B}_h,v}$, where $\psi_{\mathcal{B}_h,v}$ is a character of $\mathcal{B}_h(F_v)$. Let v be a place of F. The integral

$$\int_{\mathcal{B}_h(F_v)} f_v(g_v) \psi_{\mathcal{B}_h,v}(g_v) dg_v$$

is well defined for a smooth and compactly supported function f_v on $\mathcal{G}_h(F_v)$ and extends to a continuous linear form $f_v \mapsto \mathcal{P}_{\mathcal{B}_h,\psi_v}(f_v)$ on the space of tempered functions; see subsection 8.4. It depends on the choice of a Haar measure on $\mathcal{B}_h(F_v)$.

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Let σ_v be a tempered irreducible representation of $\mathcal{G}_h(F_v)$ equipped with an invariant inner product $(\cdot,\cdot)_v$. Let φ_v and φ'_v be vectors of σ_v. The associated matrix coefficient defined by $f_{\varphi_v,\varphi'_v}(g) = (\sigma_v(g)\varphi_v, \varphi'_v)_v$ for all $g \in \mathcal{G}_h(F_v)$ belongs to this space and we set

$$\mathcal{P}_{\mathcal{B}_h,\psi_v}(\varphi_v,\varphi'_v) = \mathcal{P}_{\mathcal{B}_h,\psi_v}(f_{\varphi_v,\varphi'_v}).$$

1.3.6. The Ichino-Ikeda conjecture for Bessel periods

Let σ be a tempered automorphic cuspidal subrepresentation σ of $\mathcal{G}_h(\mathbb{A})$. Tempered means that we have a decomposition $\sigma = \otimes'_{v\in V_F}\sigma_v$ with σ_v tempered for all v. We also assume that the weak base change of σ to $G^\flat$ is a discrete Hermitian parameter Π. As in §1.2.4, we define the ratio of L-functions $\mathcal{L}(s,\sigma)$ and its local counterparts $\mathcal{L}(s,\sigma_v)$ for $s \in \mathbb{C}$. Explicitly, we have

$$\mathcal{L}(s,\sigma_v) = \prod_{i=1}^{n+1} L(s+i-1/2,\eta_v^i)\frac{L(s,\Pi_v)}{L(s+1/2,\Pi_v,\mathrm{As}')},$$

where $\mathrm{As}' = \mathrm{As}^{(-1)^m} \otimes \mathrm{As}^{(-1)^{n+1}}$ and $\mathcal{L}(s,\sigma)$ is the product of the local factors in some half-plane. We use the local factor to define the normalized local Bessel period

$$\mathcal{P}^\natural_{\mathcal{B}_h,\psi_v}(\varphi_v,\varphi'_v) = \mathcal{L}\left(\frac{1}{2},\sigma_v\right)^{-1}\mathcal{P}_{\mathcal{B}_h,\psi,v}(f_{\varphi_v,\varphi'_v}).$$

We assume that the product of local measures on $\mathcal{B}_h(F_v)$ gives the Tamagawa measure on $\mathcal{B}_h(\mathbb{A})$. On σ, we use the Petersson inner product $(\cdot,\cdot)_{\mathrm{Pet}}$ normalized by the Tamagawa measure on $\mathcal{G}_h(\mathbb{A})$.

Theorem 1.3.6.1. *Let σ and Π as above. For every nonzero factorizable vector $\varphi = \otimes'_v\varphi_v \in \sigma$, we have*

$$\frac{|\mathcal{P}_{\mathcal{B}_h,\psi}(\varphi)|^2}{(\varphi,\varphi)_{\mathrm{Pet}}} = |S_\Pi|^{-1}\mathcal{L}\left(\frac{1}{2},\sigma\right)\prod_v \frac{\mathcal{P}^\natural_{\mathcal{B}_h,\psi_v}(\varphi_v,\varphi_v)}{(\varphi_v,\varphi_v)_v}. \tag{1.3.6.1}$$

Remarks 1.3.6.2.

1. In the right-hand side, almost all factors are equal to 1; see [Liu16, Theorem 2.2].
2. The statement has been conjectured by Y. Liu in a more general context; see [Liu16, conjecture 2.5].
3. For $m = 0$, the group $\mathcal{G}_h$ is the quasi-split unitary group U_{2r+1} of rank $2r + 1$. The theorem has been conjectured by Lapid and Mao, [LM15, conjecture 1.1].
4. The proof we give is along the same lines as for Theorem 1.3.3.1; namely, it is eventually deduce it from Theorem 1.2.4.1 in a similar fashion.

1.4. On some spectral contributions of the Jacquet-Rallis trace formulas

1.4.1.

In this subsection, we explain some new ingredients that play a role in the proof of Theorems 1.2.3.1 and 1.2.4.1. As many other contributions on the subject (among them, see [Zha14b], [Zha14a], [Xue19], [Beu21], [BP21], [BLZZ21], [BPCZ22]), we follow the strategy of the seminal paper [JR11] of Jacquet and Rallis. More precisely, besides the local harmonic analysis performed in the mentioned papers, our work is based on the geometric comparison, fully established in [CZ21], of the *relative trace formulas* constructed in [Zyd20] of the unitary groups U_h for $h \in \mathcal{H}_n$ and the corresponding group G. However, to be able to exploit this comparison, we need to obtain more tractable expressions for the spectral contributions we are interested in.

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1.4.2.
Let us first explain our result in the unitary case namely for the group $U = U_h$ and its subgroup $U' = U'_h$. Let $\mathfrak{X}(U)$ be the set of cuspidal data of U. According to the work of Zydor (see [Zyd20, section 4]), the contribution of $\chi \in \mathfrak{X}(U)$ to the relative trace formula for the group U is built upon the absolutely convergent integral

$$\int_{[U']\times[U']} K^T_{f,\chi}(x,y)\,dxdy.$$

Here, $K^T_{f,\chi}$ is a suitably modified version à la Arthur of the χ-part $K_{f,\chi}$ of the automorphic kernel $K_f(x,y) = \sum_{\gamma\in U(F)} f(x^{-1}\gamma y)$ associated to a Schwartz function f on $U(\mathbb{A})$; see (3.2.2.1) below for the precise definition. It depends on a truncation paramerer T. It turns out that the integral above is an exponential-polynomial function in T whose purely polynomial part is constant and gives by definition the χ-contribution denoted by $J^U_\chi(f)$ of the relative trace formula; see Theorem 3.2.3.1 for this slight extension of Zydor's work to Schwartz test functions. The problem, however, is to get an expression for $J_\chi(f)$ that reflects the Langlands spectral decomposition of $K_{f,\chi}$ and that is related to the periods (1.2.2.1) we are interested in. The starting point is the following new independent characterization of $J^U_\chi(f)$: the integral

$$\int_{[U']\times[U']} (K_{f,\chi}\Lambda^T_u)(x,y)\,dxdy \qquad (1.4.2.1)$$

is absolutely convergent and is asymptotic to an polynomial exponential in the variable T whose purely polynomial term is constant and equal to $J^U_\chi(f)$; see Corollary 3.3.5.2. Here, $K_{f,\chi}\Lambda^T_u$ means that we have applied the Ichino-Yamana truncation operator Λ^T_u already mentioned in §1.2.2 to the right variable of the kernel $K_{f,\chi}$. Let us now assume that the cuspidal datum χ is (U,U')-regular in the sense of §3.5.2. Then the expression (1.4.2.1) does not depend on T and thus is equal to $J^U_\chi(f)$. To state our result, we fix a representative (M_P,σ) where M_P is a Levi factor of a parabolic subgroup $P = M_P N_P$ of U and σ is a cuspidal automorphic representation of $M_P(\mathbb{A})$. Let $\mathcal{A}_{P,\sigma,\mathrm{cusp}}(U_h)$ be the space of automorphic forms on the quotient $A^\infty_P M_P(F)N_P(\mathbb{A})\backslash U(\mathbb{A})$ such that for all $g \in U(\mathbb{A})$,

$$m \in M_P(\mathbb{A}) \mapsto \delta_P(m)^{-\frac{1}{2}}\varphi(mg)$$

belongs to the σ-isotypic component of the space of cuspidal automorphic forms on the quotient $A^\infty_P M_P(F)\backslash M_P(\mathbb{A})$. Working throughout Langlands spectral decomposition of $K_{f,\chi}$, we get (see Theorem 3.5.7.1):

$$J^U_\chi(f) = \int_{i\mathfrak{a}^*_P} J^U_{P,\sigma}(\lambda,f)\,d\lambda, \qquad (1.4.2.2)$$

where the right-hand side is the absolutely convergent integral of the relative character defined by

$$J^U_{P,\sigma}(\lambda,f) = \sum_{\varphi\in\mathcal{B}_{P,\sigma}} \mathcal{P}_{U'}(I_P(\lambda,f)\varphi,\lambda)\overline{\mathcal{P}_{U'}(\varphi,\lambda)}.$$

Here, the periods $\mathcal{P}_{U'}(\cdot,\lambda)$ are those defined in (1.2.2.1), and $I_P(\lambda,f)$ denotes the induced action of f twisted by λ. The sum is over some orthonormal basis $\mathcal{B}_{P,\sigma}$ of $\mathcal{A}_{P,\sigma,\mathrm{cusp}}(U)$; see §3.5.5 for the Petersson inner product.

1.4.3.
Let us turn to the linear case – namely, $G = G_n \times G_{n+1}$. In this case, we have to consider two subgroups – namely, $H = G_n$ diagonally embedded in G and $G' = G'_n \times G'_{n+1}$, where $G'_n = \mathrm{GL}(n,F)$ is naturally

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embedded in $G_n = \mathrm{GL}(n, E)$. Let χ be a cuspidal datum of G and let f be a Schwartz function on $G(\mathbb{A})$. As before, we denote by $K_{f,\chi}$ the χ-part of the automorphic kernel. According to [BPCZ22, Theorem 1.2.4.1], the contribution $I_\chi(f)$, as defined by Zydor in [Zyd20], is also the constant term of the polynomial exponential (in the variable T) which is asymptotic to the absolutely convergent integral

$$\int_{[H]} \int_{[G']} \Lambda_r^T K_\chi(h, g)\, \eta_{G'}(g) dg dh.$$

Here, $\eta_{G'}$ is the quadratic character defined in §4.1.1 and Λ_r^T is a truncation operator (in the parameter T) introduced by Ichino-Yamana and well-suited for the study of Rankin-Selberg period. Assume that χ is represented by a pair (M, π), where M is the Levi factor of a parabolic subgroup P of G. We assume also that χ is (G, H)-regular and Hermitian in the sense of §4.1.3. Then we have (see Theorem 4.1.8.1)

$$I_\chi(f) = 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} I_{P,\pi}(\lambda, f)\, d\lambda.$$

Here, L is a Levi subgroup of G containing M and determined by π (see §4.1.4). For $\lambda \in i\mathfrak{a}_M^{L,*}$, the relative character $I_{P,\pi}(\lambda, f)$ is given by one of the two expressions

$$\begin{aligned} I_{P,\pi}(\lambda, f) &= \sum_{\varphi \in \mathcal{B}_{P,\pi}} \mathbf{P}(E(I_P(\lambda, f)\varphi, \lambda)) \cdot \overline{J(\xi, \varphi, \lambda)} \\ &= \sum_{\varphi \in \mathcal{B}_{P,\pi}} \frac{Z^{RS}(0, W(I_P(\lambda, f)\varphi, \lambda))\overline{\beta_\eta(W(\varphi, \lambda))}}{\langle W(\varphi, \lambda), W(\varphi, \lambda)\rangle_{\mathrm{Pet}}}. \end{aligned}$$

The sums are over some orthonormal bases $\mathcal{B}_{P,\pi}$ for the Petersson inner product of the space $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ (defined as above). The first expression is built upon $\mathbf{P}(E(\varphi, \lambda))$ and $J(\xi, \varphi, \lambda)$. The former is the regularized Rankin-Selberg period à la Ichino-Yamana (see [IY15]) of the Eisenstein series associated to the pair (P, π). The latter is the intertwining (Flicker-Rallis) period of Jacquet-Lapid-Rogawski [JLR99]. The second expression uses Whittaker functionals $W(\varphi, \lambda)$ associated to Eisenstein series and linear forms on the Whittaker models of $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes \lambda)$ equipped with the Petersson product $\langle \cdot, \cdot \rangle_{\mathrm{Pet}}$. The linear forms Z^{RS} and β_η are counterparts of the period and the intertwining period. In the Rankin-Selberg case, the link is recalled in Proposition 4.2.5.1: it is based on [IY15] which generalizes the classical Rankin-Selberg theory. In the Flicker-Rallis case, the precise relation is given in Proposition 4.2.7.1, which generalizes the work of Flicker [Fli88]. Besides some reductions based on [BPCZ22, section 9], the bulk of the proof of proposition is the object of Section 5. Note also that we prove in Section 5 a result that is of independent interest: we express a basic intertwining period of Jacquet-Lapid-Rogawski in terms of an integral of a Whittaker functional; see Theorem 5.5.1.1. The second expression of the relative character is better suited for the proof of Theorem 1.2.4.1. In Section 6, we give an alternative proof of this spectral expansion of $I_\chi(f)$, for (G, H)-regular χ, that is based on the theory of Zeta integrals.

Finally, let us remark that if χ is a (U, U')-regular cuspidal datum of U attached to a cuspidal representation of $M_{P_n}(\mathbb{A}) \times U(h \oplus h_0)(\mathbb{A})$ for some parabolic subgroup P_n of $U(h)$, then the modified kernel $K_{f,\chi}^T(x, y)$ coincides with the usual χ-part of the kernel. Then one can directly get the expression (1.4.2.2) in which the U'-periods are absolutely convergent. This includes, in particular, the Eisenstein series needed for the deduction of the Gan-Gross-Prasad and Ichino-Ikeda conjectures for general Bessel periods, as outlined in subsection 1.3.

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1.5. Organization of the paper

1.5.1.
The reader will find the main notations and some prelimary results in Section 2. The Section 3 is devoted to the study of some spectral contributions of the Jacquet-Rallis trace formula for unitary groups. The main general result is Theorem 3.2.3.1, which gives a more tractable expression to compute spectral contributions. The proof of Theorem 3.2.3.1 is the bulk of subsections 3.3 and 3.4. Then in subsection 3.5, the spectral contribution for some cuspidal data are explicitly given in terms of relative characters (see Theorem 3.5.7.1). In Section 4, we turn to the Jacquet-Rallis trace formula for general linear groups. The main result is Theorem 4.1.8.1, which expresses the spectral contribution for some cuspidal data in terms of relative characters. In subsection 4.2, we show that these relative characters can expressed in terms of Whittaker functionals; see Theorem 4.2.8.1. The main result of Section 5 is Theorem 5.5.1.1, which relates some basic intertwining period of Jacquet-Lapid-Rogawski to some integral of a Whittaker functional: it is used in the proof of Theorem 4.2.8.1, but it is also of independent interest. Its proof occupies the whole part of Section 5. As explained above, we provide in Section 6 an alternative proof for the description of the spectral contributions to the Jacquet-Rallis trace formula for general linear groups that can be obtained by combining Theorem 4.1.8.1 with Theorem 4.2.8.1. In Section 7, we explain the proof of Theorems 1.2.3.1 and 1.2.4.1 based on the comparison of the Jacquet-Rallis trace formulas and the results obtained before. The aim of the final Section 8 is to establish the reduction of Theorems 1.3.3.1 and 1.3.6.1 to special cases of Theorems 1.2.3.1 and 1.2.4.1, respectively. Its most technical part is in subsection 8.7, where necessary unramified computation is performed. Finally, Appendix A presents a probably well-known extension of Weyl's character formula to non-connected groups that is necessary for the unramified computation.

2. Preliminaries

2.1. Algebraic and adelic groups

2.1.1.
We shall try to follow the usual notations of Arthur and the main notations of the previous article [BPCZ22]. For the reader's convenience, we briefly recall our choices.

2.1.2.
We denote by F a number field, V_F (resp. $V_{F,\infty}$) the set of its places (resp. Archimedean places) and $\mathbb{A}$ its ring of adèles. For $v \in V_F$, let F_v be its completion at v. For any finite subset $S \subset V_F$, we set $F_S = \otimes_{v\in S} F_v$ and $F_\infty = F_{V_{F,\infty}}$. We denote by $|\cdot|$ the morphism $\mathbb{A}^\times \to \mathbb{R}_+^\times$ given by the product of normalized absolute values $|\cdot|_v$ on each F_v.

2.1.3.
Let G be a reductive group defined over F. All the subgroups of G we consider are assumed to be defined over F. We fix $P_0 \subset G$ a minimal parabolic subgroup and M_0 a Levi factor of P_0. A parabolic subgroup of G which contains P_0, resp. M_0, is said to be standard, resp. semi-standard. Let P be a semi-standard parabolic subgroup of G. It has a Levi decomposition $P = M_P N_P$ such that M_P is a semi-standard Levi factor (that is, $M_0 \subset M_P$) and N_P is the unipotent radical of P. Such a group M_P is called a semi-standard Levi subgroup of G. It is said to be standard if, moreover, P is standard. Let $X^*(P)$ be the group of rational characters of P defined over F. Attached to P are real vector spaces $\mathfrak{a}_P^* = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_P = \mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{R})$ in canonical duality:

$$\langle \cdot, \cdot \rangle : \mathfrak{a}_P^* \times \mathfrak{a}_P \to \mathbb{R}. \tag{2.1.3.1}$$

If $P \subset Q \subset G$, we have natural maps $\mathfrak{a}_Q^* \to \mathfrak{a}_P^*$ and $\mathfrak{a}_P \to \mathfrak{a}_Q$. The kernel of the second one is denoted by $\mathfrak{a}_P^Q$. We have natural decomposition $\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q$ and dually $\mathfrak{a}_P^* = \mathfrak{a}_P^{Q,*} \oplus \mathfrak{a}_Q^*$. We put a subscript

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$\mathbb{C}$ to denote the extension of scalars to $\mathbb{C}$. Then one has a decomposition

$$\mathfrak{a}_{P,\mathbb{C}}^{Q,*} = \mathfrak{a}_P^{Q,*} \oplus i\mathfrak{a}_P^{Q,*},$$

where $i^2 = -1$. We shall denote by $\Re$ and $\Im$ the associated projections. The complex conjugate is then defined by $\bar{\lambda} = \Re(\lambda) - i\Im(\lambda)$. Note that the spaces $\mathfrak{a}_P^Q$, $\mathfrak{a}_P^{Q,*}$ depend only on the Levi factors M_Q and M_P and thus are also denoted by $\mathfrak{a}_{M_P}^{M_Q}$, $\mathfrak{a}_{M_P}^{M_Q,*}$ etc.

Let Ad_P^Q be the adjoint action of M_P on the Lie algebra of $M_Q \cap N_P$. Let ρ_P^Q be the unique element of $\mathfrak{a}_P^{Q,*}$ such that for every $m \in M_P(\mathbb{A})$, we have

$$|\det(\mathrm{Ad}_P^Q(m))| = \exp(\langle 2\rho_P^Q, H_P(m)\rangle).$$

For every $g \in G(\mathbb{A})$, we set

$$\delta_P^Q(g) = \exp(\langle 2\rho_P^Q, H_P(g)\rangle).$$

If the context is clear, we set $\rho_P = \rho_P^G$ and $\delta_P = \delta_P^G$. Usually, we replace the subscript P_0 simply by 0 (for example, $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$, $\rho_0 = \rho_{P_0}$, etc.).

2.1.4.
Let P be a standard parabolic subgroups of G. Let $A_P = A_{M_P}$ be the maximal central F-split torus of M_P. Let Δ_0^P be the set of simple roots of A_0 in $M_P \cap P_0$. Let Δ_P^Q be the image of $\Delta_0^Q \setminus \Delta_0^P$ by the projection $\mathfrak{a}_0^* \to \mathfrak{a}_P^*$. It is a basis of $\mathfrak{a}_P^{Q,*}$ whose dual basis is the subset of coweights $\hat{\Delta}_P^{Q,\vee} \subset \mathfrak{a}_P^Q$. We have also the set of coroots $\Delta_P^{Q,\vee}$ which is a basis of $\mathfrak{a}_P^Q$. We have the dual basis given by the set of simple weights $\hat{\Delta}_P^Q \subset \mathfrak{a}_P^{Q,*}$. The sets Δ_P^Q and $\hat{\Delta}_P^Q$ determine open cones in $\mathfrak{a}_0$ whose characteristic functions are denoted respectively by τ_P^Q and $\hat{\tau}_P^Q$. We set

$$\mathfrak{a}_P^{*,Q+} = \left\{\lambda \in \mathfrak{a}_P^* \mid \langle\lambda, \alpha^\vee\rangle \geqslant 0,\ \forall \alpha^\vee \in \Delta_P^{Q,\vee}\right\}.$$

We define similarly $\mathfrak{a}_P^{Q+}$ using roots instead of coroots. If $Q = G$, the exponent G is omitted.

2.1.5.
Let W be the Weyl group of (G, A_0) – namely, the quotient of the normalizer of A_0 in $G(F)$ by M_0. For $P = M_P N_P$ and $Q = M_Q N_Q$ two standard parabolic subgroups of G, we denote by $W(P, Q)$ or $W(M_P, M_Q)$ the set of $w \in W$ such that $w\Delta_0^P = \Delta_0^Q$. For $w \in W(P, Q)$, we have $wM_P w^{-1} = M_Q$. When $P = Q$, the group $W(P, P)$ is simply denoted by $W(P)$ or $W(M_P)$. We will also write W^{M_P} for the Weyl group of (M_P, A_0).

2.1.6.
Let $K = \prod_{v \in V_F} K_v \subset G(\mathbb{A})$ be a 'good' maximal compact subgroup in good position relative to M_0 (called 'admissible' in [Art81, p. 9]). We write

$$K = K_\infty K^\infty,$$

where $K_\infty = \prod_{v \in V_{F,\infty}} K_v$ and $K^\infty = \prod_{v \in V_F \setminus V_{F,\infty}} K_v$. We have a homomorphism $H_P : P(\mathbb{A}) \to \mathfrak{a}_P$ such that $\langle\chi, H_P(p)\rangle = \log|\chi(p)|$ for any $p \in P(\mathbb{A})$ and $\chi \in X^*(P)$. By Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$, it extends to a map $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ which is left-invariant by $M_P(F)N_P(\mathbb{A})$ and right-invariant by K. We denote $A_P^\infty = A_{M_P}^\infty$ the neutral component of the group of real points of the maximal $\mathbb{Q}$-split torus of the Weil restriction $\mathrm{Res}_{F/\mathbb{Q}}(A_P)$. Then the restriction of H_P to A_P^∞ is an isomorphism.

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We set $[G]_P = M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$ and $[G]_{P,0} = A_P^\infty M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$. Let $[G]_P^1$ be the subset of $[G]_P$ where the map H_P vanishes. If $P = G$, we shall omit the subscript P.

2.1.7.
We fix a height $\|\cdot\|$ on $G(\mathbb{A})$ as in [BPCZ22, section 2.4]. Let $P \subset G$ be a standard parabolic subgroup. We set for $x \in [G]_P$

$$\|x\|_P = \inf_{\gamma \in M_P(F)N_P(\mathbb{A})} \|\gamma x\|.$$

2.1.8. Haar measures
Let us explain briefly our choice and notations of measures; see [BPCZ22, section 2.3] for more details. We fix a nontrivial continuous additive character $\psi' : \mathbb{A}/F \to \mathbb{C}^\times$. For each place $v \in V_F$, the local component ψ'_v of ψ' determines an autodual Haar measure on F_v. The choice of an invariant rational volume form on G then determines a Haar measure dg_v on $G(F_v)$. We have an Artin-Tate L-function $L_G(s) = \prod_{v\in V_F} L_{G,v}(s)$; see [Gro97] and more generally $L_G^S(s) = \prod_{v\in V_F\setminus S} L_{G,v}(s)$ for $S \subset V_F$ finite. Then $\Delta_G^{S,*}$ (simply Δ_G^* if S is empty) is defined to be the leading coefficient in the Laurent expansion of $L_G(s)$ at $s = 0$. We also set $\Delta_{G,v} = L_{G,v}(0)$. The Tamagawa measure dg on $G(\mathbb{A})$ is defined by $dg = dg_S \times dg^S$, where $dg_S = \prod_{v\in S} dg_v$ and $dg^S = (\Delta_G^{S,*})^{-1} \prod_{v\notin S} \Delta_{G,v} dg_v$ for $S \subset V_F$ finite.

We equip $\mathfrak{a}_P$ with the Haar measure that gives a covolume 1 to the lattice $\mathrm{Hom}(X^*(P), \mathbb{Z})$ and $i\mathfrak{a}_P^*$ with the dual Haar measure. The group A_P^∞ is equipped with the Haar measure compatible with the isomorphism $A_P^\infty \simeq \mathfrak{a}_P$ induced by the map H_P. The groups $\mathfrak{a}_P^G \simeq \mathfrak{a}_P/\mathfrak{a}_G$ and $i\mathfrak{a}_P^{G,*} \simeq i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ are provided with the quotient Haar measures.

The homogeneous space $[G]$ (resp. $[G]^1 \simeq [G]_0$) is equipped with the quotient of the Tamagawa measure on $G(\mathbb{A})$ by the counting measure on $G(F)$ (resp. by the product of the counting measure on $G(F)$ with the Haar measure we fixed on A_G^∞). For P a standard parabolic subgroup, we equip similarly $[G]_P$ with the quotient of the Tamagawa measure on $G(\mathbb{A})$ by the product of the counting measure on $M_P(F)$ with the Tamagawa measure on $N_P(\mathbb{A})$. Since the action by left translation of $a \in A_P^\infty$ on $[G]_P$ multiplies the measure by $\delta_P(a)^{-1}$, taking the quotient by the Haar measure on A_P^∞ induces a 'semi-invariant' measure on $[G]_{P,0} = A_P^\infty\backslash[G]_P$.

2.2. *Space of functions*

2.2.1.
For two positive functions f and g on a set X, we write $f(x) \ll g(x),\ x \in X$ if there exists a constant $C > 0$ such that $f(x) \leqslant Cg(x)$ for every $x \in X$. We write $f(x) \sim g(x),\ x \in X$ if $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

2.2.2.
For every $C \in \mathbb{R} \cup \{-\infty\}$ with $D > C$, we set $\mathcal{H}_{>C} = \{z \in \mathbb{C} \mid \Re(z) > C\}$.

2.2.3. Schwartz spaces
As before, G is a reductive group defined over F. We let $\mathfrak{g}_\infty$ be the Lie algebra of $G(F_\infty)$ and $\mathcal{U}(\mathfrak{g}_\infty)$ be the enveloping algebra of its complexification and $\mathcal{Z}(\mathfrak{g}_\infty) \subset \mathcal{U}(\mathfrak{g}_\infty)$ be its center.

We shall briefly review several useful locally convex topological spaces of functions; see [BPCZ22, section 2.5] for more details. Let $\mathcal{S}(G(\mathbb{A}))$ be the Schwartz space of $G(\mathbb{A})$: it contains the dense subspace $C_c^\infty(G(\mathbb{A}))$ of smooth and compactly supported functions. Let $P \subset G$ be a standard parabolic subgroup. Let $\mathcal{S}^0([G]_P)$ be the space of measurable functions $\varphi : [G]_P \to \mathbb{C}$ such that

$$\|\varphi\|_{\infty,N} = \sup_{x\in[G]_P} \|x\|_P^N |\varphi(x)| < \infty \qquad (2.2.3.1)$$

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for every $N > 0$. Let $\mathcal{S}([G]_P)$ be the Schwartz space of $[G]_P$ that is the space of smooth functions $\varphi : [G]_P \to \mathbb{C}$ such that for every $N > 0$ and $X \in \mathcal{U}(\mathfrak{g}_\infty)$, we have

$$\|\varphi\|_{N,X} = \sup_{x \in [G]_P} \|x\|_P^N |(R(X)\varphi)(x)| < \infty.$$

The space of functions of uniform moderate growth on $[G]_P$ is defined as

$$\mathcal{T}([G]_P) = \bigcup_{N>0} \mathcal{T}_N([G]_P),$$

where $\mathcal{T}_N([G]_P)$ is the space of smooth functions $\varphi : [G]_P \to \mathbb{C}$ such that for every $X \in \mathcal{U}(\mathfrak{g}_\infty)$, we have

$$\|\varphi\|_{-N,X} = \sup_{x \in [G]_P} \|x\|_P^{-N} |(R(X)\varphi)(x)| < \infty. \tag{2.2.3.2}$$

All these spaces are equipped with locally convex topologies. The space $\mathcal{S}^0([G]_P)$ is equipped with the family of semi-norms $\|\cdot\|_{\infty,N}$; it is a Fréchet space. The spaces $\mathcal{S}(G(\mathbb{A}))$, $\mathcal{S}([G]_P)$, $\mathcal{T}_N([G]_P)$ and $\mathcal{T}([G]_P)$ are LF-spaces: we refer the reader to [BPCZ22, section 2.5] for a precise description of the topology.

2.2.4. Automorphic forms

Here, we refer the reader to [BPCZ22, section 2.7] for more details. The space $\mathcal{A}_P(G)$ of automorphic forms on $[G]_P$ is the subspace of $\mathcal{Z}(\mathfrak{g}_\infty)$-finite functions in $\mathcal{T}([G]_P)$. Let $\mathcal{A}_{P,\mathrm{cusp}}(G) \subset \mathcal{A}_P(G)$ be the subspace of cuspidal automorphic forms: these are the functions $\varphi \in \mathcal{A}_P(G)$ such that $\varphi_Q = 0$ for every proper parabolic subgroup $Q \subsetneq P$. The constant term φ_Q is defined by

$$\varphi_Q(x) = \int_{[N_Q]} \varphi(ux)du$$

for all $x \in G(\mathbb{A})$. A cuspidal automorphic representation π of $M_P(\mathbb{A})$ is a topologically irreducible subrepresentation of $\mathcal{A}_{\mathrm{cusp}}(M_P)$. For every $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, we define $\pi_\lambda = \pi \otimes \lambda$ as the space of functions of the form

$$m \in [M_P] \mapsto \exp(\langle \lambda, H_P(m)\rangle)\varphi(m)$$

for $\varphi \in \pi$.

We denote by $\mathcal{A}_{\pi,\mathrm{cusp}}(M_P)$ the π-isotypic component of $\mathcal{A}_{\mathrm{cusp}}(M_P)$. The normalized smooth induction $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\mathcal{A}_{\pi,\mathrm{cusp}}(M_P))$ is denoted by $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ and is identified with the space of automorphic forms $\varphi \in \mathcal{A}_P(G)$ such that

$$m \in [M_P] \mapsto \exp(-\langle \rho_P, H_P(m)\rangle)\varphi(mg)$$

belongs to $\mathcal{A}_{\pi,\mathrm{cusp}}(M_P)$ for every $g \in G(\mathbb{A})$. The algebra $\mathcal{S}(G(\mathbb{A}))$ acts on $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ by right convolution. For every $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, we denote by $I(\lambda)$ the action on $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ we get by transport from the action of $\mathcal{S}(G(\mathbb{A}))$ on $\mathcal{A}_{P,\pi_\lambda,\mathrm{cusp}}$ and the identification $\mathcal{A}_{P,\pi,\mathrm{cusp}} \to \mathcal{A}_{P,\pi_\lambda,\mathrm{cusp}}$ given by $\varphi \mapsto \exp(\langle \lambda, H_P(.)\rangle)\varphi$. Assume that the central character of π is unitary. Then we equip $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ with the Petersson inner product given by

$$\|\varphi\|^2_{\mathrm{Pet}} = \langle \varphi, \varphi \rangle_{\mathrm{Pet}} = \int_{[G]_{P,0}} |\varphi(g)|^2 dg.$$

Note that for every cuspidal automorphic representation π of $M_P(\mathbb{A})$, there exists a unique $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ such that the central character of π_λ is trivial on A_M^∞. Unless otherwise stated, the cuspidal automorphic

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representations we consider are always normalized in the sense that they are assumed to have a central character which is trivial on A_M^∞.

Let $\hat{K}$ be the set of isomorphism classes of irreducible unitary representations of K. A K-basis $\mathcal{B}_{P,\pi}$ of $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ is by definition the union over of $\tau \in \hat{K}$ of orthonormal bases $\mathcal{B}_{P,\pi,\tau}$ for the Petersson inner product of the finite dimensional subspaces $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G,\tau)$ of functions in $\mathcal{A}_{P,\pi,\mathrm{cusp}}(G)$ which transform under K according to τ.

For any $\varphi \in \mathcal{A}_{P,\mathrm{cusp}}(G)$, $g \in G(\mathbb{A})$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ and any parabolic subgroup $Q \supset P$, we introduce the Eisenstein series

$$E^Q(g,\varphi,\lambda) = \sum_{\delta \in P(F)\backslash Q(F)} \exp(\langle \lambda, H_P(\delta g)\rangle \varphi(\delta g). \tag{2.2.4.1}$$

The right-hand side is convergent for $\Re(\lambda)$ in a suitable cone, and for more general λ, the left-hand side is given by meromorphic continuation. Let P and Q be standard parabolic subgroups of G. For any $w \in W(P,Q)$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we have the intertwining operator

$$M(w,\lambda) : \mathcal{A}_P(G) \to \mathcal{A}_Q(G).$$

For more details and continuity properties of these constructions, we refer the reader to [BPCZ22, §§2.7.3, 2.7.4].

2.2.5. Cuspidal datum

Let $P \subset G$ be a standard parabolic subgroup of G. Let $L^2([G]_P)$ be the space of square integrable functions on $[G]_P$. Let $\mathfrak{X}(G)$ be the set of cuspidal data of G. Recall that $\mathfrak{X}(G)$ is the quotient of the set of pairs (M_P,π) such that

- P is a standard parabolic subgroup of G;
- π is the isomorphism class of a cuspidal automorphic representation of $M_P(\mathbb{A})$ with central character trivial on A_P^∞

by the following equivalence relation: $(M_P,\pi) \sim (M_Q,\tau)$ if there exists $w \in W(P,Q)$ such that $w\pi w^{-1} \simeq \tau$. Then for any standard parabolic subgroup P of G, we have the Langlands decomposition (see [BPCZ22, section 2.9] for more details):

$$L^2([G]_P) = \widehat{\bigoplus_{\chi \in \mathfrak{X}(G)}} L^2_\chi([G]_P).$$

The Schwartz algebra acts on $L^2([G]_P)$ and for each $\chi \in \mathfrak{X}(G)$ on $L^2_\chi([G]_P)$ by right convolution. For each $f \in \mathcal{S}(G(\mathbb{A}))$, we get integral operators whose kernels are denoted respectively by $K_{f,P}$ and $K_{f,P,\chi}$. We shall use also the decomposition

$$L^2([G]_{P,0}) = \widehat{\bigoplus_{\chi \in \mathfrak{X}(G)}} L^2_\chi([G]_{P,0}).$$

2.3. *Around Arthur's partition*

2.3.1.

Let $T_1, T \in \mathfrak{a}_0$ and P be a standard parabolic subgroup of G. We define

$$A_{P_0}^{P,\infty}(T_1) = \{a \in A_0^\infty \mid \langle \alpha, H_0(a)\rangle \geqslant \langle \alpha, T_1\rangle,\ \forall \alpha \in \Delta_0^P\}$$

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and

$$A_{P_0}^{P,\infty}(T_1,T)=\{a\in A_0^{P,\infty}(T_1)\mid \langle\varpi,H_0(a)\rangle\leqslant\langle\varpi,T\rangle,\ \forall\varpi\in\hat{\Delta}_0^P\}.$$

2.3.2. Siegel domains

Let $T_-\in\mathfrak{a}_0^G$ and $\omega_0\subset P_0(\mathbb{A})^1$ be a compact subset such that $P_0(\mathbb{A})^1=P_0(F)\omega_0$. Let P be a standard parabolic subgroup of G. We define

$$\mathfrak{s}_P(\omega_0,T_-,K)=\omega_0A_{P_0}^{P,\infty}(T_-)K.$$

There exists $T_-\in-\mathfrak{a}_0^{G,+}$ such that for all standard parabolic subgroup of G, we have $G(\mathbb{A})=P(F)\mathfrak{s}_P(\omega_0,T_-,K)$.We will fix T_- and ω_0 and we set $\mathfrak{s}_P=\mathfrak{s}_P(\omega_0,T,K)$: this is a *Siegel domain* of $[G]_P$.

Note that for any standard parabolic subgroup P of G, we have $\|x\|\sim\|x\|_P$ for $x\in\mathfrak{s}_P$; see [BPCZ22, (2.4.1.3)].

2.3.3.

For any standard parabolic subgroup P of G and $T\in\mathfrak{a}_0^+$, we consider the characteristic function $F^P(\cdot,T)$ of $\omega_0A_0^{P,\infty}(T_-,T)K\subset\mathfrak{s}_P$. This function descends on $[G]_P$. There is a point $T_0\in\mathfrak{a}_0^{G,+}$ such that for all $T\in T_0+\mathfrak{a}_0^{G,+}$ and all standard parabolic subgroup Q of G, we have the following formula which gives a partition of $[G]_Q$ (see [Art78, Lemma 6.4] and also [LW13, Proposition 3.6.3]):

$$\sum_{P_0\subset P\subset Q}\ \sum_{\delta\in P(F)\backslash Q(F)}F^P(\delta g,T)\tau_P^Q(H_P(\delta g)-T)=1.\tag{2.3.3.1}$$

Note that the relation implies a simpler definition of $F^P(\cdot,T)$ for $T\in T_0+\mathfrak{a}_0^{G,+}$; the function $F^P(\cdot,T)$ is the characteristic function of the following set:

$$\{g\in[G]_P\mid\forall\varpi\in\hat{\Delta}_0^P,\delta\in P(F)\,\langle\varpi,H_0(\delta g)-T\rangle\leqslant 0\}.\tag{2.3.3.2}$$

Indeed, this is a straightforward consequence of [Art85, Lemma 2.1] and the definition of the operator Λ^T there.

We shall use several times the following simple lemma.

Lemma 2.3.3.1. *Let $P\subset Q$ be parabolic subgroup of G. Let $g\in G(\mathbb{A})$ be such that $F^P(g,T)\tau_P^Q(H_P(g)-T)=1$. Then we have*

$$\forall\alpha\in\Delta_0^Q\setminus\Delta_0^P,\ \langle\alpha,H_0(g)\rangle>\langle\alpha,T\rangle.$$

In particular, if $g\in\mathfrak{s}_P$, then $g\in\mathfrak{s}_Q$.

Proof. Let $\alpha\in\Delta_0^Q\setminus\Delta_0^P$. We write $\alpha=\alpha_P+\alpha^P$ according to the decomposition $\mathfrak{a}_0^*=\mathfrak{a}_P^*\oplus\mathfrak{a}_0^{P,*}$. Since we have $\langle\alpha^P,\beta^\vee\rangle=\langle\alpha,\beta^\vee\rangle\leqslant 0$ for all $\beta\in\Delta_0^P$, we know that α^P is a non-positive linear combination of elements of $\hat{\Delta}_0^P$. Thus, the condition $F^P(g,T)=1$ implies that

$$\langle\alpha^P,H_0(g)\rangle\geqslant\langle\alpha^P,T\rangle.$$

The condition $\tau_P^Q(H_P(g)-T)=1$ implies that we have $\langle\alpha_P,H_0(g)\rangle>\langle\alpha_P,T\rangle$. We conclude that $\langle\alpha,H_0(g)\rangle>\langle\alpha,T\rangle$ for all $\alpha\in\Delta_0^Q\setminus\Delta_0^P$. The last assertion is then obvious since T is positive. □

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2.3.4.
Let $\phi : \mathfrak{a}_0 \to \mathbb{C}$ be a function and let $T \in T_0 + \mathfrak{a}_0^{G,+}$. For any standard parabolic subgroup Q of G, we define for all $g \in [G]_Q$,

$$d_Q(\phi, g, T) = \sum_{P_0 \subset P \subset Q} \sum_{\delta \in P(F) \backslash Q(F)} F^P(\delta g, T) \tau_P^Q(H_P(\delta g) - T) \exp(\phi(H_P(\delta g))).$$

We will mainly use $d_Q(\lambda, g, T)$ for $\lambda \in \mathfrak{a}_0^*$. The next proposition shows that the function $d_Q(\lambda, \cdot, T)$ is an example of a function that satisfies [Fra98, Proposition 2.1] (such a kind of function is also used in [BPCZ22, §2.4.3]).

Proposition 2.3.4.1. *Let $\lambda \in \mathfrak{a}_0^*$ and $Q \subset G$ a parabolic subgroup.*

1. *Let $\mathcal{T} \subset T_0 + \mathfrak{a}_0^{G,+}$ be a compact subset. We have*

$$\exp(\langle \lambda, H_0(g) \rangle) \sim d_Q(\lambda, g, T)$$

for all $g \in \mathfrak{s}_Q$ and $T \in \mathcal{T}$. In particular, all functions $d_Q(\lambda, T)$ are equivalent for $T \in \mathcal{T}$.

2. *Let $\mathcal{K}$ be a compact subset of $G(\mathbb{A})$ and let $T \in T_0 + \mathfrak{a}_0^{G,+}$. We have*

$$d_Q(\lambda, g, T) \sim d_Q(\lambda, gc, T)$$

for all $g \in [G]_Q$, $c \in \mathcal{K}$.

Proof. 1. If we write $\lambda = \lambda_Q + \lambda^Q$ according to the decomposition $\mathfrak{a}_0^* = \mathfrak{a}_Q^* + \mathfrak{a}_0^{Q,*}$, we see that $d_Q(\lambda, g, T) = \exp(\langle \lambda_Q, H_Q(g) \rangle) d_Q(\lambda^Q, g, T)$ for all $T \in T_0 + \mathfrak{a}_0^+$ and all $g \in [G]_Q$. Since we also have $\exp(\langle \lambda, H_0(g) \rangle) = \exp(\langle \lambda_Q, H_Q(g) \rangle) \exp(\langle \lambda^Q, H_0(g) \rangle)$, we may and shall assume that $\lambda \in \mathfrak{a}_0^{Q,*}$. Let $g \in \mathfrak{s}_Q$ and $T \in \mathcal{T}$. There exists $P \subset Q$ such that $F^P(g, T) \tau_P^Q(H_P(g) - T) = 1$. Then, by definition, we get $d_Q(\lambda, g, T) = \exp(\langle \lambda, H_P(g) \rangle)$. It suffices to prove that $H_0(g) - H_P(g)$ stays in a compact subset which depends only on $\mathcal{T}$ for any $g \in \mathfrak{s}_P$ such that $F^P(g, T) = 1$. In fact, the latter condition implies $\langle \varpi, H_0(g) \rangle \leqslant \langle \varpi, T \rangle$ for all $\varpi \in \hat{\Delta}_0^P$, and $g \in \mathfrak{s}_P$ implies that $\langle \alpha, H_0(g) \rangle \geqslant \langle \alpha, T_- \rangle$ for all $\alpha \in \hat{\Delta}_0^P$. Hence, the projection of $H_0(g)$ on $\mathfrak{a}_0^P$ stays in a fixed compact subset, but this projection is nothing else but $H_0(g) - H_P(g)$.

2. First, we observe that $H_0(kc)$ stays in a fixed compact for $k \in K$ and $c \in \mathcal{K}$. By assertion 1, we may replace T by any element in $T_0 + \mathfrak{a}_0^{G,+}$. In particular, we may and shall assume that T is such that $T - H_0(kc) \in T_0 + \mathfrak{a}_0^{G,+}$ for all $k \in K$ and $c \in \mathcal{K}$. For any $g \in G(\mathbb{A})$, we shall denote $k(g)$ an element of K such that $gk(g)^{-1} \in P_0(\mathbb{A})$.

Let $g \in G(\mathbb{A})$ and $c \in \mathcal{K}$. Then there exist a unique parabolic sugroup $P \subset Q$ and $\delta \in Q(F)$ such that $F^P(\delta gc, T) \tau_P^Q(H_P(\delta gc) - T) = 1$. Observe that $H_0(\delta gc) = H_0(\delta g) + H_0(k(\delta g)c)$. We deduce on the one hand that $\tau_P^Q(H_P(\delta g) - (T - H_0(k(\delta g)c)) = 1$ is equivalent to $\tau_P^Q(H_P(\delta gc) - T) = 1$, and on the other hand, we also have $F^P(\delta g, T - H_0(k(\delta g)c)) = 1$; indeed, for all $\varpi \in \hat{\Delta}^P$, we have

$$\begin{aligned} \langle \varpi, H_0(\delta g) \rangle &= \langle \varpi, H_0(\delta gc) \rangle - \langle \varpi, H_0(k(\delta g)c) \rangle \\ &\leqslant \langle \varpi, T - H_0(k(\delta g)c) \rangle \end{aligned}$$

since $F^P(\delta gc, T) = 1$. From this, we get

$$d_Q(\lambda, g, T - H_0(k(\delta g)c)) = \exp(\langle \lambda, H_P(\delta g) \rangle) = d_Q(\lambda, gc, T) \exp(-\langle \lambda, H_0(k(\delta g)c) \rangle).$$

Using assertion 1 and the fact that $H_0(k(\delta g)c)$ stays in a fixed compact set, we can easily conclude. □

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Lemma 2.3.4.2. *Let $Q \subset R$ be standard parabolic subgroups of G. There exists $T_1 \in T_0 + \mathfrak{a}_0^{G,+}$ such that for all $T \in T_1 + \mathfrak{a}_0^{G,+}$, all $\lambda \in \mathfrak{a}_0^*$ and all $g \in G(\mathbb{A})$ such that $F^Q(g,T)\tau_Q^R(H_Q(g)-T) = 1$, we have*

$$d_Q(\lambda, g, T_0) = d_R(\lambda, g, T_0).$$

Proof. Both the hypothesis and the conclusion are invariant under left translation by $Q(F)$. It suffices to prove the result for $g \in \mathfrak{s}_Q$ such that $F^Q(g,T)\tau_Q^R(H_Q(g)-T) = 1$. First, there exists a unique standard parabolic subgroup $P \subset Q$ such that $F^P(g,T_0)\tau_P^Q(H_Q(g)-T_0) = 1$. In the proof of Proposition 2.3.4.1, we have shown that $H_0(g) - H_P(g)$ stays in a fixed compact subset for $g \in \mathfrak{s}_Q$ such that $F^P(g,T_0) = 1$. In particular, we can assume that for such elements g, we have

$$\langle \alpha, T_1 \rangle > \langle \alpha_P, T_0 \rangle + \langle \alpha, H_0(g) - H_P(g) \rangle. \tag{2.3.4.1}$$

By Lemma 2.3.3.1, we have $\langle \alpha, H_0(g) \rangle > \langle \alpha, T \rangle$ for all $\alpha \in \Delta_0^R \setminus \Delta_0^Q$. So we have for all $\alpha \in \Delta_0^R \setminus \Delta_0^Q$,

$$\begin{aligned} \langle \alpha_P, H_0(g) \rangle = \langle \alpha, H_P(g) \rangle &= \langle \alpha, H_0(g) \rangle + \langle \alpha, H_P(g) - H_0(g) \rangle \\ &> \langle \alpha, T \rangle + \langle \alpha, H_P(g) - H_0(g) \rangle \\ &\geqslant \langle \alpha, T_1 \rangle + \langle \alpha, H_P(g) - H_0(g) \rangle \\ &> \langle \alpha_P, T_0 \rangle \end{aligned}$$

by (2.3.4.1). In particular, we see that we have $\tau_P^R(H_Q(g) - T_0) = 1$. Thus, by definition, $d_R(\lambda, g, T_0) = \exp(\langle \lambda, H_P(g) \rangle) = d_Q(\lambda, g, T_0)$. □

Proposition 2.3.4.3. *Let $Q \subset R$ be standard parabolic subgroups of G such that $\Delta_0^R \setminus \Delta_0^Q = \{\alpha\}$ for some simple root α. Let c such that $c > \langle \alpha, T_{0,P} \rangle$ for all parabolic subgroups $P \subset Q$.*

For any $g \in G(\mathbb{A})$, there is at most one element $\delta \in Q(F)\backslash R(F)$ such that $d_Q(\alpha, \delta g, T_0) > \exp(c)$.

Proof. Let $g \in G(\mathbb{A})$ such that $d_Q(\alpha, g, T_0) > \exp(c)$. Using left translations by $Q(F)$, we may and shall assume that there exists a standard parabolic subgroup $P \subset Q$ such that $F^P(g,T_0)\tau_P^Q(H_P(g)-T_0) = 1$. Then the condition $d_Q(\alpha, g, T_0) > \exp(c)$ is equivalent to $\langle \alpha, H_P(g) \rangle > c$, and so $\langle \alpha, H_P(g) \rangle > \langle \alpha, T_{0,P} \rangle$. So we have also $F^P(g,T_0)\tau_P^R(H_P(g)-T_0) = 1$. Now the uniqueness follows from the partition (2.3.3.1) applies to $[G]_R$. □

3. On the spectral expansion of the Jacquet-Rallis trace formula for unitary groups

3.1. Notations

3.1.1.

Let E/F be a quadratic extension of number fields and c be the nontrivial element of the Galois group $\mathrm{Gal}(E/F)$. Let $\mathbb{A}$ be the ring of adèles of F. Let $n \geqslant 0$ be an integer. Let $\mathcal{H}_n$ be the set of isomorphism classes of pairs (V,h), where V is a E-vector space of dimension n and h a nondegenerate c-Hermitian form on V. For any $(V,h) \in \mathcal{H}_n$, we identify (V,h) with a representative, and we shall denote by $U(V,h)$ or simply $U(h)$ its automorphisms group. We will fix $(V_0, h_0) \in \mathcal{H}_1$.

3.1.2.

We attach to any $h \in \mathcal{H}_n$ the following algebraic groups over F:

- the unitary group $U_h' = U(h)$ of automorphisms of h;
- the unitary group $U_h'' = U(h \oplus h_0)$ where $h \oplus h_0$ denoted the orthogonal sum;
- the product of unitary groups $U_h = U_h' \times U_h''$.

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We have an embedding

$$U'_h \hookrightarrow U''_h \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \tag{3.1.2.1}$$

and a diagonal embedding $U'_h \hookrightarrow U_h$.

3.1.3.
Let $n \geqslant 1$ be an integer and $(V,h) \in \mathcal{H}_n$. We fix a parabolic subgroup P'_0 of U'_h and a Levi factor M'_0 of P'_0 both defined over F and minimal for these properties. As a parabolic subgroup, P'_0 is the stabilizer of a flag of totally isotropic subspaces of V. Let P''_0 be the parabolic subgroup of U''_h stabilizing the same flag. Let M''_0 be the unique Levi factor of P''_0 that contains M'_0. Then $P_0 = P'_0 \times P''_0$ is a parabolic subgroup of U_h with Levi factor $M_0 = M'_0 \times M''_0$. We fix also a pair (B''_0, T''_0) consisting of a minimal parabolic F-subgroup of U''_h and a Levi factor defined over F. We may and shall assume $B''_0 \subset P''_0$ and $T''_0 \subset M''_0$. We fix maximal good compact subgroups $K'_h \subset U'_h(\mathbb{A})$ and $K''_h \subset U''_h(\mathbb{A})$, respectively, in good position relative to M'_0 and T''_0 in the sense of §2.1.6. We set $K_h = K'_h \times K''_h \subset U_h(\mathbb{A})$. Any $g \in U_h = U'_h \times U''_h$ will be written (g', g'') without any further comment.

From now on, (V,h) is fixed, and we shall omit it in the notations; thus, we have $U = U_h$, $U' = U'_h$ and so on.

3.1.4.
Let $\mathcal{F}'_0$ be the set of parabolic subgroups of U' that contain P'_0. For any $P' \in \mathcal{F}'_0$, let P'' be the parabolic subgroup of U' which stabilizes the flag of totally isotropic subspaces of V that defines P'. Note that $P'' \cap U' = P'$. We get a one-to-one map

$$P' \mapsto P = P' \times P'' \tag{3.1.4.1}$$

from $\mathcal{F}'_0$ onto a subset, denoted by $\mathcal{F}_0$ of parabolic subgroups of U. For any standard parabolic subgroup P' of U', resp. P'' of U'', resp. P of U, we will denote by $\Delta_0^{P'}$, resp. $\Delta_0^{P''}$, resp. Δ_0^P, the set of simple roots of $A_{M'_0}$, resp. $A_{T''_0}$, resp. $A_{M'_0 \times T''_0}$ in $P'_0 \cap M_{P'}$, resp. $B''_0 \cap M_{P''}$, resp. $(P'_0 \times B''_0) \cap M_P$. We set $\mathfrak{a}'_0 = \mathfrak{a}_{M'_0} = \mathfrak{a}_{P'_0}$. The inclusion $P'_0 \subset P''_0$ gives an identification $\mathfrak{a}_{P''_0} = \mathfrak{a}_{P'_0} = \mathfrak{a}'_0$. Using the map $a^*_{B''_0} \to \mathfrak{a}^*_{P''_0} = \mathfrak{a}^*_{P'_0}$ dual to the inclusion $A_{P''_0} \subset A_{T''_0}$, we see that any $\lambda \in \mathfrak{a}^*_{B''_0}$ defined a linear map $\mathfrak{a}'_0 \to \mathbb{R}$ still denoted by λ. Let $P' \in \mathcal{F}'_0$ and $P'' \subset U''$ be the associated parabolic subgroup. Observe that if λ is a simple root $\tilde{\alpha} \in \Delta_0^{U''} \setminus \Delta_0^{P''}$, the linear map $\mathfrak{a}'_0 \to \mathbb{R}$ we get is equal, up to a positive constant, to a unique root $\alpha \in \Delta_0^{U'} \setminus \Delta_0^{P'}$.

Following §2.3.2, we have the notion of Siegel domains. They depend on auxiliary choices ω'_0, T'_- for U' and ω''_0, T''_- for U''. The Siegel domains for $U = U' \times U''$ will be the product of Siegel domains for U' and U''. We may and shall assume that T'_- and T''_- are chosen so that

$$A_{P'_0}^{P',\infty}(T'_-) \subset A_{B''_0}^{P'',\infty}(T''_-). \tag{3.1.4.2}$$

We fix a height $\|\cdot\|$ on $U(\mathbb{A})$; see §2.1.7. By (diagonal) restriction, this gives a height on $U'(\mathbb{A})$. Note that for P and P' as above, we have $\|x\|_{P'} \sim \|x\|_P$ for $x \in U'(\mathbb{A})$, as it follows from [Beu21, proposition A.1.1 (ix)].

3.1.5.
We fix a point $T_0 \in \mathfrak{a}'_0$ as in §2.3.3, and we set

$$d_{P'}(\lambda) = d_{P'}(\lambda, T_0)$$

for any $P' \in \mathcal{F}'_0$ and $\lambda \in \mathfrak{a}_0^*$. The precise choice of T_0 is irrelevant in the sequel, and we will not use this notation anymore. We proceed in the same way to define $d_{P''}(\lambda)$.

Proposition 3.1.5.1.

1. *Let $\lambda \in \mathfrak{a}^*_{B''_0}$. We have*

$$d_{P''}(\lambda, x) \sim d_{P'}(\lambda, x) \quad x \in [U']_{P'}. \tag{3.1.5.1}$$

2. *For $\tilde{\alpha} \in \Delta_0^{U''} \setminus \Delta_0^{P''}$, there exists $r > 0$ such that*

$$d_{P''}(\tilde{\alpha}, x) \sim d_{P'}(\alpha, x)^r \quad x \in [U']_{P'},$$

where $\alpha \in \Delta_0^{U'} \setminus \Delta_0^{P'}$ is deduced from $\tilde{\alpha}$ as above.

Proof. 1. Both sides of (3.1.5.1) are invariant by left $P'(F)$-translation. Thus, it suffices to prove the equivalence on $\mathfrak{s}_{P'} = \omega'_0 A_{P'_0}^{P',\infty}(T'_-)K'$. Note that if $a \in A_{P'_0}^{P',\infty}(T'_-)$, then $a^{-1}\omega'_0 a$ is included in a fixed compact subset $\mathcal{K} \subset U'(\mathbb{A})$ that does not depend on a. Thus, by Proposition 2.3.4.1 assertion 2, we have $d_{P''}(xak) \sim d_{P''}(a)$ for $x \in \omega'_0$, $a \in A_{P'_0}^{P',\infty}(T'_-)$ and $k \in K'$. By the inclusion (3.1.4.2) and Proposition 2.3.4.1 assertion 1, we have $d_{P''}(a) \sim \exp(\langle \lambda, H_{B''_0}(a)\rangle)$ for $a \in A_{P'_0}^{P',\infty}(T'_-)$. Then we have $H_{B''_0}(a) = H_{P'_0}(a) = H_{P'_0}(xak)$ and $\exp(\langle \lambda, H_{B''_0}(a)\rangle) \sim d_{P'}(\lambda, xak)$ by Proposition 2.3.4.1 assertion 1 applied to P'. 2. is a consequence of 1 and the fact that the restriction of $\tilde{\alpha}$ to $\mathfrak{a}'_0$ is $r\alpha$ for some $r > 0$. □

3.1.6. Sufficiently positive *T*

We will fix an euclidean norm $\|.\|$ on $\mathfrak{a}'_0$, invariant by the Weyl group. For $T \in \mathfrak{a}'_0$, let

$$d(T) = \inf_{\alpha \in \Delta_0^{U'}} \langle \alpha, T\rangle.$$

We will fix $C > 0$ and $\varepsilon > 0$, respectively, large and small enough constants. We shall throughout assume that T is 'sufficiently positive'; that is, we assume T satisfies the inequality $d(T) \geqslant \max(\varepsilon\|T\|, C)$. In particular, we shall assume that Lemma 2.3.4.2 holds for any sufficiently positive T and our functions $d_{P'}(\lambda)$ and $d_{P''}(\lambda)$.

3.2. *Truncated kernel*

3.2.1.

Let $f \in \mathcal{S}(U(\mathbb{A}))$. For any cuspidal datum $\chi \in \mathfrak{X}(U)$ and any standard parabolic subgroup P of U, we get kernels $K_{f,P}$, resp. $K_{f,P,\chi}$; see §2.2.5. If $P = U$, we shall omit the subscript U.

3.2.2.

For any $T \in \mathfrak{a}'_0$, $x, y \in U'(\mathbb{A})$ and $\chi \in \mathfrak{X}(U)$, we set

$$K^T_{f,\chi}(x, y) = \sum_{P' \in \mathcal{F}'_0} \epsilon_{P'} \sum_{\gamma \in P'(F)\backslash U'(F)} \sum_{\delta \in P'(F)\backslash U'(F)} \hat{\tau}_{P'}(H_{P'}(\delta y) - T) K_{f,P,\chi}(\gamma x, \delta y), \tag{3.2.2.1}$$

where we set

$$\epsilon_{P'} = (-1)^{\dim(\mathfrak{a}_{P'})} \tag{3.2.2.2}$$

and $K_{f,P,\chi}$ is the kernel attached to the parabolic subgroup P of U image of P' by (3.1.4.1). Recall that we set $\hat{\tau}_{P'} = \hat{\tau}^{U'}_{P'}$.

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Remark 3.2.2.1. This is the kernel used in [Zyd20] for compactly supported functions. Since we consider more general test functions, we need a comment here. Let P and P' be as above. First, the sum over $\delta \in P'(F)\backslash U'(F)$ may be taken in a *finite* set which depends on y (see [Art78] Lemma 5.1). Second, by [BPCZ22, Lemma 2.10.1.1], there exists $N_0 > 0$ such that for any $N > 0$ and any continuous semi-norm $\|\cdot\|_{N'}$ on $\mathcal{T}_{N'}([U]_P)$ with $N' = N + N_0$, there exists a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that we have

$$\sum_{\chi \in \mathfrak{X}(U)} \|K_{f,P,\chi}(x,\cdot)\|_{N'} \leqslant \|f\|_{\mathcal{S}} \|x\|_P^{-N} \tag{3.2.2.3}$$

for all $x \in [U]_P$ and $f \in \mathcal{S}(U(\mathbb{A}))$. It follows that the sum over $\gamma \in P'(F)\backslash U'(F)$ is absolutely convergent.

3.2.3.

The next theorem is an extension to Schwartz test functions of the work of Zydor in [Zyd20, section 4]. The proof will be given in subsection 3.4 below.

Theorem 3.2.3.1. *Let $T \in \mathfrak{a}_0'$ be sufficiently positive.*

1. *We have*

$$\sum_{\chi \in \mathfrak{X}(U)} \int_{[U']\times[U']} |K^T_{f,\chi}(x,y)|\, dxdy < \infty.$$

2. *Let $\chi \in \mathfrak{X}(U)$. As a function of T, the integral*

$$J^{U,T}_\chi(f) = \int_{[U']\times[U']} K^T_{f,\chi}(x,y)\, dxdy \tag{3.2.3.1}$$

   coincides with an exponential-polynomial function in T whose purely polynomial part is constant and denoted by $J^U_\chi(f)$.
3. *For any $\chi \in \mathfrak{X}(U)$, the distribution J^U_χ is continuous, left and right $U'(\mathbb{A})$-invariant.*
4. *The sum*

$$J^U(f) = \sum_{\chi \in \mathfrak{X}(U)} J^U_\chi(f) \tag{3.2.3.2}$$

   is absolutely convergent and defines a continuous distribution J^U.

3.3. *Truncation operator and distributions J^U_χ*

3.3.1.

The goal of this section is to state Theorem 3.3.5.1, which gives the asymptotics of the distributions $J^{U,T}_\chi(f)$ defined in Theorem 3.2.3.1 when the parameter goes to infinity. The theorem will be useful for subsequent computations in Section 3.5.

3.3.2. The Ichino-Yamana truncation operator

Let $T \in \mathfrak{a}_0'$ sufficiently positive. In [IY19], Ichino-Yamana defined a truncation operator which transforms functions of uniform moderate growth on $[U'']$ into rapidly decreasing functions on $[U']$; see [IY19, lemma 2.2]. By applying it to the *right* component of $[U] = [U'] \times [U'']$, we get a truncation operator which we denote by Λ^T_u. It associates to any function φ on $[U]$ the function on $[U']$ defined by the following formula: for any $x \in [U']$,

$$(\Lambda^T_u \varphi)(x) = \sum_{P' \in \mathcal{F}_0'} \epsilon_{P'} \sum_{\delta \in P'(F)\backslash U'(F)} \hat{\tau}_{P'}(H_{P'}(\delta x) - T)\varphi_{U'\times P''}(\delta x), \tag{3.3.2.1}$$

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where we follow notations involved in (3.2.2.1) and in §3.1.4. Moreover, $\varphi_{U'\times P''}$ is the constant term of φ along the parabolic subgroup $U'\times P''$ of U, where $P'\times P''$ is the image of P' by (3.1.4.1). We shall recall and precise the main properties of Λ_u^T.

Remark 3.3.2.1. To avoid confusions, we emphasize that in (3.3.2.1), the map $\varphi_{U'\times P''}$ is evaluated at $\delta x\in U'(\mathbb{A})$ where U' is viewed as a diagonal subgroup of U.

3.3.3. Properties of Λ_u^T

Proposition 3.3.3.1.

1. *The map $\varphi\mapsto\Lambda_u^T\varphi$ induces a linear continuous map from $\mathcal{T}([U])$ to $\mathcal{S}^0([U'])$.*
2. *For every $N_1,N_2>0$ and $r>0$, there exists a continuous semi-norm $\|.\|$ on $\mathcal{T}_{N_1}([U])$ such that*

$$|(\Lambda_u^T\varphi)(x)-F^{U'}(x,T)\varphi(x)|\leqslant e^{-r\|T\|}\|x\|_{U'}^{-N_2}\|\varphi\|$$

for all $x\in[U']$, $\varphi\in\mathcal{T}_{N_1}([U])$ and $T\in\mathfrak{a}_0'$ sufficiently positive.

Proof. The assertion 1 can be easily extracted from [IY19, (proof of) lemma 2.2]. For the convenience of the reader, we give some details. First, it suffices to show that for any $N>0$, there exists a continuous semi-norm $\|.\|$ on $\mathcal{T}([U])$ such that for all $\varphi\in\mathcal{T}([U])$, we have

$$\|\Lambda_u^T\varphi\|_{\infty,N}\leqslant\|\varphi\|.$$

We recall that $\|\psi\|_{\infty,N}=\sup_{x\in[U']}\|x\|_{U'}^N|\psi(x)|$ for any function ψ on $[U']$. We can write for $x\in[U']$,

$$(\Lambda_u^T\varphi)(x)=\sum_{P_1'\subset P_2'}\sum_{\delta\in P_1'(F)\backslash U'(F)}F^{P_1'}(\delta x,T)\sigma_1^2(H_0(\delta x)-T)\varphi_{1,2}(\delta x),\tag{3.3.3.1}$$

where $H_0=H_{P_0'}$ and $\sigma_1^2=\sigma_{P_1'}^{P_2'}$ is the eponymous function with values in $\{0,1\}$ introduced by Arthur in [Art78, section 6] for the group U' and

$$\varphi_{1,2}=\sum_{P_1'\subset P'\subset P_2'}\epsilon_{P'}\varphi_{U'\times P''}\tag{3.3.3.2}$$

with $\varphi_{U'\times P''}$ as above. Note that if $P_1'=P_2'$, then $\sigma_1^2=0$ unless $P_1'=P_2'=U'$. In this case, the corresponding term is $F^{U'}(x,T)\varphi(x)$ for which the result is obvious. The other cases are deduced from the next lemma, which also gives assertion 2. □

Lemma 3.3.3.2. *Assume $P_1'\subsetneq P_2'$. For every $N_1,N_2>0$ and $r>0$, there exists a continuous semi-norm $\|.\|$ on $\mathcal{T}_{N_1}([U])$ such that*

$$\sum_{\delta\in P_1'(F)\backslash U'(F)}F^{P_1'}(\delta x,T)\sigma_1^2(H_{P_0'}(\delta x)-T)|\varphi_{1,2}(\delta x)|\leqslant e^{-r\|T\|}\|x\|_{U'}^{-N_2}\|\varphi\|$$

for all $x\in[U']$, $\varphi\in\mathcal{T}_{N_1}([U])$ and $T\in\mathfrak{a}_0'$ sufficiently positive.

3.3.4. Proof of Lemma 3.3.3.2

Let g be an element in $U(\mathbb{A})$ which we write $g=(g',g'')$ with $g'\in U'(\mathbb{A})$ and $g''\in U''(\mathbb{A})$. Let $N>0$. By [MW94, preuves du lemme I.2.10 et du corollaire I.2.11], for any λ which is a linear combination of elements of $\Delta_0^{P_2''}\setminus\Delta_0^{P_1''}$ with positive coefficients, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_N([U])$ such that

$$|\varphi_{1,2}(g)|\leqslant\exp(-\langle\lambda,H_{B_0''}(g'')\rangle)\|g\|^N\|\varphi\|\tag{3.3.4.1}$$

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for all $g \in U(\mathbb{A})$ such that $g' \in U'(\mathbb{A})$, $g'' \in \mathfrak{s}_{P''_2}$ and all $\varphi \in \mathcal{T}_N([U])$. We want to apply this majorization to an element $g = (x, x)$ where $x \in U'(\mathbb{A})$ satisfies $F^{P'_1}(x, T)\sigma_1^2(H_{P'_0}(x) - T) = 1$. Since this condition and the map $x \in U'(\mathbb{A}) \mapsto \varphi_{1,2}(x, x)$ are $P'_1(F)$-left invariant, we may assume that $x \in \mathfrak{s}_{P'_1}$. By [Art78, lemma 6.1], x also satisfies $F^{P'_1}(y, T)\tau_{P'_1}^{P'_2}(H_{P'_1}(y) - T) = 1$, so by Lemma 2.3.3.1, we have $x \in \mathfrak{s}_{P'_2}$. We can deduce from the proof of Proposition 3.1.5.1 and the majorization (3.3.4.1) that there exists a fixed compact subset $\mathcal{K} \subset U'(\mathbb{A})$ such that $\mathfrak{s}_{P'_2} \subset \mathfrak{s}_{P''_2}\mathcal{K}$ and that there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_N([U])$ such that

$$|\varphi_{1,2}(x)| \leqslant \exp(-\langle\lambda, H_{P'_0}(x)\rangle)\|x\|^N\|\varphi\| \tag{3.3.4.2}$$

for all $x \in \mathfrak{s}_{P'_2}$ and $\varphi \in \mathcal{T}_N([U])$. Here, we wrote x instead of (x, x) in the left-hand side. Since the restriction to $\mathfrak{a}_{P'_0}$ of elements of $\Delta_0^{P''_2} \setminus \Delta_0^{P''_1}$ are (up to some positive constants) the restriction of elements of $\Delta_0^{P'_2} \setminus \Delta_0^{P'_1}$, we can conclude by the next Lemma 3.3.4.1 and the fact that, for all $N > 0$, there exists $c, N' > 0$ such that for all $x \in [U']$,

$$\sum_{\delta \in P'_1(F)\backslash U'(F)} \|\delta x\|_{P_1}^{-N'} \leqslant c\|x\|_{U'}^{-N}.$$

Lemma 3.3.4.1. *For every $N \geqslant 0$ and $r > 0$, there exists $t > 0$ and $C > 0$ such that for any*

$$\lambda = \sum_{\alpha \in \Delta_0^{P'_2}\setminus\Delta_0^{P'_1}} x_\alpha \alpha \text{ with } x_\alpha > t, \tag{3.3.4.3}$$

we have

$$F^{P'_1}(x, T)\sigma_1^2(H_{P'_0}(x) - T)\exp(-\langle\lambda, H_{P'_0}(x)\rangle) \leqslant Ce^{-r\|T\|}\|x\|_{P'_1}^{-N}$$

for all $x \in \mathfrak{s}_{P'_1}$ and $T \in \mathfrak{a}'_0$ sufficiently positive.

Proof. Any $H_1 \in \mathfrak{a}_{P'_1}$ can be written $H_1 = H_1^2 + H_2$ according to the decomposition $\mathfrak{a}_{P'_1} = \mathfrak{a}_{P'_1}^{P'_2} \oplus \mathfrak{a}_{P'_2}$. By [Art80, Corollary 6.2], there is $c_1 > 0$ such that

$$\|H_1\| \leqslant c_1(1 + \|H_1^2\|) \tag{3.3.4.4}$$

for all $H_1 \in \mathfrak{a}_{P'_1}$ such that $\sigma_1^2(H_1) \neq 1$. There exist $c_2, c_3, c_4 > 0$ such that for all $x \in \mathfrak{s}_{P'_1}$, we have

$$\|x\|_{P'_1} \leqslant c_2\|x\| \leqslant c_3\exp(c_4\|H_0(x)\|),$$

where we set $H_0 = H_{P'_0}$. We assume from now on that we have

$$F^{P'_1}(x, T)\sigma_1^2(H_0(x) - T) \neq 0. \tag{3.3.4.5}$$

In particular, we have $x \in \mathfrak{s}_{P'_2}$ as we have already seen it in §3.3.4 – discussion below (3.3.4.1). According to (3.3.4.4), the norm of $H_0(x)$ is bounded in terms of the norm of the projection of $H_0(x)$ on $\mathfrak{a}_{P'_0}^{P'_2}$, so up to some positive constant, it is bounded by

$$1 + \sum_{\alpha \in \Delta_{P'_0}^{P'_2}} |\langle\alpha, H_0(x)\rangle|.$$

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Since we assume $F^{P'_1}(x,T) = 1$ and $x \in \mathfrak{s}_{P'_1}$, the norm of the projection of $H_0(x)$ on $\mathfrak{a}_{P'_0}^{P'_1}$ is bounded by some multiple of $\|T\|$. For all $\alpha \in \Delta_{P'_0}^{P'_2} \setminus \Delta_{P'_0}^{P'_1}$, we have $\langle \alpha, H_0(x)\rangle > \langle \alpha, T\rangle \geqslant 0$ since we assume $\sigma_1^2(H_0(x) - T) \neq 0$; see [Art78, lemma 6.1]. Hence, we deduce that there exist c_5, c_6 such that for all $x \in \mathfrak{s}_{P'_1}$ that satisfy (3.3.4.5), we have

$$\|x\|_{P'_1} \leqslant c_2\|x\| \leqslant c_5 \exp(c_6(\|T\| + \sum_{\alpha\in\Delta_{P'_0}^{P'_2}\setminus\Delta_{P'_0}^{P'_1}} \langle \alpha, H_0(x)\rangle)).$$

The lemma is then obvious since for such x, and all $\alpha \in \Delta_{P'_0}^{P'_2} \setminus \Delta_{P'_0}^{P'_1}$ and any sufficiently positive T (see §3.1.6), we have $\langle \alpha, H_0(x)\rangle > \langle \alpha, T\rangle \geqslant \varepsilon\|T\|$. □

3.3.5.
We can apply the operator Λ_u^T to the right variable of the kernel $K_{f,\chi}(x,y)$: we get a function on $[U] \times [U']$ denoted by $K_{f,\chi}\Lambda_u^T$.

Theorem 3.3.5.1.

1. *For all $T \in \mathfrak{a}'_0$ sufficiently positive, there exists a continuous semi-norm $\|.\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that for all $f \in \mathcal{S}(U(\mathbb{A}))$, we have*

$$\sum_{\chi\in\mathfrak{X}(U)} \int_{[U']\times[U']} \left|(K_{f,\chi}\Lambda_u^T)(x,y)\right| dxdy \leqslant \|f\|. \tag{3.3.5.1}$$

2. *For all $r > 0$, there exists a continuous semi-norm $\|.\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that for any $T \in \mathfrak{a}'_0$ sufficiently positive and $f \in \mathcal{S}(U(\mathbb{A}))$, we have*

$$\sum_{\chi\in\mathfrak{X}(U)} \left| J_\chi^{U,T}(f) - \int_{[U']\times[U']} (K_{f,\chi}\Lambda_u^T)(x,y)\, dxdy \right| \leqslant e^{-r\|T\|}\|f\|. \tag{3.3.5.2}$$

From Theorem 3.2.3.1 assertion 2 and from Theorem 3.3.5.1, we get the following:

Corollary 3.3.5.2. *The absolutely convergent integral*

$$\int_{[U']\times[U']} (K_{f,\chi}\Lambda_u^T)(x,y)\, dxdy$$

is asymptotic to an exponential-polynomial in the variable T whose purely polynomial term is constant and equal to $J_\chi^U(f)$.

3.4. *Proof of main theorems*

3.4.1. Proof of Theorem 3.3.5.1 assertion 1

Let $N_1, N_2 > 0$. Let $N_0 > 0$ be as in Remark 3.2.2.1 (for $P = U$). We set $N'_1 = N_1 + N_0$. Let $T \in \mathfrak{a}'_0$ be sufficiently positive. By Proposition 3.3.3.1, there exists a continuous semi-norm $\|\cdot\|_{N'_1}$ on $\mathcal{T}_{N'_1}([U])$ such that for all $\chi \in \mathfrak{X}(U)$, all $x \in [U]$ and $y \in [U']$, we have

$$\|y\|_{U'}^{N_2}|(K_{f,\chi}\Lambda_u^T)(x,y)| \leqslant \|K_{f,\chi}(x,\cdot)\|_{N'_1}.$$

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Then, by Remark 3.2.2.1, more particularly the majorization (3.2.2.3) for $P = U$, there exists a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that for all $x \in [U]$ and $y \in [U']$,

$$\sum_{\chi \in \mathfrak{X}(U)} |(K_{f,\chi}\Lambda_u^T)(x,y)| \leqslant \|f\|_{\mathcal{S}} \|x\|_U^{-N_1} \|y\|_{U'}^{-N_2}.$$

Now the map $(x,y) \mapsto \|x\|_U^{-N_1}\|y\|_{U'}^{-N_2}$ is integrable on $[U'] \times [U']$ for large N_1, N_2; see [Beu21, Proposition A.1.1 (ix)].

3.4.2. Asymptotics of several truncated kernels

We shall use the two following results.

Theorem 3.4.2.1. *For every $N_1, N_2, r > 0$, there exists a continuous semi-norm $\|.\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that*

$$\sum_{\chi \in \mathfrak{X}(G)} \left| K_{f,\chi}^T(x,y) - K_{f,\chi}(x,y) F^{U'}(y,T) \right| \leqslant e^{-r\|T\|} \|x\|_{U'}^{-N_1} \|y\|_{U'}^{-N_2} \|f\| \qquad (3.4.2.1)$$

for $f \in \mathcal{S}(U(\mathbb{A}))$, $(x,y) \in [U'] \times [U']$ and $T \in \mathfrak{a}_0'$ sufficiently positive.

The proof of Theorem 3.4.2.1 will be given in §3.4.5 below.

Corollary 3.4.2.2. *For every $N_1, N_2, r > 0$, there exists a continuous semi-norm $\|.\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that*

$$\sum_{\chi \in \mathfrak{X}(G)} \left| K_{f,\chi}^T(x,y) - (K_{f,\chi}\Lambda_u^T)(x,y) \right| \leqslant e^{-r\|T\|} \|x\|_{U'}^{-N_1} \|y\|_{U'}^{-N_2} \|f\| \qquad (3.4.2.2)$$

for $f \in \mathcal{S}(U(\mathbb{A}))$, $(x,y) \in [U'] \times [U']$ and $T \in \mathfrak{a}_0'$ sufficiently positive.

Proof. Let $N_1, N_1', N_2 > 0$ be as in §3.4.1. By assertion 2 of Proposition 3.3.3.1, for every $r > 0$, there exists a continuous semi-norm $\|\cdot\|_{N_1'}$ on $\mathcal{T}_{N_1'}([U])$ such that for all $f \in \mathcal{S}(U(\mathbb{A}))$, $\chi \in \mathfrak{X}(U)$, $x \in [U]$, $y \in [U']$ and $T \in \mathfrak{a}_0'$ sufficiently positive, we have

$$|(K_{f,\chi}\Lambda_u^T)(x,y) - K_{f,\chi}(x,y)F^{U'}(y,T)| \leqslant e^{-r\|T\|} \|y\|_{U'}^{-N_2} \|K_{f,\chi}(x,\cdot)\|_{N_1'}.$$

From this and the majorization (3.2.2.3) (for $P = G$), we deduce that there exists a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that for all $f \in \mathcal{S}(U(\mathbb{A}))$, $x \in [U]$, $y \in [U']$ and $T \in \mathfrak{a}_0'$ sufficiently positive, we have

$$\sum_{\chi \in \mathfrak{X}(U)} |(K_{f,\chi}\Lambda_u^T)(x,y) - K_{f,\chi}(x,y)F^{U'}(y,T)| \leqslant e^{-r\|T\|} \|f\|_{\mathcal{S}} \|x\|_U^{-N_1} \|y\|_{U'}^{-N_2}.$$

By [Beu21, Proposition A.1.1 (ix)], we have $\|x\|_U \sim \|x\|_{U'}$ for $x \in [U']$. The corollary is then a straightforward consequence of the inequality above and Theorem 3.4.2.1. □

3.4.3. Proof of Theorem 3.2.3.1

First, we mention that all the statements but the continuity are stated and proved for compactly supported functions; see [Zyd20, theorems 4.1, 4.5 and 4.7]. We just need the extension to Schwartz functions. The assertion 1 of Theorem 3.2.3.1 is a direct consequence of Corollary 3.4.2.2 and assertion 1 of Theorem 3.3.5.1. This gives also the continuity of the distributions $J_\chi^{U,T}$ and their sum over $\chi \in \mathfrak{X}(U)$.

The assertion 2 of Theorem 3.2.3.1 can be proved as in [Zyd20, proof of theorems 4.5]. We take for granted the obvious extension of Corollary 3.4.2.2 and Theorem 3.3.5.1 assertion 1 to the auxiliary kernels of [Zyd20, section 4.1] relative to some parabolic subgroups of U. Indeed, it can be proved with the same technics and is left to the reader. In particular, the formula of [Zyd20, proposition 4.4] holds for Schwartz functions, and each term in right-hand side of the formula (among them $J^U_\chi(f)$) is a continuous distribution. From this, we deduce assertion 3 of Theorem 3.2.3.1 as in [Zyd20, theorem 4.7]. Finally, assertion 4 follows from assertion 1.

3.4.4. Proof of Theorem 3.3.5.1 assertion 2

It is an obvious application of Corollary 3.4.2.2.

3.4.5. Proof of Theorem 3.4.2.1

We shall use the notation of the proof of Proposition 3.3.3.1. We start from the expression for all $x, y \in [U']$

$$\begin{aligned} &K^T_{f,\chi}(x,y) - K_{f,\chi}(x,y)F^{U'}(y,T) \\ &\quad = \sum_{P'_1 \subsetneq P'_2} \sum_{\delta \in P'_1(F)\backslash U'(F)} \sum_{\gamma \in P'_2(F)\backslash U'(F)} F^{P'_1}(\delta y, T)\sigma_1^2(H_{P'_1}(\delta y) - T) K_{1,2,f,\chi}(\gamma x, \delta y), \end{aligned}$$

where $\sigma_1^2 = \sigma_{P'_1}^{P'_2}$ is as in (3.3.3.1), the sum is over $P'_1, P'_2 \in \mathcal{F}'_0$ and we set for $x \in P'_2(F)\backslash U'(\mathbb{A})$ and $y \in P'_1(F)\backslash U'(\mathbb{A})$,

$$K_{1,2,f,\chi}(x,y) = \sum_{P'_1 \subset P \subset P'_2} \epsilon_{P'} \sum_{\gamma \in P'(F)\backslash P'_2(F)} K_{f,P,\chi}(\gamma x, y).$$

From now on, we fix $P'_1 \subsetneq P'_2$. Let $\alpha \in \Delta_0^{P'_2} \setminus \Delta_0^{P'_1}$ and let $P'_1 \subsetneq {}^\alpha P'_1 \subset P'_2$ be defined by $\Delta_0^{{}^\alpha P'_1} = \Delta_0^{P'_1} \cup \{\alpha\}$. For any ${}^\alpha P'_1 \subset P' \subset P'_2$, we denote by P'_α the parabolic subgroup $P'_1 \subset P'_\alpha \subsetneq P'$ defined by $\Delta_0^{P'_\alpha} = \Delta_0^{P'} \setminus \{\alpha\}$.

We denote by P and P_α the parabolic subgroups associated to P' and P'_α (see the map (3.1.4.1)). For any ${}^\alpha P'_1 \subset P' \subset P'_2$, we set for $x \in P'(F)\backslash U'(\mathbb{A})$ and $y \in P'_1(F)\backslash U'(\mathbb{A})$,

$$K^\alpha_{f,P,\chi}(x,y) = K_{f,P,\chi}(x,y) - \sum_{\gamma \in P'_\alpha(F)\backslash P'(F)} K_{f,P_\alpha,\chi}(\gamma x, y).$$

Note that we have for all $x \in P'_2(F)\backslash U'(\mathbb{A})$ and $y \in P'_1(F)\backslash U'(\mathbb{A})$,

$$K_{1,2,f,\chi}(x,y) = \sum_{{}^\alpha P'_1 \subset P' \subset P'_2} \epsilon_{P'} \sum_{\delta \in P'(F)\backslash P'_2(F)} K^\alpha_{f,P,\chi}(\delta x, y). \tag{3.4.5.1}$$

We fix P' as in the sum above, and we start by majorizing each $K^\alpha_{P,\chi,f}(x,y)$.

Lemma 3.4.5.1. *There exists $n_0 > 0$ such that for any $n_1, n_2 > 0$, there is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that*

$$\sum_{\chi \in \mathfrak{X}(U)} |K^\alpha_{f,P,\chi}(x,y)| \leqslant \|f\| d_{P'_1}(\alpha, y)^{-n_1} \|y\|_{P'_1}^{n_2+n_0} \|x\|_{P'}^{-n_2}.$$

for all $x \in P'(F)\backslash U'(\mathbb{A})$, $y \in P'_1(F)\backslash U'(\mathbb{A})$ and $T \in \mathfrak{a}_0^+$ sufficiently positive such that

$$F^{P'_1}(y,T)\sigma_1^2(H_{P'_1}(y) - T) = 1. \tag{3.4.5.2}$$

Proof. Let $y \in U'(\mathbb{A})$. By using left translation by $P'_1(F)$, we may and shall assume that $y \in \mathfrak{s}_{P'_1}$. Assume that y satisfies also the Condition (3.4.5.2): by [Art78, lemma 6.1], y thus satisfies $F^{P'_1}(y,T)\tau_{P'_1}^{P'_2}(H_{P'_1}(y) - T) = 1$. By Lemma 2.3.3.1 and then Lemma 2.3.4.2, we have $y \in \mathfrak{s}_{P'_2}$ and $d_{P'_\alpha}(\alpha, y) = d_{P'_1}(\alpha, y)$. Since

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we have $\|y\|_{P_1'} \sim \|y\|$ for $y \in \mathfrak{s}_{P_1'}$ and $d_{P_\alpha'}(\alpha, y) \sim \exp(\langle \alpha, H_{P_0'}(y)\rangle)$ for $y \in \mathfrak{s}_{P_\alpha'}$, we may and shall freely replace $d_{P_1'}(\alpha, y)$ by either $d_{P_\alpha'}(\alpha, y)$ or $\exp(\langle \alpha, H_{P_0'}(y)\rangle)$ and $\|y\|_{P_1'}$ by $\|y\|$ in the inequality we have to prove.

Let us observe that we have

$$\sum_{\gamma\in P_\alpha(F)\backslash P(F)} K_{f,P_\alpha,\chi}(\gamma x, y) = \int_{[N_\alpha]} K_{f,P,\chi}(x, ny)\, dn,$$

where $N_\alpha = N_{P_\alpha}$. We have $P_\alpha = P_\alpha' \times P_\alpha''$, where $P_\alpha' \subset P'$ and $P_\alpha'' \subset P''$ are maximal parabolic subgroups. We set $N_\alpha'' = N_{P_\alpha''}$. We denote by $\tilde{\alpha}$ the unique element of $\Delta_0^{P''} \setminus \Delta_0^{P_\alpha''}$.

Thus, we see that $K^\alpha_{P,\chi,f}(x, y)$ is the sum of three terms:

$$K^{1,\alpha}_{f,P,\chi}(x, y) = K_{f,P,\chi}(x, y) - \int_{[N_\alpha'']} K_{f,P,\chi}(x, ny)\, dn \tag{3.4.5.3}$$

$$K^{2,\alpha}_{f,P,\chi}(x, y) = \int_{[N_\alpha'']} K_{f,P,\chi}(x, ny)\, dn - \int_{[N_\alpha]} K_{f,P,\chi}(x, ny)\, dn \tag{3.4.5.4}$$

$$K^{3,\alpha}_{f,P,\chi}(x, y) = \sum_{\gamma\in\Omega_{P,\alpha}} K_{f,P_\alpha,\chi}(\gamma x, y), \tag{3.4.5.5}$$

where $\Omega_{P,\alpha}$ is the complement in $P_\alpha(F)\backslash P(F)$ of the diagonal image of $P'(F)$.

By a variant of (3.3.4.2), for all $n_1, n_2 > 0$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_{n_2}([U])$ such that for all $f \in \mathcal{S}(U(\mathbb{A}))$, $\chi \in \mathfrak{X}(U)$, $x \in [U]_P$ and $y \in \mathfrak{s}_{P'}$, we have

$$|K^{1,\alpha}_{f,P,\chi}(x, y)| \leqslant \exp(-n_1\langle \alpha, H_{P_0'}(y)\rangle)\|y\|^{n_2}\|K_{f,P,\chi}(x, \cdot)\|.$$

By [BPCZ22, Lemma 2.10.1.1], there exists $n_0 > 0$ such that for any $n_2 > 0$ and any continuous semi-norm $\|\cdot\|$ on $\mathcal{T}_{n_2}([U])$, there is a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that

$$\sum_{\chi\in\mathfrak{X}(U)} \|K_{f,P,\chi}(x, \cdot)\| \leqslant \|f\|_{\mathcal{S}}\|x\|_P^{-(n_2-n_0)}$$

for all $f \in \mathcal{S}(U(\mathbb{A}))$ and $x \in [U]_P$. We conclude that there exists n_0 such for all $n_1, n_2 > 0$, there is a semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that

$$\sum_{\chi\in\mathfrak{X}(U)} |K^{1,\alpha}_{f,P,\chi}(x, y)| \leqslant \exp(-n_1\langle \alpha, H_{P_0'}(y)\rangle)\|f\|_{\mathcal{S}}\|y\|^{n_2+n_0}\|x\|_{P'}^{-n_2} \tag{3.4.5.6}$$

for all $f \in \mathcal{S}(U(\mathbb{A}))$, $x \in [U']_{P'}$ and $y \in \mathfrak{s}_{P'}$. In the same way, one proves that the bounds (3.4.5.6) holds for $K^{2,\alpha}_{f,P,\chi}(x, y)$.

We introduce the weight function

$$w(x) = \min(d_{P_\alpha''}(\tilde{\alpha}, x'), d_{P_\alpha''}(\tilde{\alpha}, x''))$$

for $x = (x', x'') \in [U]_{P_\alpha} = [U]_{P_\alpha'} \times [U]_{P_\alpha''}$. In the following, we shall view $\alpha + \tilde{\alpha}$ as an element of $\mathfrak{a}^*_{P_0'} \oplus \mathfrak{a}^*_{B_0''}$. By [BPCZ22, Lemma 2.10.1.1], there exists $n_0 > 0$ such that for any $n_1, n_2 > 0$, there is a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that

$$\sum_{\chi\in\mathfrak{X}(U)} |K_{f,P_\alpha,\chi}(x, z)| \leqslant \|f\|_{\mathcal{S}} w(z)^{-n_1}\|z\|_{P_\alpha}^{n_2+n_0} w(x)^{n_1}\|x\|_{P_\alpha}^{-n_2}$$

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for all $f \in \mathcal{S}(U(\mathbb{A}))$ and $x, z \in [U]_{P_\alpha}$. Thus, for all $x \in P'(F)\backslash U'(\mathbb{A})$ and $z \in [U']_{P'_\alpha}$, we have

$$\sum_{\chi \in \mathfrak{X}(U)} \sum_{\gamma \in \Omega_{P,\alpha}} |K_{f,P_\alpha,\chi}(\gamma x, z)| \leqslant \|f\|_{\mathcal{S}} w(z)^{-n_1} \|z\|_{P_\alpha}^{n_2+n_0} \sum_{\gamma \in \Omega_{P,\alpha}} w(\gamma x)^{n_1} \|\gamma x\|_{P_\alpha}^{-n_2}. \quad (3.4.5.7)$$

By Proposition 2.3.4.3, there is $c > 0$ such that for all $\gamma = (\gamma', \gamma'') \in P(F) = P'(F) \times P''(F)$ and any $x \in U'(\mathbb{A})$ (viewed as a diagonal element of $U(\mathbb{A})$) such that $w(\gamma x) > c$, we have $\gamma'' \in P''_\alpha(F)\gamma'$. Thus, such a γ cannot belong to the subset $\Omega_{P,\alpha}$. In this way, for $x \in U'(\mathbb{A})$, we have

$$\sum_{\gamma \in \Omega_{P,\alpha}} w(\gamma x)^{n_1} \|\gamma x\|_{P_\alpha}^{-n_2} \leqslant c^{n_1} \sum_{\gamma \in P_\alpha(F)\backslash P(F)} \|\gamma x\|_{P_\alpha}^{-n_2}. \quad (3.4.5.8)$$

By Proposition 3.1.5.1, we have $w(z) \sim d_{P'_\alpha}(\alpha, z)^r$ for some $r > 0$ and all $z \in [U']_{P'_\alpha}$. We deduce from (3.4.5.7) and (3.4.5.8) that there exists $n_0 > 0$ such that for any $n_1, n_2 > 0$, there is a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that

$$\sum_{\chi \in \mathfrak{X}(U)} |K^{3,\alpha}_{f,P,\chi}(x, y)| \leqslant \|f\|_{\mathcal{S}} d_{P'_\alpha}(\alpha, y)^{-n_1} \|y\|^{n_2+n_0} \|x\|_{P'}^{-n_2}$$

for $y \in \mathfrak{s}_{P'_\alpha}$ and $x \in P'(F)\backslash U'(\mathbb{A})$. The conclusion is clear. □

Lemma 3.4.5.2. *There exists $n_0 > 0$ such that for any $n_2 > 0$ and any λ in the open convex cone generated by $\Delta_0^{P'_2} \setminus \Delta_0^{P'_1}$, there is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that*

$$\sum_{\chi \in \mathfrak{X}(U)} |K_{1,2,f,\chi}(x, y)| \leqslant \|f\| d_{P'_1}(-\lambda, y) \|y\|_{P'_1}^{n_2+n_0} \|x\|_{P'_2}^{-n_2}.$$

for all $f \in \mathcal{S}(U(\mathbb{A}))$, $T \in \mathfrak{a}'_0$ sufficiently positive, $x \in P'_2(F)\backslash U'(\mathbb{A})$ and $y \in P'_1(F)\backslash U'(\mathbb{A})$ such that $F^{P'_1}(y, T)\sigma_1^2(H_{P'_1}(y) - T) = 1$.

Proof. Let $P'_1 \subset P' \subset P'_2$ and $\alpha \in \Delta_0^{P'_2} \setminus \Delta_0^{P'_1}$. By Lemma 3.4.5.1, there exists $n_0 > 0$ such that for all $n_1, n_2 > 0$, there is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that for all f, T and y as in the statement and all $x \in P'(F)\backslash U'(\mathbb{A})$, we have

$$\sum_{\chi \in \mathfrak{X}(U)} |K^{\alpha}_{f,P,\chi}(x, y)| \leqslant \|f\| d_{P'_1}(\alpha, y)^{-n_1} \|y\|_{P'_1}^{n_2+n_0} \|x\|_{P'}^{-n_2}.$$

Using the decomposition (3.4.5.1), we see that there exists $n_0 > 0$ such that for any $n_1, n_2 > 0$, there is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that

$$\sum_{\chi \in \mathfrak{X}(U)} |K_{1,2,f,\chi}(x, y)| \leqslant \|f\| d_{P'_1}(\alpha, y)^{-n_1} \|y\|_{P'_1}^{n_2+n_0} \|x\|_{P'_2}^{-n_2}$$

for all f, T, x and y as in the statement. The result follows easily. □

Let n_0 be as in Lemma 3.4.5.2. For any $n_1, n_2, r > 0$, there are λ and $C > 0$ as in Lemma 3.3.4.1 such that for all $y \in P'_1(F)\backslash U'(\mathbb{A})$ and $T \in \mathfrak{a}'_0$ sufficiently positive, we have

$$F^{P'_1}(y, T)\sigma_1^2(H_{P'_1}(y) - T) d_{P'_1}(-\alpha, y) \|y\|_{P'_1}^{n_0+n_2+n_1} \leqslant C \exp(-r\|T\|).$$

This indeed follows from Lemma 3.3.4.1: we may take $y \in \mathfrak{s}_{P'_1}$ and use the fact that $d_{P'_1}(-\lambda, y)$ is equivalent to $\exp(\langle -\lambda, H_{P'_0}(y)\rangle)$ for $y \in \mathfrak{s}_{P'_1}$ (see Proposition 2.3.4.1).

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As a consequence, we get by Lemma 3.4.5.2 that for any $n_1, n_2 > 0$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(U(\mathbb{A}))$ such that

$$F^{P_1'}(y,T)\sigma_1^2(H_{P_1'}(y)-T)\sum_{\chi\in\mathfrak{X}(U)}|K_{1,2,f,\chi}(x,y)| \leqslant \|f\|e^{-r\|T\|}\|x\|_{P_2'}^{-n_2}\|y\|_{P_1'}^{-n_1}$$

for all T sufficiently positive, all $x \in P_2'(F)\backslash U'(\mathbb{A})$ and $y \in P_1'(F)\backslash U'(\mathbb{A})$ and all $f \in \mathcal{S}(U(\mathbb{A}))$. It is then straightforward to get Theorem 3.4.2.1.

3.5. The (U,U')-regular contribution in the Jacquet-Rallis trace formula

3.5.1.
The goal of the section is to get Theorem 3.5.7.1 below which gives a computation of the distributions J_χ^U of Theorem 3.2.3.1 in terms of relative characters for some specific cuspidal data which we are going to define.

3.5.2.
Recall that we have $U = U' \times U''$. Let $\chi = (\chi', \chi'') \in \mathfrak{X}(U) = \mathfrak{X}(U') \times \mathfrak{X}(U'')$ be a cuspidal datum. Let $(M = M' \times M'', \pi = \pi' \boxtimes \pi'')$ be a representative where $M = M_P$ is standard Levi subgroup of a standard parabolic subgroup P of U. For any integer r, we set $G_r = \mathrm{Res}_{E/F}\,\mathrm{GL}(r,E)$. We can find hermitian forms h' and h'' respectively of rank n' and n'', integers $n_1', \ldots, n_{r'}'$ and $n_1'', \ldots, n_{r''}''$ and for $1 \leqslant i \leqslant r'$ cuspidal representations π_i' of $G_{n_i'}(\mathbb{A})$ (with central character trivial on $A_{G_{n_i'}}^\infty$) and for $1 \leqslant i \leqslant r''$ cuspidal representations π_i'' of $G_{n_i''}(\mathbb{A})$ (with central character trivial on $A_{G_{n_i''}}^\infty$), cuspidal representations σ' and σ'' respectively of $U(h')(\mathbb{A})$ and $U(h'')(\mathbb{A})$ such that such that

- $n' + 2(n_1' + \ldots + n_{r'}') = n$ and $n'' + 2(n_1'' + \ldots + n_{r''}'') = n+1$;
- $M' \simeq G_{n_1'} \times \ldots \times G_{n_{r'}'} \times U(h')$ and $M'' \simeq G_{n_1''} \times \ldots \times G_{n_{r''}''} \times U(h'')$;
- $\pi' = \pi_1' \boxtimes \ldots \boxtimes \pi_{r'}' \boxtimes \sigma'$ and $\pi'' = \pi_1'' \boxtimes \ldots \boxtimes \pi_{r''}'' \boxtimes \sigma''$ accordingly.

We shall say that χ is

- *U-regular* (or simply regular) if both χ' and χ'' are regular. We say that χ' is regular if the representations $\pi_1', \ldots, \pi_{r'}', (\pi_1')^*, \ldots, (\pi_{r'}')^*$ are two by two distinct (the same definition applies to χ''). Here, π^* means the conjugate dual of the representation π.
- *U'-regular* if for each $1 \leqslant i \leqslant r'$ and $1 \leqslant j \leqslant r''$ such that $n_i' = n_j''$, the representations $(\pi_i')^\vee$ (the contragredient of π_i') and π_j'' are neither isomorphic nor conjugate dual;
- *(U,U')-regular* if it is both U-regular and U'-regular.

3.5.3.
Let $\varphi \in \mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$. Let $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ such that the Eisenstein series $E(\varphi,\lambda)$ on $[U]$ is regular at λ. For $T \in \mathfrak{a}_0'$, we denote by $\Lambda_u^T E(\varphi,\lambda)$ the function on $[U']$ obtained from $E(\varphi,\lambda)$ by truncation by the operator defined in (3.3.2.1). The following proposition is basic to our calculation.

Proposition 3.5.3.1. *Let $T \in \mathfrak{a}_0'$ sufficiently positive.*

1. *The integral*

$$\int_{[U']} \Lambda_u^T E(x,\varphi,\lambda)\,dx \tag{3.5.3.1}$$

   is absolutely convergent.
2. *If the spectral datum χ defined by (M,π) is U'-regular, then the integral (3.5.3.1) does not depend on T.*

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3. *Write $P = P' \times P''$. If $P' = U'$ or $P'' = U''$, then we have*

$$\int_{[U']} \Lambda_u^T E(x,\varphi,\lambda)\,dx = \int_{[U']} E(x,\varphi,\lambda)\,dx, \tag{3.5.3.2}$$

where the left-hand side is absolutely convergent.

Proof. 1. The absolute convergence follows from the uniform moderate growth of Eisenstein series and the basic properties of the truncation operator Λ_u^T recalled in Proposition 3.3.3.1.

2. To analyse the dependence on T, we shall use the following formula (adapted from [IY19, eq. (2.2)]): for any $T' \in \mathfrak{a}_0'$ and any smooth function φ on $[U]$, we have for any $x \in U'(\mathbb{A})$,

$$(\Lambda_u^{T+T'}\varphi)(x) = \sum_{R' \in \mathcal{F}_0'} \sum_{\delta \in R'(F)\backslash U'(F)} \Gamma_{R'}(H_{R'}(\delta x) - T_{R'}, T')(\Lambda_u^{T,R'}\varphi)(\delta x), \tag{3.5.3.3}$$

where the function $\Gamma_{R'}(\cdot, T')$ is compactly supported on $\mathfrak{a}_{R'}$ (for the precise definition, see p.9 of [IY19]); moreover, we set for any $x \in U'(\mathbb{A})$,

$$(\Lambda_u^{T,R'}\varphi)(x) = \sum_{Q' \subset R'} (-1)^{\dim(\mathfrak{a}_{Q'}^{R'})} \sum_{\delta \in Q'(F)\backslash R'(F)} \hat{\tau}_{Q'}^{R'}(H_{Q'}(\delta x) - T)\varphi_{U' \times Q''}(\delta x),$$

where the sum is over the set of standard parabolic subgroups Q' of R' and $\varphi_{U'\times Q''}$ is the constant term along $U' \times Q''$, where $Q' \times Q''$ is the image of Q' by the map (3.1.4.1). For $R' = U'$, we recover the operator Λ_u^T.

Using (3.5.3.3) and some Iwasawa decomposition, we see that assertion 2 follows from the following vanishing statement: for any *proper* standard parabolic subgroups R' of U', we have for some Haar measure on K,

$$\int_{[M_{R'}]} \int_K \exp(-\langle 2\rho_{R'}, H_{R'}(x)\rangle)\Gamma_{R'}(H_{R'}(x) - T, T')\Lambda_u^{T,R'} E_R(x,\varphi,\lambda)\,dx = 0,$$

where $E_R(\varphi,\lambda)$ is the constant term of $E(\varphi,\lambda)$ along the parabolic subgroup R, the image of R' by the map (3.1.4.1).

Using the usual computation of the constant term of (cuspidal) Eisenstein series, we are reduced to prove that

$$\int_{[M_{R'}]^1} \Lambda_u^{T,M_R'} E^R(x,\varphi,\lambda)\,dx = 0 \tag{3.5.3.4}$$

for all parabolic subgroup $P \subset R$, all $\varphi \in \mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$ such that the class of (M_P, π) is U'-regular. Here, $E^R(\varphi,\lambda)$ denotes the Eisenstein series relative to R.

Let us prove this last claim. The reasoning here is very similar to that of [BPCZ22, proof of proposition 5.1.4.1], so we will be quite brief. There exist a hermitian form h' of rank m and integers $n_1, \ldots, n_r$ such that $m + 2(n_1 + \ldots + n_r) = n$

$$M_R \simeq (G_{n_1} \times \ldots \times G_{n_r}) \times U(h') \times (G_{n_1} \times \ldots \times G_{n_r}) \times U(h' \oplus h_0)$$
$$M_{R'} \simeq (G_{n_1} \times \ldots \times G_{n_r}) \times U(h').$$

The embedding $M_{R'} \subset M_R$ is given by the product of the diagonal embeddings $G_{n_1} \times \ldots \times G_{n_r} \subset (G_{n_1} \times \ldots \times G_{n_r})^2$ and $U(h') \subset U(h') \times U(h' \oplus h_0)$. Then the operator $\Lambda_u^{T,M_R'}$ is the product of

- the usual Arthur operator attached to $G_{n_1} \times \ldots \times G_{n_r}$ viewed as an operator on functions on $[G_{n_1} \times \ldots \times G_{n_r}] \times [G_{n_1} \times \ldots \times G_{n_r}]$ acting on the second factor;
- the operator $\Lambda_u^{T,U(h')}$ defined relatively to the embedding $U(h') \subset U(h') \times U(h' \oplus h_0)$.

We see that in (3.5.3.4), each integral over G_{n_i} can be interpreted as a scalar product of an Eisenstein series and a truncated Eisenstein series. By Langlands' computation of this scalar product and our U'-regularity assumption, we get the expected vanishing.

3. If $P' = U'$ or $P'' = U''$, then on the one hand, the cuspidal datum defined by (M, π) is automatically U'-regular. Thus, the left-hand side of (3.5.3.2) does not depend on T. On the other hand, the restriction of $E(\varphi, \lambda)$ to $[U']$ is rapidly decreasing. So the right-hand side of (3.5.3.2) is absolutely convergent. Now the equality (3.5.3.2) is a straightforward consequence of assertion 2 of Proposition 3.3.3.1 and the dominated convergence theorem. □

3.5.4. Ichino-Yamana regularized period

Assume that the cuspidal datum defined by (M, π) is U'-regular. Following [IY19], we define the regularized period of $E(\varphi, \lambda)$ for $\varphi \in \mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ by

$$\mathcal{P}_{U'}(\varphi, \lambda) = \int_{[U']} \Lambda_u^T E(x, \varphi, \lambda)\, dx, \tag{3.5.4.1}$$

where the right-hand side is the absolutely convergent integral (3.5.3.1) attached to the Eisenstein series $E(\varphi, \lambda)$ for any $T \in \mathfrak{a}'_0$ sufficiently positive. It does not depend on T by Proposition 3.5.3.1 assertion 2. It is meromorphic in λ and holomorphic when $E(\varphi, \lambda)$ is regular, in particular on $i\mathfrak{a}^*_P$. The map $\varphi \mapsto \mathcal{P}_{U'}(\varphi, \lambda)$ is continuous.

3.5.5. Relative character

We keep our assumption on (M, π). Let $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. We define the relative character $J^U_{P,\pi}(\lambda, f)$ by

$$J^U_{P,\pi}(\lambda, f) = \sum_{\varphi \in \mathcal{B}_{P,\pi}} \mathcal{P}_{U'}(I_P(\lambda, f)\varphi, \lambda)\overline{\mathcal{P}_{U'}(\varphi, \lambda)}, \tag{3.5.5.1}$$

where $f \in \mathcal{S}(U(\mathbb{A}))$ and $\mathcal{B}_{P,\pi}$ is a K-basis of $\mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$ in the sense of §2.2.4. Outside the singularities of the involved Eisenstein series, the sum is absolutely convergent. It is holomorphic on $i\mathfrak{a}^*_P$. It does not depend on the choice of $\mathcal{B}_{P,\pi}$ and it defines a continuous linear form on $\mathcal{S}(U(\mathbb{A}))$ (see [BPCZ22, proposition 2.8.4.1])). For further use, we observe the following simple functional equation.

Proposition 3.5.5.1. *Let P and P_1 be standard parabolic subgroup of U of respective standard Levi factors M and M_1. Assume that the pairs (M, π) and (M_1, π_1) define the same U'-regular cuspidal datum. Then for $w \in W(M, M_1)$ such that $\pi_1 = w\pi$ and $\lambda \in i\mathfrak{a}^*_P$, we have*

$$J^U_{P,\pi}(\lambda, f) = J^U_{P_1,\pi_1}(w \cdot \lambda, f).$$

Proof. This is an immediate consequence of the functional equation $E(\varphi, \lambda) = E(M(w, \lambda)\varphi, w\lambda)$ of Eisenstein series and the fact that for $\lambda \in i\mathfrak{a}^*_P$, the intertwining operator sends a K-basis of $\mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$ to a K-basis of $\mathcal{A}_{P,\pi_1,\mathrm{cusp}}(U)$. □

3.5.6. Some auxiliary results

For further use, we state and prove some useful results.

Lemma 3.5.6.1. *Let $\chi \in \mathfrak{X}(U)$ be a U-regular cuspidal datum. For any standard parabolic subgroup P of U and any representative (M_P, π) of χ, we have for all $f \in \mathcal{S}(U(\mathbb{A}))$ and $x, y \in [U]$,*

$$K_{f,\chi}(x, y) = \int_{i\mathfrak{a}^*_P} \sum_{\varphi \in \mathcal{B}_{P,\pi}} E(x, I_P(\lambda, f)\varphi, \lambda)\overline{E(y, \varphi, \lambda)}\, d\lambda, \tag{3.5.6.1}$$

where $\mathcal{B}_{P,\pi}$ is a K-basis as above.

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Proof. Let $\chi \in \mathfrak{X}(U)$ and $x, y \in [U]$. We start from the general spectral expansion (see [BPCZ22, lemma 2.10.2.1])

$$K_{f,\chi}(x,y) = \sum_Q n_Q^{-1} \int_{i\mathfrak{a}_Q^*} \sum_{\varphi\in\mathcal{B}_{Q,\chi}} E(x, I_Q(\lambda, f)\varphi, \lambda)\overline{E(y,\varphi,\lambda)}\, d\lambda, \tag{3.5.6.2}$$

where the sum is over the set of standard parabolic subgroups Q of U; the integer n_Q is the number of semi-standard parabolic subgroups of U which admits the same semi-standard Levi component as Q. The set $\mathcal{B}_{Q,\chi}$ is a K-basis of the space $\mathcal{A}^0_{Q,\chi,\mathrm{disc}}(U)$ (we refer the reader to [BPCZ22, §2.10.2] for the notation). From now on, we assume moreover that χ is U-regular. Let (M_P, π) be a representative of χ with P a standard parabolic subgroup of G. Let Q be a standard parabolic subgroup of G. Then the space $\mathcal{A}^0_{Q,\chi,\mathrm{disc}}(U)$ has the following simple description:

$$\mathcal{A}^0_{Q,\chi,\mathrm{disc}}(U) = \hat{\oplus}_{w\in W(P,Q)} \mathcal{A}_{Q,w\pi,\mathrm{cusp}}(U). \tag{3.5.6.3}$$

In particular, this space is zero unless Q and P are associated, which we assume from now on. As a consequence, we may and shall assume that $\mathcal{B}_{Q,\chi}$ is a union over $w \in W(P,Q)$ of K-bases $\mathcal{B}_{Q,w\pi}$ of $\mathcal{A}_{Q,w\pi,\mathrm{cusp}}(U)$. In this way, by the same argument as in the proof of Proposition 3.5.5.1, we see that we have for all $w \in W(P,Q)$ and $\lambda \in i\mathfrak{a}_Q^*$,

$$\sum_{\varphi\in\mathcal{B}_{Q,w\pi}} E(x, I_Q(\lambda, f)\varphi, \lambda)\overline{E(y,\varphi,\lambda)} = \sum_{\varphi\in\mathcal{B}_{P,\pi}} E(x, I_P(w^{-1}\lambda, f)\varphi, w^{-1}\lambda)\overline{E(y,\varphi,w^{-1}\lambda)},$$

where $\mathcal{B}_{P,\pi}$ is a K-basis of $\mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$. So we get by the change of variables $\lambda \mapsto w^{-1}\lambda$,

$$\begin{aligned}
&\int_{i\mathfrak{a}_Q^*} \sum_{\varphi\in\mathcal{B}_{Q,\chi}} E(x, I_Q(\lambda, f)\varphi, \lambda)\overline{E(y,\varphi,\lambda)}\, d\lambda \\
&= \sum_{w\in W(P,Q)} \int_{i\mathfrak{a}_Q^*} \sum_{\varphi\in\mathcal{B}_{Q,w\pi}} E(x, I_Q(\lambda, f)\varphi, \lambda)\overline{E(y,\varphi,\lambda)}\, d\lambda \\
&= |W(P,Q)| \int_{i\mathfrak{a}_P^*} \sum_{\varphi\in\mathcal{B}_{P,\pi}} E(x, I_P(\lambda, f)\varphi, \lambda)\overline{E(y,\varphi,\lambda)}\, d\lambda.
\end{aligned}$$

This gives the result since we have $\sum_Q |W(P,Q)| n_Q^{-1} = 1$ where the sum is taken over standard parabolic subgroups Q. Note that in the second line above the integral over $i\mathfrak{a}_Q^*$ is absolutely convergent. Let us check this. By Dixmier-Malliavin theorem, we may and shall assume that we have $f = f_1 * f_2^*$ for some $f_1, f_2 \in \mathcal{S}(U(\mathbb{A}))$, where $*$ denotes the convolution product and we have set $f_2^*(g) = \overline{f_2(g^{-1})}$. By a standard change of basis argument, we have for all $x_1, x_2 \in U(\mathbb{A})$,

$$\sum_{\varphi\in\mathcal{B}_{Q,w\pi}} E(x_1, I_Q(\lambda, f)\varphi, \lambda)\overline{E(x_2,\varphi,\lambda)} = \sum_{\varphi\in\mathcal{B}_{Q,w\pi}} E(x_1, I_Q(\lambda, f_1)\varphi, \lambda)\overline{E(x_2, I_Q(\lambda, f_2)\varphi,\lambda)}.$$

By Cauchy-Schwarz inequality, the expression

$$\int_{i\mathfrak{a}_Q^*} \Big| \sum_{\varphi\in\mathcal{B}_{Q,w\pi}} E(x_1, I_Q(\lambda, f)\varphi, \lambda)\overline{E(x_2,\varphi,\lambda)}\Big|\, d\lambda$$

is bounded above by the square-root of the product over $i = 1, 2$ of

$$\int_{i\mathfrak{a}_Q^*} \sum_{\varphi\in\mathcal{B}_{Q,w\pi}} |E(x_i, I_Q(\lambda, f_i)\varphi, \lambda)|^2\, d\lambda = \int_{i\mathfrak{a}_Q^*} \sum_{\varphi\in\mathcal{B}_{Q,w\pi}} E(x_i, I_Q(\lambda, g_i)\varphi, \lambda)\overline{E(x_i,\varphi,\lambda)}\, d\lambda,$$

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where we have set $g_i = f_i * f_i^*$. By the same argument, we see that the spectral expansion of $K_{g_i}(x,x)$ deduced from (3.5.6.2) is a sum of non-negative terms, and thus, the last expression above is bounded above by $K_{g_i}(x_i,x_i)$ times n_Q. □

Let T_1, T_2 be two sufficiently positive points in $\mathfrak{a}_0'$. Let $\Lambda_u^{T_1} K_{f,\chi} \Lambda_u^{T_2}$ be the function of two variables we get when we apply on $K_{f,\chi}$ the truncation operators $\Lambda_u^{T_1}$ and $\Lambda_u^{T_2}$, respectively, on the left and right variables.

Lemma 3.5.6.2. *Let $\chi \in \mathfrak{X}(U)$ be a U-regular cuspidal datum. For any standard parabolic subgroup P of U and any representative (M_P,π) of χ, we have for all $f \in \mathcal{S}(U(\mathbb{A}))$ and $x,y \in [U']$,*

$$(\Lambda_u^{T_1} K_{f,\chi} \Lambda_u^{T_2})(x,y) = \int_{i\mathfrak{a}_P^*} \sum_{\varphi\in\mathcal{B}_{P,\pi}} (\Lambda_u^{T_1} E)(x, I_P(\lambda,f)\varphi,\lambda)\overline{\Lambda_u^{T_2} E(y,\varphi,\lambda)}\, d\lambda, \tag{3.5.6.4}$$

where $\mathcal{B}_{P,\pi}$ is a K-basis as above.

Proof. This is a variant of the proof of Lemma 3.5.6.1. Simply instead of (3.5.6.2), we start from the spectral expansion for any $\chi \in \mathfrak{X}(U)$:

$$(\Lambda_u^{T_1} K_{f,\chi} \Lambda_u^{T_2})(x,y) = \sum_Q n_Q^{-1} \int_{i\mathfrak{a}_Q^*} \sum_{\varphi\in\mathcal{B}_{Q,\chi}} \Lambda_u^{T_1} E(x, I_Q(\lambda,f)\varphi,\lambda)\overline{\Lambda_u^{T_2} E(y,\varphi,\lambda)}\, d\lambda. \tag{3.5.6.5}$$

The proof of this expansion is similar to that of [BPCZ22, proof of lemma 4.2.3.1] and thus is left to the reader. □

Remark 3.5.6.3. Assume we have $f = f_1 * f_2^*$ as in the proof of Lemma 3.5.6.1; one has the following bound (proved by the same method as in the proof of Lemma 3.5.6.1): there exists $c > 0$ (independent of f_1 and f_2) such that for all $x_1, x_2 \in [U']$,

$$\begin{aligned}\sum_{(M,\pi)} \int_{i\mathfrak{a}_P^*} \left| \sum_{\varphi\in\mathcal{B}_{P,\pi}} \Lambda_u^{T_1} E(x_1, I_Q(\lambda,f)\varphi,\lambda)\overline{\Lambda_u^{T_2} E(x_2,\varphi,\lambda)} \right| d\lambda \\ \leqslant c\left((\Lambda_u^{T_1} K_{g_1}\Lambda_u^{T_1})(x_1,x_1)\right)^{1/2}\left((\Lambda_u^{T_2} K_{g_2}\Lambda_u^{T_2})(x_2,x_2)\right)^{1/2},\end{aligned} \tag{3.5.6.6}$$

where the first sum is over a set of representatives (M,π) of U-regular cuspidal data of U and we have set $g_i = f_i * f_i^*$.

Proposition 3.5.6.4. *There exists a semi-norm on $\mathcal{S}(U(\mathbb{A}))$ such that for all $f \in \mathcal{S}(U(\mathbb{A}))$,*

$$\sum_{(M,\pi)} \int_{i\mathfrak{a}_P^*} |J_{P,\pi}^U(\lambda,f)|\, d\lambda \leqslant \|f\|, \tag{3.5.6.7}$$

where the sum is over a set of representatives (M,π) of (U,U')-regular cuspidal data of U and P is the standard parabolic subgroup of which M is the standard Levi factor.

Proof. By uniform boundedness principle, it suffices to prove that the expression (3.5.6.7) is finite for each f. Let T be a sufficiently positive point in $\mathfrak{a}_0'$ such that for all representatives (M_P,π) of (U,U')-regular cuspidal data of U with P a standard parabolic subgroup of U and for all $\varphi \in \mathcal{A}_{P,\pi,\mathrm{cusp}}(U)$, we have

$$\mathcal{P}_{U'}(\varphi,\lambda) = \int_{[U']} \Lambda_u^T E(x,\varphi,\lambda)\, dx. \tag{3.5.6.8}$$

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Recall that the map $\varphi \mapsto \mathcal{P}_{U'}(\varphi, \lambda)$ is continuous. It follows from [BPCZ22, Proposition 2.8.41] that we have for $\lambda \in i\mathfrak{a}_P^*$,

$$J^U_{P,\pi}(\lambda, f) = \int_{[U']\times[U']} \sum_{\varphi\in\mathcal{B}_{P,\pi}} \Lambda^T_u E(h_1, I_P(\lambda, f)\varphi, \lambda)\overline{\Lambda^T_u E(h_2, \varphi, \lambda)} dh_1 dh_2.$$

Now it suffices to show that we have

$$\sum_{(M,\pi)} \int_{i\mathfrak{a}_P^*} \int_{[U']\times[U']} \left| \sum_{\varphi\in\mathcal{B}_{P,\pi}} \Lambda^T_u E(h_1, I_P(\lambda, f)\varphi, \lambda)\overline{\Lambda^T_u E(h_2, \varphi, \lambda)} \right| dh_1 dh_2 d\lambda < \infty, \tag{3.5.6.9}$$

where the outer sum is as in (3.5.6.7). To show the convergence of (3.5.6.9) for a fixed function $f \in \mathcal{S}(U(\mathbb{A}))$, we may and shall assume, by Dixmier-Malliavin theorem, that $f = f_1 * f_2^*$ (the notations are those used in the proof of lemma 3.5.6.1). But then we can apply the bound (3.5.6.6). In this way, we see that the left-hand side of (3.5.6.9) is bounded above by (up to some irrelevant constant) by

$$\mathrm{vol}([U']) \left(\prod_{i=1}^2 \int_{[U']} (\Lambda^T_u K_{g_i} \Lambda^T_u)(x, x)\, dx \right)^{1/2}.$$

The volume of $[U']$ is finite, and the integral

$$\int_{[U']} (\Lambda^T_u K_{g_i} \Lambda^T_u)(x, x)\, dx$$

is absolutely convergent by properties of truncation operator (see Proposition 3.3.3.1) and the fact that the kernel is slowly increasing; see [BPCZ22, Lemma 2.10.1.1]. Note that the derivatives of the kernel K_{g_i} are related to the kernel associated to derivatives of g_i; see also the proof of Theorem 3.5.7.1 where this fact is used. So we can conclude. □

3.5.7.
We can now state and prove the main result of this section.

Theorem 3.5.7.1. *Let $\chi \in \mathfrak{X}(U)$ be a (U, U')-regular cuspidal datum. For any standard parabolic subgroup P of U and any representative (M_P, π) of χ, we have*

$$J^U_\chi(f) = \int_{i\mathfrak{a}_P^*} J^U_{P,\pi}(\lambda, f)\, d\lambda,$$

where the integral in the right-hand side is absolutely convergent.

Proof. Let (M_P, π) be a representative of a (U, U')-regular cuspidal datum χ with P a standard parabolic subgroup of U. Let T_1, T_2 be two sufficiently positive points in $\mathfrak{a}_0'$ such that (3.5.6.8) holds for $T = T_1, T_2$. We shall use the spectral expansion given by Lemma 3.5.6.2. Using (3.5.6.6), we get (as in the proof of Proposition 3.5.6.4)

$$\int_{i\mathfrak{a}_P^*} \int_{[U']\times[U']} | \sum_{\varphi\in\mathcal{B}_{P,\pi}} \Lambda^{T_1}_u E(x, I_P(\lambda, f)\varphi, \lambda)\overline{\Lambda^{T_1}_u E(y, \varphi, \lambda)}|\, dx dy < \infty.$$

Using this and Fubini theorem, we see that we can integrate the spectral expansion (3.5.6.4) over $[U'] \times [U']$ and permute the adelic and the complex integrals. Using the definition of the relative

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character (3.5.5.1) built upon the periods (3.5.4.1), we have by [BPCZ22, Proposition 2.8.41],

$$J^U_{P,\pi}(\lambda, f) = \int_{[U']\times[U']} \sum_{\varphi\in\mathcal{B}_{P,\pi}} \Lambda^{T_1}_u E(h_1, I_P(\lambda, f)\varphi, \lambda)\overline{\Lambda^{T_2}_u E(h_2, \varphi, \lambda)}dh_1dh_2.$$

So we can conclude that we have

$$\int_{[U']\times[U']} (\Lambda^{T_1}_u K_{f,\chi}\Lambda^{T_2}_u)(x, y)\, dxdy = \int_{i\mathfrak{a}^*_P} J^U_{P,\pi}(\lambda, f)\, d\lambda, \tag{3.5.7.1}$$

where both sides are absolutely convergent. Note that the right-hand side of (3.5.7.1), and hence the left-hand side, depend neither on T_1 nor on T_2.

We fix $T_2 \in \mathfrak{a}'_0$ sufficiently positive. By Proposition 3.3.3.1 assertion 2, $\Lambda^{T_1}_u K_{f,\chi}\Lambda^{T_2}_u$ converges to $K_{f,\chi}\Lambda^{T_2}_u$ pointwise when $d(T_1) \to +\infty$ (see §3.1.6). We want to apply the dominated convergence theorem. Let $N, N' > 0$. By Proposition 3.3.3.1 assertion 2, there exists a continuous semi-norm $\|\cdot\|_{N'}$ on $\mathcal{T}_{N'}([U])$ (which does not depend on T_1) such that we have for all $x, y \in [U']$ and all $f \in \mathcal{S}(U)$,

$$|(\Lambda^{T_1}_u K_{f,\chi}\Lambda^{T_2}_u)(x, y)| \leqslant \|x\|^{-N}_{U'} \|(K_{f,\chi}\Lambda^{T_2}_u)(\cdot, y)\|_{N'} + |(K_{f,\chi}\Lambda^{T_2}_u)(x, y)|.$$

Let $J \subset U(\mathbb{A}_f)$ be a compact open subgroup and $S(U(\mathbb{A}))^J \subset S(U(\mathbb{A}))$ be the subspace of right-J-invariant functions. There are finite families $(N_i)_{i\in I}$ of integers and $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{u}_\infty)$ such that we have for all $x, y \in [U']$ and $f \in S(U(\mathbb{A}))^J$, we have

$$\begin{aligned}\|(K_{f,\chi}\Lambda^{T_2}_u)(\cdot, y)\|_{N'} &\leqslant \sum_{i\in I} \sup_{h\in[U]} (\|h\|^{-N_i}_U |(R(X_i)K_{f,\chi}\Lambda^{T_2}_u)(h, y)|)\\ &= \sum_{i\in I} \sup_{h\in[U]} (\|h\|^{-N_i}_U |(K_{L(X_i)f,\chi}\Lambda^{T_2}_u)(h, y)|).\end{aligned}$$

Here, $R(X_i)K_{f,\chi}$ means that we apply the differential operator $R(X_i)$ on the left variable of $K_{f,\chi}$. We also use the fact that we have $R(X_i)K_{f,\chi} = K_{L(X_i)f,\chi}$ for some right-invariant differential operator $L(X_i)$. Now we can use (3.2.2.3) and Proposition 3.3.3.1 assertion 1 applied to the operator $\Lambda^{T_2}_u$ to see that for all $n_1 > 0$, there exists a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(U(\mathbb{A}))$ such that for all $x \in [U]$, $y \in [U']$ and all $f \in \mathcal{S}(U(\mathbb{A}))$, we have

$$|(K_{f,\chi}\Lambda^{T_2}_u)(x, y)| \leqslant \|f\|_{\mathcal{S}}\|x\|^{-n_1}_U \|y\|^{-n_1}_{U'}.$$

We deduce that for any $n_2 > 0$, there exists $c > 0$ (which depends on T_2, n_2 and $f \in \mathcal{S}(U(\mathbb{A}))$ but not on T_1) such that for all $x, y \in [U']$,

$$|(\Lambda^{T_1}_u K_{f,\chi}\Lambda^{T_2}_u)(x, y)| \leqslant c\|x\|^{-n_2}_{U'} \|y\|^{-n_2}_{U'}.$$

By choosing n_2 large enough so that $(x, y) \mapsto \|x\|^{-n_2}_{U'} \|y\|^{-n_2}_{U'}$ is integrable over $[U'] \times [U']$, we can apply the dominated convergence theorem to get

$$\lim_{d(T_1)\to+\infty} \int_{[U']\times[U']} (\Lambda^{T_1}_u K_{f,\chi}\Lambda^{T_2}_u)(x, y)\, dxdy = \int_{[U']\times[U']} (K_{f,\chi}\Lambda^{T_2}_u)(x, y)\, dxdy.$$

We have seen that the left-hand side depends neither on T_1 nor on T_2. So the right-hand side is also independent of T_2. We can conclude from Corollary 3.3.5.2 that the right-hand side is in fact equal to $J^U_\chi(f)$. □

4. The (G,H)-regular contribution in the Jacquet-Rallis trace formula

4.1. Statement and proof

4.1.1.

Let E/F be a quadratic extension of number fields. Let $\mathbb{A}$ be the ring of adèles of F and $\eta = \eta_{E/F}$ be the quadratic character of the group $\mathbb{A}^\times$ attached to E/F. Let $n \geqslant 1$ and $G'_n = \mathrm{GL}_{n,F}$ be the algebraic group of F-linear automorphisms of F^n. We view as an F-subgroup of $G_n = \mathrm{Res}_{E/F}(G'_n \times_F E)$. We denote by c the Galois involution of G_n. Let $\eta_{G'_n}$ be the character of $G'_n(\mathbb{A})$ given by

$$\eta_{G'_n}(h) = \eta(\det(h))^{n+1}$$

for all $h \in G'_n(\mathbb{A})$. Let (B'_n, T'_n) be a pair where B'_n is the Borel subgroup G'_n of upper triangular matrices and T'_n is the maximal torus of G'_n of diagonal matrices. Let (B_n, T_n) be the pair deduced from (B'_n, T'_n) by extension of scalars to E and restriction to F: it is a pair of a minimal parabolic subgroup of G_n and its Levi factor. Let $K_n \subset G_n(\mathbb{A})$ and $K'_n = K_n \cap G'_n(\mathbb{A}) \subset G'_n(\mathbb{A})$ be the 'standard' maximal compact subgroups. We set

$$\mathfrak{a}^+_{n+1} = \mathfrak{a}^{G_{n+1}+}_{B_{n+1}},$$

where the right-hand side is defined in §2.1.4.

We set $G = G_n \times G_{n+1}$ and $G' = G'_n \times G'_{n+1}$ (see §4.1.1). Let c be the Galois involution of G whose fixed points set if G'. The reductive groups G and G' are equipped with the pairs $(B_n \times B_{n+1}, T_n \times T_{n+1})$ and $(B'_n \times B'_{n+1}, T'_n \times T'_{n+1})$. Let $K = K_n \times K_{n+1} \subset G(\mathbb{A})$ and $K' = K \cap G'(\mathbb{A})$. We denote by $\eta_{G'}$ the character $\eta_{G'_n} \boxtimes \eta_{G'_{n+1}}$ of $G'(\mathbb{A})$. Let H be the image of the diagonal embedding

$$G_n \hookrightarrow G_n \times G_{n+1}.$$

Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with central character trivial on A^∞_G. As in §3.5.2, we denote by π^* the conjugate-dual representation of $G(\mathbb{A})$. We shall say that π is self conjugate-dual if $\pi \simeq \pi^*$ and that π is G'-distinguished, resp. (G', η)-distinguished, if the linear form (called the Flicker-Rallis period)

$$\varphi \mapsto \int_{[G']_0} \varphi(h)\, dh, \text{ resp. } \int_{[G']_0} \varphi(h)\eta(\det(h))\, dh \tag{4.1.1.1}$$

does not vanish identically on $\mathcal{A}_{\pi,\mathrm{cusp}}(G)$. Then π is self conjugate-dual if and only if π is either G'-distinguished or (G', η)-distinguished and it cannot be both (see [Fli88]). Note that since we are working with general linear groups, we shall omit the subscript cusp and we shall write simply $\mathcal{A}_\pi(G)$ for the space $\mathcal{A}_{\pi,\mathrm{cusp}}(G)$. The same rule is applied to the space $\mathcal{A}_{P,\pi}(G)$ if π is a cuspidal automorphic representation of $M_P(\mathbb{A})$.

4.1.2. Caution

Since it is easy to get confused, we want to emphasize some consequences of our choices of measures. Let P be a standard parabolic subgroup of G and let $P' = P \cap G'$. The restriction map $X^*(P) \to X^*(P')$ identifies $X^*(P)$ with a subgroup of $X^*(P')$ of index $2^{\dim(\mathfrak{a}_P)}$. It induces an isomorphism $\mathfrak{a}_{P'} \to \mathfrak{a}_P$ which *does not preserve* the Haar measures: the pullback to $\mathfrak{a}_{P'}$ of the Haar measure on $\mathfrak{a}_P$ is $2^{\dim(\mathfrak{a}_P)}$ times the Haar measure on $\mathfrak{a}_{P'}$. In the same way, the groups A^∞_P and $A^\infty_{P'}$ are canonically identified, but the Haar measure on A^∞_P is $2^{\dim(\mathfrak{a}_P)}$ times the Haar measure on $A^\infty_{P'}$.

Note also that for all $x \in G'(\mathbb{A})$, we have

$$\langle \rho^Q_P, H_P(x)\rangle = 2\langle \rho^{Q'}_{P'}, H_{P'}(x)\rangle. \tag{4.1.2.1}$$

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4.1.3. The (G,H)-regular and Hermitian cuspidal data

Let $\chi = (\chi_n, \chi_{n+1}) \in \mathfrak{X}(G) = \mathfrak{X}(G_n) \times \mathfrak{X}(G_{n+1})$ be a cuspidal datum. Let (M,π) be a representative in the class of χ with $M = M_P$ for a standard parabolic subgroup P of G. We write $M = M_n \times M_{n+1}$ and $\pi = \pi_n \boxtimes \pi_{n+1}$ accordingly. Let $j \in \{n, n+1\}$. We write $M_j = G_{n_{1,j}} \times \ldots \times G_{n_{r_j,j}}$ for some integer $r_j \geqslant 1$ and $\pi_j = \sigma_{1,j} \boxtimes \ldots \boxtimes \sigma_{r_j,j}$ accordingly.

We shall say that χ is

- G-regular (or simply regular) if for all $j \in \{n, n+1\}$ and $1 \leqslant i, i' \leqslant r_j$ such that $n_{i,j} = n_{i',j}$ and $\sigma_{i,j} = \sigma_{i',j}$, we have $i' = i$.
- H-regular if for $1 \leqslant i \leqslant r_n$ and all $1 \leqslant j \leqslant r_{n+1}$, if $n_{i,n} = n_{j,n+1}$, the representation $\sigma_{i,n}$ is not isomorphic to the contragredient of $\sigma_{j,n+1}$;
- (G,H)-regular if it is both G-regular and H-regular;
- Hermitian if $\pi = \pi^*$ and if the representation $\sigma_{i,j}$ is $\eta_{G'_j}$-distinguished for all $1 \leqslant i \leqslant r_j$ and $j \in \{n, n+1\}$ such that $\sigma_{i,j} = \sigma^*_{i,j}$.

4.1.4.

We fix a (G,H)-regular and Hermitian cuspidal datum χ. With the notations as above, we may and shall choose the representative (M,π) such that for all $j \in \{n, n+1\}$, there exists an integer $s_j \geqslant 0$ such that the following conditions are satisfied:

1. $2s_j \leqslant r_j$ and for all odd i such that $1 \leqslant i < 2s_j$, we have $\sigma_{i+1,j} = \sigma^*_{i,j}$ (in particular, $\sigma_{i,j} \neq \sigma^*_{i,j}$);
2. for all $i > 2s_j$, we have $\sigma_{i,j} = \sigma^*_{i,j}$.

Let $L = L_n \times L_{n+1}$ be the standard Levi subgroup of G such that for all $j \in \{n, n+1\}$, we have

$$L_j = G_{n_{1,j}+n_{2,j}} \times \ldots \times G_{n_{2s_j-1,j}+n_{2s_j,j}} \times G_{n_{2s_j+1,j}} \times \ldots \times G_{n_{r,j}}. \tag{4.1.4.1}$$

Let $\xi \in W(M)$ such that $\xi^2 = 1$ and $\mathfrak{a}^L_M$ is the kernel of $\xi + \mathrm{Id}$ for the natural action of $W(M)$ on $\mathfrak{a}_M$. We denote by Q the standard parabolic subgroup of Levi L.

4.1.5. Intertwining period

We identify the Weyl group of G with the group of permutation matrix. In this way, we identify ξ with an element of $L(F)$. Let $\tilde{\xi} \in L(F)$ such that $\tilde{\xi} c(\tilde{\xi})^{-1} = \xi$, where c is the Galois involution of G. We define the F-subgroups $P_{\tilde{\xi}}$, $M_{\tilde{\xi}}$ and $N_{\tilde{\xi}}$ of G', respectively, by

$$P_{\tilde{\xi}} = G' \cap \tilde{\xi}^{-1} P \tilde{\xi}. \tag{4.1.5.1}$$

$$M_{\tilde{\xi}} = G' \cap \tilde{\xi}^{-1} M \tilde{\xi}. \tag{4.1.5.2}$$

$$N_{\tilde{\xi}} = G' \cap \tilde{\xi}^{-1} N \tilde{\xi}. \tag{4.1.5.3}$$

We have the Levi decomposition $P_{\tilde{\xi}} = M_{\tilde{\xi}} N_{\tilde{\xi}}$, where $M_{\tilde{\xi}}$ is reductive and $N_{\tilde{\xi}}$ is unipotent. Let $Q' = Q \cap G'$: this is a standard parabolic subgroup of G' of Levi factor $L' = L \cap G$. Observe that we have $A^\infty_{M_{\tilde{\xi}}} = A^\infty_L$, $M_{\tilde{\xi}} \subset L'$ and $N_{\tilde{\xi}} = N_{Q'}$.

The map $a \mapsto H_P(\tilde{\xi} a \tilde{\xi}^{-1})$ identifies $A^\infty_{M_{\tilde{\xi}}}$ with the subspace $\mathfrak{a}_L$. In particular, we have $\langle \lambda, H_P(\tilde{\xi} a \tilde{\xi}^{-1}) \rangle = 0$ for any $a \in A^\infty_{M_{\tilde{\xi}}}$ and $\lambda \in \mathfrak{a}^{L,*}_{M,\mathbb{C}}$. We define the intertwining period for all $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $\lambda \in \mathfrak{a}^{L,*}_{M,\mathbb{C}}$ by

$$J(\xi, \varphi, \lambda) = \int_{A^\infty_{M_{\tilde{\xi}}} M_{\tilde{\xi}}(F) N_{\tilde{\xi}}(\mathbb{A}) \backslash G'(\mathbb{A})} \exp(\langle \lambda, H_P(\tilde{\xi} h) \rangle) \varphi(\tilde{\xi} h) \eta_{G'}(h) \, dh. \tag{4.1.5.4}$$

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Here, the group $N_{\tilde{\xi}}(\mathbb{A})A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F)$ is equipped with a right-invariant Haar measure. However, this measure is not left-invariant: the modular character is given by

$$\delta_{P,\tilde{\xi}} : x \mapsto \exp(\langle \rho_P, H_P(\tilde{\xi}x\tilde{\xi}^{-1})\rangle);$$

see [JLR99, VII p.221]. The integral in (4.1.5.4) is understood as a right-$G'(\mathbb{A})$-invariant linear form on the space of $(N_{\tilde{\xi}}(\mathbb{A})A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F), \delta_{P,\tilde{\xi}})$-equivariant functions. This space contains

$$\exp(\langle \lambda, H_P(\tilde{\xi}\cdot)\rangle)\varphi(\tilde{\xi}\cdot)$$

for $\lambda \in \mathfrak{a}^{L,*}_{M,\mathbb{C}}$. The integral (4.1.5.4) makes sense at least formally. It is, in fact, absolutely convergent for λ such that $\langle \lambda, \alpha^\vee\rangle$ is large enough for any $\alpha^\vee \in \Delta^{Q,\vee}_P$ and it admits a meromorphic continuation to $\mathfrak{a}^{L,*}_{M,\mathbb{C}}$. It does not depend on a specific choice of $\tilde{\xi}$. For all these properties, we refer the reader to [JLR99, theorem 23 and (proof of) lemma 32]. In the same way, for any $\phi \in \mathcal{A}_{L\cap P,\pi}(L)$, one defines the intertwining period $J^L(\xi, \phi, \lambda)$ by analytic continuation of the integral

$$\int_{A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F)\backslash L'(\mathbb{A})} \exp(\langle \lambda, H_P(\tilde{\xi}h)\rangle)\phi(\tilde{\xi}h)\eta_{G'}(h)\, dh,$$

which is convergent for λ in some cone in $\mathfrak{a}^{L,*}_{M,\mathbb{C}}$. Let dk' be the Haar measure on K' such that the Iwasawa decomposition $G'(\mathbb{A}) = L'(\mathbb{A})N_{Q'}(\mathbb{A})K'$ is compatible with the various choices of measures. The following 'parabolic descent' will be useful:

Lemma 4.1.5.1. *For all $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $\lambda \in \mathfrak{a}^{L,*}_{M,\mathbb{C}}$, we have*

$$J(\xi, \varphi, \lambda) = J^L(\xi, \varphi^{K'}, \lambda),$$

where we set

$$\varphi^{K'}(g) = \exp(-\langle \rho_Q, H_Q(g)\rangle) \int_{K'} \varphi(gk')\eta_{G'}(k')\, dk'. \tag{4.1.5.5}$$

Proof. We refer the reader to [JLR99, (proof of) lemma 32]. □

However, by [Lap06, lemma 8.1 case 2], we have in our situation

$$\begin{aligned} J(\xi, \varphi, \lambda) &= J^L(\xi, \varphi^{K'}, \lambda) \\ &= \int_{[L']_0} \Lambda^T_m E^Q(x, \varphi^{K'}, \lambda)\eta_{G'}(x)\, dx, \end{aligned} \tag{4.1.5.6}$$

where the truncation operator Λ^T_m is essentially that introduced in [JLR99]. More precisely, we decompose $L = L_n \times L_{n+1}$ as in (4.1.4.1): on a factor $G_{n_r+n_{r+1},j}$ with $j = n, n+1$ and r an odd integer such that $1 \leqslant r < 2s_j$, the truncation operator is exactly that of [JLR99, IV] and it is trivial on a factor $G_{n_r,j}$ with $r_j \geqslant r > 2s_j$. The Eisenstein series $E^Q(x, \varphi^{K'}, \lambda)$ is holomorphic on $i\mathfrak{a}^{L,*}_M$. The basic properties of the truncation operator Λ^T_m (see [JLR99, IV]) implies that the integral of the truncated Eisenstein series (4.1.5.6) is meromorphic on $\mathfrak{a}^{L,*}_{M,\mathbb{C}}$ and even holomorphic on $i\mathfrak{a}^{L,*}_M$. We deduce that the meromorphic continuation of $J(\xi, \varphi, \lambda)$ is holomorphic on $i\mathfrak{a}^{L,*}_M$ and that, for $\lambda \in i\mathfrak{a}^{L,*}_M$, the map $\varphi \mapsto J(\xi, \varphi, \lambda)$ is continuous on $\mathcal{A}_{P,\pi}(G)$.

4.1.6. Rankin-Selberg period

Let $T \in \mathfrak{a}^+_{n+1}$ be a sufficiently positive parameter. Let $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $\lambda \in \mathfrak{a}^*_M$. Let $\mathbf{P}(E(\varphi, \lambda))$ be the regularized Rankin-Selberg period of the Eisenstein series $E(\varphi, \lambda)$ defined by Ichino-Yamana in [IY15].

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Because we assume that (M,π) is H-regular, the period is given by the truncated integral

$$\mathbf{P}(E(\varphi,\lambda)) = \int_{[H]} \Lambda_r^T E(h,\varphi,\lambda)\, dh, \tag{4.1.6.1}$$

where Λ_r^T is the Ichino-Yamana truncation operator (whose definition is recalled in [BPCZ22, eq. (3.3.2.1)]). In fact, as the notation suggests, the right-hand side of (4.1.6.1) does not depend on T (the proof of this property is the same as the proof of [BPCZ22, proposition 5.1.4.1]). In particular, $\mathbf{P}(E(\varphi,\lambda))$ inherits analytic properties of the Eisenstein series $E(\varphi,\lambda)$. Thus, it is holomorphic on $i\mathfrak{a}_M^*$ and the map $\varphi \mapsto \mathbf{P}(E(\varphi,\lambda))$ is continuous.

4.1.7. The relative character

For any ψ and $\varphi \in \mathcal{A}_{P,\pi}(G)$, the expression

$$\mathbf{P}(E(\psi,\lambda)) \cdot J(\xi,\bar{\varphi},-\lambda)$$

is holomorphic on $i\mathfrak{a}_M^{L,*}$ and gives a continuous pairing on $\mathcal{A}_{P,\pi}(G)$. By [BPCZ22, proposition 2.8.4.1], for any $f \in \mathcal{S}(G(\mathbb{A}))$ and any $\lambda \in i\mathfrak{a}_M^{L,*}$, we can define the relative character

$$I_{P,\pi}(\lambda,f) = \sum_{\varphi\in\mathcal{B}_{P,\pi}} \mathbf{P}(E(I_P(\lambda,f)\varphi,\lambda)) \cdot J(\xi,\bar{\varphi},-\lambda).$$

where the sum is over a K-basis $\mathcal{B}_{P,\pi}$. We get a continuous map $f \mapsto (\lambda \mapsto I_{P,\pi}(\lambda,f))$ from $\mathcal{S}(G(\mathbb{A}))$ to the space of Schwartz functions on $i\mathfrak{a}_M^{L,*}$: this is a slight extension of [BPCZ22, proposition 4.1.10.1] which basically relies on bounds of Eisenstein series due to Lapid in [Lap06, proposition 6.1]. Note also that this is also an easy consequence of the inequality given in [Cha22, proof of proposition 7.2.3.2].

4.1.8. The (G,H)-regular contribution to the Jacquet-Rallis trace formula

The contribution to the Jacquet-Rallis trace formula of a cuspidal datum χ is a distribution denoted by I_χ and defined in [BPCZ22, theorem 3.2.4.1]. In the case of a Hermitian (G,H)-regular cuspidal datum, the next theorem relates this contribution to the relative character defined above.

Theorem 4.1.8.1. *Let $\chi \in \mathfrak{X}(G)$ be a (G,H)-regular cuspidal datum.*

1. *If χ is not Hermitian, we have $I_\chi = 0$.*
2. *Assume that χ is, moreover, Hermitian and let (M_P,π) and L be as in §4.1.4. For all $f \in \mathcal{S}(G(\mathbb{A}))$, we have*

$$I_\chi(f) = 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} I_{P,\pi}(\lambda,f)\, d\lambda, \tag{4.1.8.1}$$

where the integral in the right-hand side is absolutely convergent.

4.1.9. Proof of Theorem 4.1.8.1

Let $\chi \in \mathfrak{X}(G)$ be a (G,H)-regular cuspidal datum. Let $f \in \mathcal{S}(G(\mathbb{A}))$ and let K_χ be the kernel of the right convolution by f on $L^2_\chi([G])$. By [BPCZ22, proposition 3.3.8.1 and theorem 3.3.9.1], the contribution $I_\chi(f)$ is the constant term in the asymptotic expansion in $T \in \mathfrak{a}_{n+1}^+$ of the absolutely convergent integral

$$\int_{[H]} \int_{[G']} \Lambda_r^T K_\chi(h,g)\, \eta_{G'}(g) dg dh. \tag{4.1.9.1}$$

By [BPCZ22, lemma 5.2.2.1], we have for all $h \in [H]$,

$$\int_{[G']} (\Lambda_r^T K_\chi)(h,g)\eta_{G'}(g)\,dg = \Lambda_r^T \left(\int_{[G']} K_\chi(\cdot,g)\eta_{G'}(g)\,dg\right)(h). \tag{4.1.9.2}$$

Then we have the following lemma:

Lemma 4.1.9.1. *Let $\chi \in \mathfrak{X}(G)$ be a (G,H)-regular cuspidal datum. For all $x \in G(\mathbb{A})$, the absolutely convergent integral*

$$\int_{[G']} K_\chi(x,g)\eta_{G'}(g)\,dg \tag{4.1.9.3}$$

vanishes unless χ is Hermitian. If χ is Hermitian and represented by (M_P,π), we have

$$\int_{[G']} K_\chi(x,g)\,\eta_{G'}(g)dg = 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} (\sum_{\varphi\in\mathcal{B}_{P,\pi}} E(x, I_P(\lambda,f)\varphi,\lambda)\cdot J(\xi,\bar{\varphi},-\lambda)\,d\lambda, \tag{4.1.9.4}$$

where the Levi subgroup L is that defined in §4.1.4 and $\mathcal{B}_{P,\pi}$ is a K-basis.

Proof. For brevity reasons, we shall get the lemma as a simple application of a much more difficult result – namely, [Cha22, Theorem 7.2.4.1]. Our situation is essentially that of [Cha22, Theorem 7.2.4.1] except that we are working here with a product of two general linear groups and our Flicker-Rallis periods are twisted by the caracter $\eta_{G'}$. Note that the absolute convergence of (4.1.9.3) results from [BPCZ22, Lemma 2.10.1.1]. Let (M,π) be a representative of the (G,H)-regular cuspidal datum χ with P a standard parabolic subgroup of G and $M = M_P$. Let P_1 be standard parabolic subgroup of G and set $M_1 = M_{P_1}$. There is a finite set $\Pi_\chi(M_1)$ used in the statement of [Cha22, theorem 7.2.4.1]. There is no need to recall the general definition given in [Cha22, §7.1.1]: indeed, because χ is G-regular, either P_1 and P are not associated in which case $\Pi_\chi(M_1)$ is empty or $\Pi_\chi(M_1)$ is the set of $w\pi$ with $w \in W(P,P_1)$.

We can extract from [Cha22, theorem 7.2.4.1] that (4.1.9.3) is equal to the absolutely convergent expression

$$\sum_{P_1} |\mathcal{P}(M_1)|^{-1} \sum_{L_1\in\mathcal{L}_2(M_1)} 2^{-\dim(\mathfrak{a}_{L_1})} \sum_{w\in W(P,P_1)} \int_{i\mathfrak{a}_{M_1}^{L_1,*}} \mathcal{I}_{L_1,w\pi}(x,f,\lambda)\,d\lambda, \tag{4.1.9.5}$$

where the sum is over the set of standard parabolic sugroups P_1 of G and $M_1 = M_{P_1}$, the number of parabolic subgroups of Levi M is denoted by $|\mathcal{P}(M)|$, the set $\mathcal{L}_2(M_1)$ is a subset of the set of Levi subgroups of G containing M_1 (see [Cha22, §2.2.3]) and $\mathcal{I}_{L_1,w\pi}(x,f,\lambda)$ is *mutatis mutandis* the relative character that essentially appears in the statement of [Cha22, theorem 7.2.4.1]. The main difference is that we are working here on a product of two linear groups and that the relative character is built upon Eisenstein series and the intertwining periods defined in [Cha22, §5.1.4] but twisted by the character $\eta_{G'}$. Let M_1, L_1 and $w \in W(P,P_1)$ be as in (4.1.9.5) and let $x \in G(\mathbb{A})$ and $\lambda \in i\mathfrak{a}_{M_1}^{L_1,*}$. Rather than spell out the exact definition of $\mathcal{I}_{L_1,w\pi}(x,f,\lambda)$, we will explain it in the particular case that concerns us. Before doing that, we owe a detailed explanation to the careful reader who will certainly have noticed the apparent discrepancy of a factor $2^{\dim(\mathfrak{a}_G)}$. The reason is the following: [Cha22, theorem 7.2.4.1] gives in fact the spectral expansion of

$$\int_{[G']_0}\int_{A_G^\infty} K_\chi(ax,g)\eta_{G'}(g)\,dadg,$$

whereas we are working with

$$\int_{[G']} K_\chi(x,g)\eta_{G'}(g)\,dg = 2^{-\dim(\mathfrak{a}_G)} \int_{[G']_0}\int_{A_G^\infty} K_\chi(ax,g)\eta_{G'}(g)\,dadg.$$

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The equality above comes from the difference between the respective measures on A_G^∞ and $A_{G'}^\infty$; see §4.1.2.

By the fact that χ is G-regular and by the basic properties of the cuspidal intertwining periods (see [JLR99, theorem 23]), we see that the relative character $\mathcal{I}_{L_1,w\pi}(x,f,\lambda)$ vanishes unless χ is Hermitian. So we get the first assertion, and we assume from now on that χ is moreover Hermitian. Note that, by definition, we have for all $\mu\in i\mathfrak{a}_M^{L,*}$,

$$\mathcal{I}_{L,\pi}(x,f,\mu)=\sum_{\varphi\in\mathcal{B}_{P,\pi}} E(x,I_P(\mu,f)\varphi,\mu)\cdot J(\xi,\bar{\varphi},-\mu). \tag{4.1.9.6}$$

We claim that we have

$$\forall\lambda\in i\mathfrak{a}_{M_1}^{L_1,*}\ \ \mathcal{I}_{L_1,w\pi}(x,f,\lambda)=\mathcal{I}_{w^{-1}L_1w,\pi}(x,f,w^{-1}\lambda). \tag{4.1.9.7}$$

Moreover, it follows from the fact that χ is G-regular and Hermitian that for all Levi subgroup $L_1\in\mathcal{L}_2(M)$ and all $\mu\in\mathfrak{a}_M^{L_1,*}$, we have

$$\mathcal{I}_{L_1,w\pi}(x,f,\lambda)=0 \text{ if } L_1\neq wLw^{-1}. \tag{4.1.9.8}$$

Let us assume (4.1.9.7) for the moment, and let us finish the proof of (4.1.9.4). By (4.1.9.7) and (4.1.9.8) and the change of variables $\lambda\mapsto w^{-1}\lambda$, we see that (4.1.9.5) is equal to

$$\begin{aligned}
&2^{-\dim(\mathfrak{a}_L)}\sum_{P_1}|\mathcal{P}(M_1)|^{-1}\sum_{w\in W(P,P_1)}\int_{i\mathfrak{a}_{M_1}^{wLw^{-1},*}}\mathcal{I}_{L,\pi}(x,f,w^{-1}\lambda)\,d\lambda\\
&=2^{-\dim(\mathfrak{a}_L)}\Big(\sum_{P_1}|\mathcal{P}(M_1)|^{-1}|W(P,P_1)|\Big)\int_{i\mathfrak{a}_M^{L,*}}\mathcal{I}_{L,\pi}(x,f,\lambda)\,d\lambda\\
&=2^{-\dim(\mathfrak{a}_L)}\int_{i\mathfrak{a}_M^{L,*}}\mathcal{I}_{L,\pi}(x,f,\lambda)\,d\lambda
\end{aligned}$$

since $\sum_{P_1}|\mathcal{P}(M_1)|^{-1}|W(P,P_1)|=1$ where the various sums are over standard parabolic subgroups P_1.

Let us prove the claim (4.1.9.7). We start from M_1,L_1,λ,w as in (4.1.9.5). By (4.1.9.8), we may and shall assume that we have $L_1=wLw^{-1}$. Let Q_1 be a parabolic subgroup of Levi L_1. Let P_2 be a standard parabolic subgroup of G and let $w_1\in W(P_1,P_2)$ such that $P_2\subset w_1Q_1w_1^{-1}$. Using the definition of intertwining periods given in [Cha22, §5.1.4], the functional equation of Eisenstein series and a standard basis change relying on the fact that the intertwining operator $M(w_1,\lambda)$ induces a unitary isomorphism from $\mathcal{A}_{P_1,w\pi}(G)$ onto $\mathcal{A}_{P_1,w_1w\pi}(G)$, we see that we have

$$\mathcal{I}_{L_1,w\pi}(x,f,\lambda)=\mathcal{I}_{L_2,w_1w\pi}(x,f,w_1\lambda)$$

with $L_2=w_1L_1w_1^{-1}$. Thus, we are reduced to prove (4.1.9.7) in the special case where the Levi subgroup $L_1=wLw^{-1}$ is standard. But then (4.1.9.7) follows from the same arguments as before, namely the functional equation of Eisenstein series and a standard basis change and the functional equation of intertwining periods, namely [JLR99, theorem 31] completed with [Lap06, lemma 8.1]. For a similar statement, see [Lap06, proposition 8.2]. This finishes the proof. □

It follows from Lemma 4.1.9.1 that, for $h\in[H]$, we have

$$\Lambda_r^T\left(\int_{[G']}K_\chi(\cdot,g)\eta_{G'}(g)\,dg\right)(h)=2^{-\dim(\mathfrak{a}_L)}\int_{i\mathfrak{a}_M^{L,*}}\sum_{\varphi\in\mathcal{B}_{P,\pi}}(\Lambda_r^T E(h,I_P(\lambda,f)\varphi,\lambda)\cdot J(\xi,\bar{\varphi},-\lambda)\,d\lambda. \tag{4.1.9.9}$$

To get this, we have to permute the truncation operator with the sum and the integral. However, one observes that for a fixed $h \in [H]$, the truncation operator is a finite sum of constant terms. So to get (4.1.9.9), one has basically to permute the integral over $i\mathfrak{a}_M^{L,*}$ and the integral that gives a constant term. So, by Fubini theorem, it suffices to show that for any parabolic subgroup Q of G, the sum

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}} \int_{N_Q} |E(nh, I_P(\lambda, f)\varphi, \lambda)|\, dn \cdot |J(\xi, \bar{\varphi}, -\lambda)|$$

is a Schwartz function in the variable $\lambda \in i\mathfrak{a}_M^{L,*}$. But this is a variant of [Lap06, Lemma 7.4]; it is also an easy consequence of [Cha22, proposition 7.2.3.2]. The last step is to observe that

$$\int_{[H]} \int_{i\mathfrak{a}_M^{L,*}} \sum_{\varphi \in \mathcal{B}_{P,\pi}} \Lambda_r^T E(h, I_P(\lambda, f)\varphi, \lambda) \cdot J(\xi, \bar{\varphi}, -\lambda)\, d\lambda dh \tag{4.1.9.10}$$

$$= \int_{i\mathfrak{a}_M^{L,*}} \sum_{\varphi \in \mathcal{B}_{P,\pi}} \int_{[H]} \Lambda_r^T E(h, I_P(\lambda, f)\varphi, \lambda)\, dh \cdot J(\xi, \bar{\varphi}, -\lambda)\, d\lambda$$

and to use (4.1.6.1) to recognize the relative character $I_{P,\pi}(\lambda, f)$ in the inner sum. Once again by a variant of [Lap06, Lemma 7.4] (or by [Cha22, proposition 7.2.3.2]) and the basic properties of truncation operator (see [BPCZ22, proposition 3.3.2.1]), we have

$$\int_{i\mathfrak{a}_M^{L,*}} \sum_{\varphi \in \mathcal{B}_{P,\pi}} \int_{[H]} |\Lambda_r^T E(h, I_P(\lambda, f)\varphi, \lambda)|\, dh \cdot |J(\xi, \bar{\varphi}, -\lambda)|\, d\lambda < \infty.$$

We can again conclude with Fubini's theorem.

4.2. The relative character in terms of Whittaker functions

4.2.1.
We keep the notations of the previous subsection. Let $N = N_n \times N_{n+1}$ and $N_H = N_n$ be viewed as a diagonal subgroup of N. Let $N' = N \cap G'$.

4.2.2.
We fix a nontrivial additive character $\psi' : \mathbb{A}/F \to \mathbb{C}^\times$. We deduce a character $\psi : \mathbb{A}_E/E \to \mathbb{C}^\times$ trivial on $\mathbb{A}$ by $\psi(z) = \psi'(\mathrm{Tr}_{E/F}(\tau z))$, where $\tau \in E^\times$ is such that $c(\tau) = -\tau$. We define a regular character $\psi_n : [N_n] \to \mathbb{C}^\times$ by

$$\psi_n(u) = \psi\left((-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right)$$

for any $u \in [N_n]$. In the same way, we get a character ψ_{n+1} of $[N_{n+1}]$. Thus, we have a character $\psi_N = \psi_n \boxtimes \psi_{n+1}$ of $[N]$. By construction, ψ_N is trivial on the subgroups N' and N_H.

4.2.3.
Recall that we have fixed a pair (M, π) with $M = M_P$ (see §§4.1.3 and 4.1.4). Let $\lambda \in i\mathfrak{a}_P^{G,*}$. Let π_λ be the representation π twisted by the character $m \mapsto \exp(\langle \lambda, H_M(m)\rangle)$ and let $\Pi_\lambda = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi_\lambda)$ be the induced representation. The representation Π_λ is irreducible, unitary and generic. Let $\mathcal{W}(\Pi_\lambda, \psi_N)$ be its Whittaker model with respect to the character ψ_N.

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For any $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $g \in G(\mathbb{A})$, let

$$W(g,\varphi,\lambda) = \int_{[N]} E(ug,\varphi,\lambda)\psi_N^{-1}(u)\,du.$$

We may and shall identify $g \mapsto W(g,\varphi,\lambda)$ with an element of $\mathcal{W}(\Pi_\lambda,\psi_N)$.

4.2.4.
Let $W \in \mathcal{W}(\Pi_\lambda,\psi_N)$. By [JPSS83] and [Jac09], the integral

$$Z^{\mathrm{RS}}(s,W) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} W(h)|\det h|_{\mathbb{A}_E}^s\,dh$$

converges for $\Re(s) \gg 0$ and extends to a meromorphic function on $\mathbb{C}$ which is holomorphic at $s=0$. Let $\mathcal{P} = \mathcal{P}_n \times \mathcal{P}_{n+1}$ be the product of the respective mirabolic subgroups $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ of G_n and G_{n+1}. Let $\mathcal{P}' = \mathcal{P} \cap G'$. We put $\mathrm{As}_G = \mathrm{As}^{(-1)^{n+1}} \boxtimes \mathrm{As}^{(-1)^n}$. For any finite set $S \subset V_F$, we set

$$\beta_\eta(W) = (\Delta_{G'}^{S,*})^{-1}L^{S,*}(1,\Pi_\lambda,\mathrm{As}_G)\int_{N'(F_S)\backslash\mathcal{P}'(F_S)} W(p_S)\eta_{G'}(p_S)dp_S$$

and

$$\langle W,W\rangle_{\mathrm{Whitt}} = (\Delta_G^{S,*})^{-1}L^{S,*}(1,\Pi_\lambda,\mathrm{Ad})\int_{N(F_S)\backslash\mathcal{P}(F_S)} |W(p_S)|^2 dp_S.$$

Here and hereafter, $L^*(1)$ means the leading coefficient in the Laurent expansion of the meromorphic function $L(s)$ at $s=1$. The above expressions converge and are independent of S as soon as it is chosen sufficiently large according to the level of W (see [Fli88] and [JS81b]).

4.2.5.
Proposition 4.2.5.1. *For any $\varphi \in \mathcal{A}_{P,\pi}(G)$, we have*

$$Z^{\mathrm{RS}}(0,W(\varphi,\lambda)) = \mathbf{P}(E(\varphi,\lambda)).$$

Proof. This is a straightforward application of results of Ichino-Yamana and the fact that (M,π) is H-regular. First, for any $T \in \mathfrak{a}_{n+1}^+$ and any $s \in \mathbb{C}$, the integral

$$\int_{[H]} \Lambda_r^T E(h,\varphi,\lambda)|\det(h)|^s\,dh \tag{4.2.5.1}$$

converges and defines a holomorphic function in the variable s. Moreover, since (M,π) is H-regular, it does not depend on T (see [BPCZ22, proof of proposition 5.1.4.1]). Because of this, (4.2.5.1) is the regularized Rankin-Selberg period of $h \mapsto E(h,\varphi,\lambda)|\det(h)|^s$ defined in [IY15]. By [IY15, theorem 1.1], we deduce that (4.2.5.1) is equal to $Z^{\mathrm{RS}}(s,W(\varphi,\lambda))$. It suffices to take $s=0$ to get the result. □

4.2.6.
Proposition 4.2.6.1. *For any $\varphi \in \mathcal{A}_{P,\pi}(G)$, we have*

$$\langle\phi,\phi\rangle_{\mathrm{Pet}} = \langle W(\varphi,\lambda),W(\varphi,\lambda)\rangle_{\mathrm{Whitt}}.$$

Proof. This is the proof of [BPCZ22, proposition 8.1.2.1]), the main assumption there being that (M,π) is regular. □

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4.2.7.

Proposition 4.2.7.1. *For any $\varphi \in \mathcal{A}_{P,\pi}(G)$, we have*

$$J(\xi, \varphi, \lambda) = \beta_\eta(W(\varphi, \lambda)). \tag{4.2.7.1}$$

Proof. We shall follow the notations of §§4.1.4 and 4.1.5. Let $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^{L,*}$. For all $\varphi \in \mathcal{A}_{P,\mathrm{cusp}}(G)$, we have the Eisenstein series $E^Q(\varphi, \lambda)$ defined in (2.2.4.1), and we set for all $g \in G(\mathbb{A})$,

$$W_L(g, \varphi, \lambda) = \int_{[N\cap L]} E^Q(ug, \varphi, \lambda)\psi_N^{-1}(u)\,du.$$

Let $\mathcal{P}_L$ the mirabolic subgroup of L (defined as the product of the mirabolic subgroups of its factors in the decomposition into a product of general linear groups). We set $\mathcal{P}_{L'} = \mathcal{P}_L \cap L'$. Let As_{L_n} (resp. $\mathrm{As}_{L_{n+1}}$) be the tensor product of $\mathrm{As}^{(-1)^{n+1}}$ (resp. $\mathrm{As}^{(-1)^n}$) for each factor in the decomposition of L_n, resp. L_{n+1}, into a product of general linear groups. Let $\mathrm{As}_L = \mathrm{As}_{L_n} \boxtimes \mathrm{As}_{L_{n+1}}$. Let $\Pi_\lambda^Q = \mathrm{Ind}_{P(\mathbb{A})}^{Q(\mathbb{A})}(\pi_\lambda)$ be the induced representation. It follows from our hypothesis on π and from [Fli92, proposition 2.6] that Π_λ^Q is $(L', \eta_{G'})$-distinguished.

Let $\varphi \in \mathcal{A}_{P,\pi}(G)$. First, by Lemma 4.1.5.1, we have

$$J(\xi, \varphi, \lambda) = J^L(\xi, \varphi^{K'}, \lambda),$$

where $\varphi^{K'}$ is defined in (4.1.5.5). Then using [Zha14a, proposition 3.2], [Fli88] and Theorem 5.5.1.1 in section 5 below, we can compute the intertwining period $J^L(\xi, \varphi^{K'}, \lambda)$ in terms of W_L: we deduce that, for S a large enough set of places, $J(\xi, \varphi, \lambda)$ is equal to

$$\begin{aligned}&(\Delta_{L'}^{S,*})^{-1} L^{S,*}(1, \Pi_\lambda^Q, \mathrm{As}_L) \int_{(N'\cap L)(F_S)\backslash \mathcal{P}'_L(F_S)} W_L(q_S, \varphi^{K'} \lambda)\eta_{G'}(q_S)\,dq_S \\ &= (\Delta_{L'}^{S,*})^{-1} L^{S,*}(1, \Pi_\lambda^Q, \mathrm{As}_L) \int_{K'} \int_{(N'\cap L)(F_S)\backslash \mathcal{P}'_L(F_S)} W_L(q_S k', \varphi_{-\rho_Q}, \lambda)\eta_{G'}(q_S k')\,dq_S dk',\end{aligned}$$

where $\varphi(g) = \exp(-\rho_Q, H_Q(g)\rangle)\varphi(g)$. For $g \in G(F_S)$, set

$$\Phi(g) = \int_{(N'\cap L)(F_S)\backslash \mathcal{P}'_L(F_S)} W_L(q_S g, \varphi_{-\rho_Q}, \lambda)\eta_{G'}(q_S g)\,dq_S.$$

The map Φ is left-invariant by $L'(F_S)$ as follows from the first assertion of [BPCZ22, theorem 9.1.7.1]. We deduce that it is also left-invariant by $Q'(F_S)$. Using the equality (4.1.2.1), we get that $J(\xi, \varphi, \lambda)$ is equal to

$$(\Delta_{G'}^{S,*})^{-1} L^{S,*}(1, \Pi_\lambda^Q, \mathrm{As}_L) \int_{Q'(F_S)\backslash G'(F_S)} \exp(\rho_Q, H_Q(g_S)\rangle)\Phi(g_S) dg_S.$$

Now we can appeal to the equality of [BPCZ22, theorem 9.1.7.1] to get that the last line is equal to

$$(\Delta_{G'}^{S,*})^{-1} L^{S,*}(1, \Pi_\lambda^Q, \mathrm{As}_L) \int_{N'(F_S)\backslash \mathcal{P}'(F_S)} \mathbb{W}_S(p_S, \lambda, \varphi)\eta_{G'}(p_S)\,dp_S,$$

where $\mathbb{W}_S(g_S, \lambda, \varphi)$ stands for the Jacquet integral given by the value at $s = 0$ of the holomorphic continuation of the integral:

$$\int_{(w_L^{-1} L w_L \cap N)(F_S)\backslash N(F_S)} \delta_Q(w_L u g_S)^s W_L(w_L u g_S, \varphi, \lambda)\psi_N(u)^{-1}\,du, \ \Re(s) \gg 1,$$

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where w_L is the permutation matrix that represents the product of the longest elements respectively of the Weyl groups of L and of G. To conclude, it suffices to observe that for any $g_S \in G(\mathbb{A}_S)$,

$$L^{S,*}(1, \Pi_\lambda^Q, \mathrm{As}_L)\mathbb{W}_S(g_S, \lambda, \varphi) = L^{S,*}(1, \Pi_\lambda, \mathrm{As})W(g_S, \lambda, \varphi).$$

Indeed, this follows from computations of [Sha81, section 4] and the fact that the functions $L^S(s, \Pi_\lambda^Q, \mathrm{As}_L)$ and $L^S(s, \Pi_\lambda, \mathrm{As})$ have a pole of same order at $s = 1$ since the cuspidal datum associated to (M, π) is G-regular and Hermitian. □

4.2.8.
Let $\mathcal{B}_{P,\pi}$ be a K-basis of $\mathcal{A}_{P,\pi}(G)$. For any $f \in \mathcal{S}(G(\mathbb{A}))$, we define

$$I_{\Pi_\lambda}(f) = \sum_{\varphi \in \mathcal{B}_{P,\pi}} \frac{Z^{RS}(0, W(I_P(\lambda, f)\varphi, \lambda))\overline{\beta_\eta(W(\varphi, \lambda))}}{\langle W(\varphi, \lambda), W(\varphi, \lambda)\rangle_{\mathrm{Pet}}}. \tag{4.2.8.1}$$

Theorem 4.2.8.1.

1. *The series (4.2.8.1) converges, does not depend on the choice of $\mathcal{B}_{P,\pi}$ and defines a continuous distribution on $\mathcal{S}(G(\mathbb{A}))$.*
2. *We have $I_{\Pi_\lambda}(f) = I_{P,\pi}(\lambda, f)$.*

Proof. The two assertions follow from the fact that the relative characters I_{Π_λ} and $I_{\pi,\lambda}$ can be identified term by term by the combination of Propositions 4.2.5.1, 4.2.6.1 and 4.2.7.1. □

5. Intertwining periods and Whittaker functions

5.1. Notations

5.1.1.
We follow the notations of §4.1.1. However, in this section, we set $G = G_{2n}$ and $G' = G'_{2n}$. Let $P = MN$ be the maximal standard parabolic subgroup of G where its standard Levi factor M is $G_n \times G_n$. Let σ be an irreducible cuspidal automorphic representation of G_n with central character trivial on $A_{G_n}^\infty$. Let $\pi = \sigma \boxtimes \sigma^*$: this a cuspidal representation of M.

5.1.2.
We set

$${}_PW_P = \{w \in W \mid M \cap w^{-1}B_{2n}w = M \cap B_{2n} = M \cap wB_{2n}w^{-1}\}$$

and the subset of involutions

$${}_PW_{P,2} = \{w \in {}_PW_P \mid w^2 = 1\}.$$

We have the following lemma:

Lemma 5.1.2.1. *(Jacquet-Lapid-Rogawski, see [JLR99, proposition 20]). Any double coset in $P(F)\backslash G(F)/G'(F)$ has a representative $\tilde{\xi}$ such that $\xi = \tilde{\xi}c(\tilde{\xi})^{-1}$ belongs to ${}_PW_{P,2}$. The map $P(F)\tilde{\xi}G'(F) \mapsto \xi$ is well defined and induces a bijection from $P(F)\backslash G(F)/G'(F)$ onto ${}_PW_{P,2}$.*

For any $\tilde{\xi} \in G(F)$ such that $\tilde{\xi}c(\tilde{\xi})^{-1}$ belongs to ${}_PW_{P,2}$, we set $P_{\tilde{\xi}} = G' \cap \tilde{\xi}^{-1}P\tilde{\xi}$ and $M_{\tilde{\xi}} = G' \cap \tilde{\xi}^{-1}M\tilde{\xi}$. Note that $M_{\tilde{\xi}}$ is a Levi factor of $P_{\tilde{\xi}}$.

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5.1.3.
We fix $\tau \in E$ such that $c(\tau) = -\tau$. Let

$$\tilde{\xi}_0 = \begin{pmatrix} I_n & \tau I_n \\ I_n & -\tau I_n \end{pmatrix} \text{ and } \xi_0 = \tilde{\xi}_0 c(\tilde{\xi}_0)^{-1}.$$

We have $\xi_0 \in {}_P W_{P,2}$ and $P_{\tilde{\xi}_0} = M_{\tilde{\xi}_0}$.

Let $\varphi \in \mathcal{A}_{P,\pi}(G)$. For any $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$, we consider the intertwining period (due to Jacquet-Lapid-Rogawski; see [JLR99])

$$J(\xi_0, \varphi, \lambda) = \int_{A^\infty_{M_{\tilde{\xi}_0}} M_{\tilde{\xi}_0}(F)\backslash G'(\mathbb{A})} \exp(\langle \lambda, H_P(\tilde{\xi} h)\rangle)\varphi(\tilde{\xi} h)\, dh. \qquad (5.1.3.1)$$

Note that we have $A^\infty_{M_{\tilde{\xi}_0}} = A^\infty_{G'}$. Let α be the unique root in Δ_P. The integral above is absolutely convergent if $\Re(\langle \alpha, \lambda\rangle) \gg 0$. Moreover, it admits a meromorphic continuation to $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$; see [JLR99, theorem 23].

5.2. Epstein series and intertwining periods

5.2.1.
The group G acts on the right on the space of rows of size $2n$ (identified to E^{2n}). Let $\mathcal{P}$ be the stabilizer of $e_{2n} = (0, \dots, 0, 1)$. Let $\mathcal{P}' = \mathcal{P} \cap G'$.

Let $\Phi \in \mathcal{S}(\mathbb{A}^{2n})$. For any $s \in \mathbb{C}$ such that $\Re(s) > 1$ and any $h \in [G']$, the Epstein series is defined by the following absolutely convergent integral:

$$E(h, \Phi, s) = \int_{A^\infty_{G'}} \sum_{\gamma \in \mathcal{P}'(F)\backslash G'(F)} \Phi(e_{2n}\gamma a h)|\det(ah)|^s\, da.$$

Here, $|\cdot|$ is the product over all places v of F of normalized absolute values of the completions F_v. The map $s \mapsto E(\Phi, s)$ extends to a meromorphic function valued in $\mathcal{T}([G'])$ with simple poles at $s = 0, 1$ of respective residues $\Phi(0)$ and $\widehat{\Phi}(0)$ (cf. [JS81b, Lemma 4.2]).

5.2.2.
Let $\tilde{\xi} \in G(F)$ such that $\xi = \tilde{\xi} c(\tilde{\xi})^{-1}$ belongs to ${}_P W_{P,2}$. We assume also $\xi \neq 1$. Let $\varphi \in \mathcal{A}_{P,\pi}(G)$. We define

$$J(\xi, \varphi, \lambda, \Phi, s) = \int_{A^\infty_{G'} P_{\tilde{\xi}}(F)\backslash G'(\mathbb{A})} \exp(\langle \lambda, H_P(\tilde{\xi} h)\rangle)\varphi(\tilde{\xi} h) E(h, \Phi, s)\, dh. \qquad (5.2.2.1)$$

If it is well defined, this integral does not depend on $\tilde{\xi}$ provided that we have $\xi = \tilde{\xi} c(\tilde{\xi})^{-1}$, hence the notation.

Proposition 5.2.2.1. *Assume* $\xi \neq 1$.

1. *There exists* $r > 0$ *such that for each for any* λ *in the domain,*

$$\mathcal{D}_r = \{\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{G,*} \mid \Re(\langle \lambda, \alpha^\vee\rangle) > r\},$$

   and any $s \in \mathbb{C} \setminus \{0, 1\}$ *the integral (5.2.2.1) converges absolutely. It also converges uniformly for* $\Re(\lambda)$ *in a compact subset of* $\mathcal{D}_r$ *and* s *in a compact subset of* $\mathbb{C} \setminus \{0, 1\}$.
2. *The map* $(\lambda, s) \mapsto J(\xi, \varphi, \lambda, \Phi, s)$ *is holomorphic on* $\mathcal{D}_r \times \mathbb{C} \setminus \{0, 1\}$.
3. *If* $\xi \neq \xi_0$*, the map has a holomorphic extension to* $\mathcal{D}_r \times \mathbb{C}$.

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4. *If $\xi = \xi_0$, the map has simple poles at $s = 0, 1$ with respective residues:*

$$\Phi(0)J(\xi_0, \varphi, \lambda) \text{ and } \hat{\Phi}(0)J(\xi_0, \varphi, \lambda).$$

Proof. We fix a compact subset Ω of $\mathbb{C}$. We can write

$$J(\xi, \varphi, \lambda, \Phi, s) = \int_{[G']_0} \left(\sum_{\delta \in P_{\tilde{\xi}}(F) \backslash G'(F)} \exp(\langle \lambda, H_P(\tilde{\xi}\delta h)\rangle) \varphi(\tilde{\xi}\delta h) \right) E(h, \Phi, s)\, dh.$$

We shall use the following two facts:

- there exist $C, t > 0$ such that for all $h \in G'(\mathbb{A})^1$ and $s \in \Omega$,

$$|s(s-1)E(h, \Phi, s)| \leqslant C\|h\|_{G'}^t. \tag{5.2.2.2}$$

- there exists $r > 0$ such that for any $t > 0$ and any $\lambda \in \mathcal{D}_r$, there exists $C > 0$ such that for all $h \in G'(\mathbb{A})^1$,

$$\sum_{\delta \in P_{\tilde{\xi}}(F) \backslash G'(F)} |\exp(\langle \lambda, H_P(\tilde{\xi}\delta h)\rangle) \varphi(\tilde{\xi}\delta h)| \leqslant C\|h\|_{G'}^{-t}. \tag{5.2.2.3}$$

Moreover, in (5.2.2.3), the constant C may be chosen uniformly for λ such that $\Re(\lambda)$ belongs to a compact subset of $\mathcal{D}_r$. This can be extracted from the proof of [JLR99, theorem 23], more precisely from the combination of proposition 24 and the lines following (58) of [JLR99].

We get the holomorphic continuation with at most simple poles at $s = 0, 1$. Up to a factor $\Phi(0)$ or $\hat{\Phi}(0)$, the residue is given by

$$\int_{A_{G'}^{\infty} P_{\tilde{\xi}}(F) \backslash G'(\mathbb{A})} \exp(\langle \lambda, H_P(\tilde{\xi}h)\rangle) \varphi(\tilde{\xi}h)\, dh.$$

This integral converges absolutely thanks to the majorization (5.2.2.3). According to the proof of [JLR99, theorem 23], the integral vanishes unless $\xi = \xi_0$. In this case, the integral is nothing else but $J(\xi_0, \varphi, \lambda)$. □

5.3. Period of a pseudo-Eisenstein series: first computation

5.3.1.
Let ω be the central character of π. By restriction, it induces a unitary character of $Z'_G(\mathbb{A})$. Let

$$\tilde{\Phi}(h, \omega, s) = \int_{Z_{G'}(\mathbb{A})} \Phi(e_{2n} z h) \omega(z) |\det(zh)|^s dz.$$

The integral is convergent for $\Re(s) > \frac{1}{n}$. Let $P'_1 = \mathcal{P}' Z_{G'}$. This is a parabolic subgroup of G' of type $(2n-1, 1)$. Then we have

$$\int_{A_{G'}^{\infty} Z_{G'}(F) \backslash Z_{G'}(\mathbb{A})} E(zh, \Phi, s) \omega(z)\, dz = E(h, \tilde{\Phi}(\omega, s)),$$

where

$$E(h, \tilde{\Phi}(\omega, s)) = \sum_{\gamma \in P'_1(F) \backslash G'(F)} \tilde{\Phi}(\gamma h, \omega, s)$$

is a usual Eisenstein series, which is convergent for $\Re(s) > 1$. By the classical computation of the constant term of an Eisenstein series, there exist a finite set I and families $(\varphi_{i,s})_{i\in I} \in \mathcal{A}_{P'}(G')^I$ and $(\mu_{i,s})_{i\in I} \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$ for any $s \in \mathbb{C}$ such that the following conditions are satisfied for any $i \in I$:

- the map $s \mapsto \mu_{i,s}$ is affine;
- the map $s \mapsto \varphi_{i,s}$ is a meromorphic function;
- we have $\varphi_{i,s}(ah) = \varphi_{i,s}(h)$ for any $h \in G'(\mathbb{A})$ and $a \in A_{M'}^\infty$;
- we have for any $a \in A_{M'}^\infty$, $m \in M'(\mathbb{A})^1$ and $k \in K'$,

$$\int_{[N_{P'}]} E(namk, \tilde{\Phi}(\omega, s))dn = \sum_{i\in I} \varphi_{i,s}(mk)\exp(\langle \mu_{i,s}, H_P(a)\rangle). \tag{5.3.1.1}$$

5.3.2.
We shall simply say that a map $\beta : \mathfrak{a}_{P,\mathbb{C}}^{G,*} \to \mathbb{C}$ is a Paley-Wiener function if it is given by the Fourier-Laplace transform of a compactly supported smooth function on $\mathfrak{a}_P^{G,*}$.

5.3.3.
In the following, we consider the following objects:

- β a Paley-Wiener function;
- $f \in C_c^\infty(G(F_\infty))$ a decomposable function;
- $\kappa \in \mathfrak{a}_P^*$;
- $\varphi \in \mathcal{A}_{P,\pi}(G)$.

From these, one defines the pseudo-Eisenstein series

$$\theta(g) = \sum_{\delta\in P(F)\backslash G(F)} B(\delta g),$$

where B is the function on $A_G^\infty N(\mathbb{A})M(F)\backslash G(\mathbb{A})$ given by

$$B(g) = \int_{\kappa+i\mathfrak{a}_P^{G,*}} \exp(\langle\lambda, H_P(g)\rangle)(I_P(\lambda, f)\varphi)(g)\beta(\lambda)\, d\lambda.$$

In the following, we shall assume that κ is in the region of convergence of the Eisenstein series $E(\varphi,\lambda)$. Then we have

$$\theta(g) = \int_{\kappa+i\mathfrak{a}_P^{G,*}} E(g, I_P(\lambda, f)\varphi, \lambda)\beta(\lambda)\, d\lambda$$

for any $g \in G(\mathbb{A})$.

We fix $r > 2$ that satisfies the conditions of Proposition 5.2.2.1. In the following, we assume moreover that κ is such that $\langle\kappa, \alpha^\vee\rangle > r$.

5.3.4.

Proposition 5.3.4.1. *Let $s \in \mathbb{C}\setminus\{0,1\}$. Assume that β vanishes at the points $-\mu_{i,s}$ for $i \in I$. Then we have*

$$\int_{[G']_0} \theta(h)E(h,\Phi,s)\, dh = \sum_{\xi \in {}_PW_{P,2},\xi\neq 1} \int_{\kappa+i\mathfrak{a}_P^{G,*}} J(\xi, I_P(\lambda, f)\varphi, \lambda, \Phi, s)\beta(\lambda)\, d\lambda, \tag{5.3.4.1}$$

where both sides are absolutely convergent.

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Proof. Because $\theta(h)$ is rapidly decreasing, the left-hand side is absolutely convergent. Using the majorizations (5.2.2.2) and (5.2.2.3) given in the proof of Proposition 5.2.2.1, we see that the right-hand side of (5.3.4.1) is also absolutely convergent.

By the Lemma 5.1.2.1, the left-hand side of (5.3.4.1) is given by

$$\sum_{\xi \in {}_P W_{P,2}} \int_{A_{G'}^\infty P_{\tilde\xi}(F)\backslash G'(\mathbb{A})} B(\tilde\xi h) E(h,\Phi,s)\, dh, \tag{5.3.4.2}$$

where $\tilde\xi \in G(F)$ is any element such that $\xi = \tilde\xi c(\tilde\xi)^{-1}$.

Assume $\xi \neq 1$. Using the definition of B and permuting the adelic and the complex integrals, we get that

$$\int_{A_{G'}^\infty P_{\tilde\xi}(F)\backslash G'(\mathbb{A})} B(\tilde\xi h) E(h,\Phi,s)\, dh = \int_{\kappa + i\mathfrak{a}_P^{G,*}} J(\xi, I_P(\lambda,f)\varphi,\lambda,\Phi,s)\beta(\lambda)\, d\lambda.$$

This permutation is easily justified by the majorization (5.2.2.3) and the fact that β is a Paley-Wiener function. We have to compute the term corresponding to $\xi = 1$ (for which we take $\tilde\xi = 1$), namely

$$\int_{A_{G'}^\infty P'(F)\backslash G'(\mathbb{A})} B(h) E(h,\Phi,s)\, dh.$$

We will show that this integral vanishes. Using Iwasawa decomposition, we can write it as follows:

$$\int_{K'} \int_{M'(F)Z_{G'}(\mathbb{A})^1\backslash M'(\mathbb{A})^1} \int_{A_{G'}^\infty\backslash A_{M'}^\infty} \exp(\langle -\rho_P, H_P(a)\rangle) B(amk) \int_{[N_{P'}]} E(namk, \tilde\Phi(\omega,s))\, dn dm dk. \tag{5.3.4.3}$$

According to the shape of the constant term in (5.3.1.1), we are reduced to fix $i \in I$, $m \in M'(\mathbb{A})^1$, $k \in K'$ and to show the vanishing of

$$\int_{A_{G'}^\infty\backslash A_{M'}^\infty} \exp(\langle \mu_{i,s}, H_P(a)\rangle) \int_{\kappa + i\mathfrak{a}_P^{G,*}} (I_P(\lambda,f)\varphi)(mk)\beta(\lambda)\exp(\langle \lambda, H_P(a)\rangle)\, d\lambda da$$

for any $m \in M'(\mathbb{A})^1$ and $k \in K'$. By Fourier inversion, this is, up to a constant,

$$(I_P(-\mu_{i,s}, f)\varphi)(mk)\beta(-\mu_{i,s}),$$

and we are done. □

5.4. Period of a pseudo-Eisenstein series: computation in terms of the Whittaker functional

5.4.1.

We fix a nontrivial character $\psi' : \mathbb{A}/F \to \mathbb{C}^\times$. We define $\psi : [N_{2n}] \to \mathbb{C}^\times$ by

$$\psi(u) = \psi'(\mathrm{Tr}_{E/F}(\tau \sum_{i=1}^{2n-1} u_{i,i+1}))$$

for $u = (u_{i,j}) \in N_{2n}(\mathbb{A})$. Note that ψ is trivial on $N'_{2n}(\mathbb{A})$, where $N'_{2n} = N_{2n} \cap G'$.

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5.4.2. A variant of mirabolic subgroups

For all $1 \leqslant i \leqslant 2n$, we define the following subgroup of G_{2n}:

$$\mathcal{P}_i = \{\begin{pmatrix} g & * \\ 0 & u \end{pmatrix} \mid g \in G_{2n-i}, u \in N_i\}.$$

Note that $\mathcal{P}_1$ is the mirabolic subgroup $\mathcal{P}$ defined in §5.2.1 and $\mathcal{P}_{2n} = B_{2n}$. We denote by $N_{\mathcal{P}_i}$ the unipotent radical of $\mathcal{P}_i$. Let P_i be the standard parabolic subgroup of G of type $(2n-i, i)$. We have $\mathcal{P}_i \subset P_i$ and P_n is the parabolic subgroup P defined in §5.1.1. We denote by an upper script $'$ the subgroups obtained by intersection with G' that is $\mathcal{P}_i' = \mathcal{P}_i \cap G'$.

For any smooth function ϕ on $\mathcal{P}_i(F)\backslash G(\mathbb{A})$, we put for $g \in G(\mathbb{A})$,

$$W_i(g, \phi) = \int_{[N_{\mathcal{P}_i}]} \phi(ug)\psi(u)^{-1}\, du.$$

When $i = 2n$, we have $N_{\mathcal{P}_i} = N_{2n}$ and we omit the subscript: we set $W = W_{2n}$.

5.4.3.

For any $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$ such that $\Re(\langle \lambda, \alpha^\vee \rangle)$ is large enough, we define

$$\mathbb{W}(g, \varphi, \lambda) = \int_{(N_{2n} \cap M)(F)\backslash N_{2n}(\mathbb{A})} \exp(\langle \lambda, H_P(\xi_0 ug)\rangle)\varphi(\xi_0 ug)\psi(u)^{-1}\, du.$$

One has in the convergence region

$$\mathbb{W}(g, \varphi, \lambda) = W(g, E(\varphi, \lambda)), \tag{5.4.3.1}$$

and so the integral has a meromorphic continuation to $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$. It factors through the Fourier coefficient

$$W_M^\psi(g, \varphi) = \int_{[N_{2n} \cap M]} \varphi(ug)\psi(u)^{-1}\, du.$$

5.4.4.

We view $W_M^\psi(1, \cdot)$ as a Whittaker functional on π. Let $\pi = \otimes'_{v \in V_F} \pi_v$ be a decomposition of π as a restricted tensor product of representations π_v of $M(F_v)$. According to this decomposition, we fix a Whittaker functionnal $W_{M,v}^\psi$ on π_v such that $W_M^\psi(1, \cdot) = \otimes_{v \in V_F} W_{M,v}^\psi$. Let $V_{F,\infty} \subset S \subset V_F$ be a finite set such that all objects π, Φ, ψ, E/F are unramified outside S. We put $W_{M,S}^\psi = \otimes_{v \in S} W_{M,v}^\psi$. For any $\lambda \in i\mathfrak{a}_{P,\mathbb{C}}^{G,*}$, let $\pi_{v,\lambda}$ be the representation of $M(F_v)$ given by $\pi_{v,\lambda}(m) = \exp(\langle \lambda, H_M(m)\rangle)\pi_v(m)$ for $m \in M(F_v)$. Let $\Pi_{v,\lambda} = \mathrm{Ind}_{P(F_v)}^{G(F_v)}(\pi_{v,\lambda})$ and $\Pi_v = \Pi_{v,\lambda=0}$. We put $\Pi_S = \otimes_{v \in S} \Pi_v$.

For any $g_S \in G(F_S)$ and $\phi \in \Pi_S$, we define the Jacquet integral by the analytic continuation of

$$\mathbb{W}_S(g_S, \phi, \lambda) = \int_{(N_{2n} \cap M)(F_S)\backslash N_{2n}(F_S)} W_{M,S}^\psi(\Pi_\lambda(\xi_0 u g_S)\phi)\, du. \tag{5.4.4.1}$$

For any $v \notin S$, let $W_v^\psi(\cdot, \pi_\lambda)$ be the K_v-invariant Whittaker function such that $W_v^\psi(1, \pi_\lambda) = 1$. We shall identify φ with $\varphi_S \otimes \varphi^S$, where $\varphi_S \in \Pi_S$ and $\varphi^S \in \otimes'_{v \notin S} \Pi_v$ is K^S-invariant. Then we have

$$\mathbb{W}(g, I_P(\lambda, f)\varphi, \lambda) = \frac{1}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma \times \sigma^c)} \mathbb{W}_S(g_S, I_P(\lambda, f)\varphi_S, \lambda) \prod_{v \notin S} W_v^\psi(g_v, \pi_\lambda) \tag{5.4.4.2}$$

for all $g = (g_S, (g_v)_{v \notin S}) \in G(\mathbb{A})$.

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Proposition 5.4.4.1. *There is $s_0 \in \mathbb{R}$ such that for any $\lambda \in \kappa + i\mathfrak{a}_P^{G,*}$ and any $s \in \mathcal{H}_{>s_0}$, the integral*

$$\int_{N'_{2n}(\mathbb{A})\backslash G'(\mathbb{A})} |\mathbb{W}(h, I_P(\lambda, f)\varphi, \lambda)\Phi(e_{2n}h)| \det(h)|^s| \, dh$$

is absolutely convergent and uniformly bounded on compact subsets of $\mathcal{H}_{>s_0}$.

Proof. We shall use (5.4.4.2). First, the factor $L^S(1 + \langle\lambda, \alpha^\vee\rangle, \sigma \times \sigma^c)^{-1}$ is bounded uniformy for λ such $\langle\Re(\lambda), \alpha^\vee\rangle \geqslant 2$. We have $\Phi = \Phi_S \otimes \Phi^S$, where Φ^S is the characteristic function of $(\mathcal{O}^S)^{2n}$. By (the proof of) [BP18, proposition 2.6.1 and lemma 3.3.1], there exists $s_0 \in \mathbb{R}$ such that for any $\lambda \in \kappa + i\mathfrak{a}_P^{G,*}$ and any $s \in \mathcal{H}_{>s_0}$, the integral

$$\int_{N'_{2n}(F_S)\backslash G'(F_S)} |\mathbb{W}_S(h_S, I_P(\lambda, f)\varphi_S, \lambda)\Phi_S(e_{2n}h_S)| \det(h_S)|^s| \, dh_S$$

is convergent and uniformly bounded on compact subsets of $\mathcal{H}_{>s_0}$. So we are left with

$$\prod_{v \notin S} \int_{N'_{2n}(F_v)\backslash G'(F_v)} |W_v^\psi(h_v, \pi_\lambda)|\mathbf{1}_{\mathcal{O}_v^{2n}}(e_{2n}h_v)| \det(h_v)|_v^s| \, dh_v.$$

Then we can use the Iwasawa decomposition and the bound of [JPSS79, proposition 2.4.1] since we may assume that S is large enough so that for $v \notin S$ the cardinality of the residue field of F_v is bigger than n. The details are left to the reader (see also [IY15, proof of lemma 4.5]). □

5.4.5.

We consider the situation of §5.3.3.

Proposition 5.4.5.1. *There is $s_0 \in \mathbb{R}$ such that for any $s \in \mathcal{H}_{>s_0}$ and any Paley-Wiener function β that vanishes at the points $\pm s\alpha/2$, we have*

$$\int_{[G']_0} \theta(h)E(h, \Phi, s) \, dh = \int_{\kappa + i\mathfrak{a}_P^{G,*}} \left(\int_{N'_{2n}(\mathbb{A})\backslash G'(\mathbb{A})} \mathbb{W}(h, I_P(\lambda, f)\varphi, \lambda)\Phi(e_{2n}h)| \det(h)|^s \, dh \right) \beta(\lambda) \, d\lambda, \tag{5.4.5.1}$$

where the two sides are given by absolutely convergent integrals.

Before giving the proof (which is to find in §§5.4.6-5.4.8 below), we shall give a corollary.

Corollary 5.4.5.2. *There is $s_0 \in \mathbb{R}$ such that for any $\lambda \in \kappa + i\mathfrak{a}_P^{G,*}$ and $s \in \mathcal{H}_{>s_0}$, we have*

$$\sum_{\xi \in {}_P W_{P,2}, \xi \neq 1} J(\xi, \varphi, \lambda, \Phi, s) = \int_{N'_{2n}(\mathbb{A})\backslash G'(\mathbb{A})} \mathbb{W}(h, \varphi, \lambda)\Phi(e_{2n}h)| \det(h)|^s \, dh,$$

where the right-hand side is given by an absolutely and uniformly convergent integral on compact subsets of $\mathcal{H}_{>s_0}$.

Proof. The combination of Propositions 5.3.4.1 and 5.4.5.1 implies (see [LR03, lemma 9.1.2] for a simple argument)

$$\sum_{\xi \in {}_P W_{P,2}, \xi \neq 1} J(\xi, I_P(\lambda, f)\varphi, \lambda, \Phi, s) = \int_{N'_{2n}(\mathbb{A})\backslash G'(\mathbb{A})} \mathbb{W}(h, I_P(\lambda, f)\varphi, \lambda)\Phi(e_{2n}h)| \det(h)|^s \, dh$$

for any $\lambda \in \kappa + i\mathfrak{a}_P^{G,*}$. Since we can find f such that $I_P(\lambda, f)\varphi = \varphi$, we get the result. □

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5.4.6. Proof of Proposition 5.4.5.1

We first recall the well-known computation of the constant term of a pseudo Eisenstein series.

Lemma 5.4.6.1. *For $1 \leqslant i \leqslant 2n$, the constant term of θ along P_i, defined by*

$$\forall g \in G(\mathbb{A}) \quad \theta_{P_i}(g) = \int_{[N_{P_i}]} \theta(ng)\, dn,$$

vanishes unless $i \in \{n, 2n\}$. Moreover, we have

$$\theta_P(g) = \sum_{w \in W(M)} \int_{\kappa + i\mathfrak{a}_P^{G,*}} \exp(w\lambda, H_P(g)\rangle (M(w,\lambda) I_P(\lambda, f)\varphi)(g)\beta(\lambda)\, d\lambda.$$

Unfolding the Epstein series, we get

$$\begin{aligned}\int_{[G']_0} \theta(h) E(h, \Phi, s)\, dh &= \int_{\mathcal{P}'_1(F)\backslash G'(\mathbb{A})} \theta(h)\Phi(e_{2n}h)|\det(h)|^s\, dh \\ &= \int_{\mathcal{P}'_1(F) N'_{\mathcal{P}_1}(\mathbb{A})\backslash G'(\mathbb{A})} \left(\int_{[N'_{\mathcal{P}_1}]} \theta(nh)\, dn\right) \Phi(e_{2n}h)|\det(h)|^s\, dh.\end{aligned}$$

The Fourier expansion of the map $n \in [N_{\mathcal{P}_1}] \mapsto \theta(nh)$ gives

$$\theta(h) = \theta_{P_1}(h) + \sum_{\gamma \in \mathcal{P}_2(F)\backslash \mathcal{P}_1(F)} W_1(\gamma h, \theta)$$

from which we deduce

$$\int_{[N'_{\mathcal{P}_1}]} \theta(nh)\, dn = \theta_{P_1}(h) + \sum_{\gamma \in \mathcal{P}'_2(F)\backslash \mathcal{P}'_1(F)} W_1(\gamma h, \theta).$$

If $n > 1$, then $\theta_{P_1} = 0$ (see Lemma 5.4.6.1). In particular, we get

$$\begin{aligned}\int_{[G']_0} \theta(h) E(h, \Phi, s)\, dh &= \int_{\mathcal{P}'_2(F) N'_{\mathcal{P}_1}(\mathbb{A})\backslash G'(\mathbb{A})} W_1(h, \theta)\Phi(e_{2n}h)|\det(h)|^s\, dh \\ &= \int_{\mathcal{P}'_2(F) N'_{\mathcal{P}_2}(\mathbb{A})\backslash G'(\mathbb{A})} \left(\int_{[N'_{\mathcal{P}_1}\backslash N'_{\mathcal{P}_2}]} W_1(nh, \theta)\, dn\right) \Phi(e_{2n}h)|\det(h)|^s\, dh.\end{aligned}$$

Next, using the Fourier expansion of $n \in [N_{\mathcal{P}_2} \cap M_{P_1}] \mapsto W_1(nh, \theta)$, we get

$$\int_{[N'_{\mathcal{P}_1}\backslash N'_{\mathcal{P}_2}]} W_1(nh, \theta)\, dn = W_1(h, \theta_{P_2}) + \sum_{\gamma \in \mathcal{P}'_3(F)\backslash \mathcal{P}'_2(F)} W_2(\gamma h, \theta).$$

If $n > 2$, we have $\theta_{P_2} = 0$ (see Lemma 5.4.6.1). By recursion, we get that the left-hand side of (5.4.5.1) is the sum of

$$\int_{\mathcal{P}'_n(F) N'_{\mathcal{P}_n}(\mathbb{A})\backslash G'(\mathbb{A})} W_{n-1}(h, \theta_{P_n})\Phi(e_{2n}h)|\det(h)|^s\, dh \tag{5.4.6.1}$$

and

$$\int_{\mathcal{P}'_{n+1}(F) N'_{\mathcal{P}_n}(\mathbb{A})\backslash G'(\mathbb{A})} W_n(h, \theta)\Phi(e_{2n}h)|\det(h)|^s\, dh. \tag{5.4.6.2}$$

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The manipulation is justified as in [IY15, corollary 4.3 and bottom of p. 697]. The next step is to compute both expressions. Let us start with the second.

5.4.7. Computation of (5.4.6.2)
We can continue the process. Since $\theta_{P_k} = 0$ for $k > n$, $W = W_{2n}$ and $\mathcal{P}'_{2n}(F)N'_{\mathcal{P}_{2n}}(\mathbb{A}) = N'_{2n}(\mathbb{A})$, we get that (5.4.6.2) is equal to

$$\int_{N_{2n}(\mathbb{A})\backslash G'(\mathbb{A})} W(h,\theta)\Phi(e_{2n}h)|\det(h)|^s\, dh.$$

Using (5.4.3.1), we also have

$$W_{2n}(h,\theta) = \int_{\kappa+i\mathfrak{a}_P^{G,*}} \mathbb{W}(h, I_P(\lambda, f)\varphi, \lambda)\beta(\lambda)\, d\lambda.$$

To get the right-hand side of (5.4.5.1),we just need to permute the adelic integral and the integral over λ. This is justified by Proposition 5.4.4.1. To conclude, it suffices to show that (5.4.6.1) vanishes: this is done in the next §.

5.4.8. Vanishing of (5.4.6.1)
Recall that $P_n = P$. Using the Iwasawa decomposition, we get that the expression (5.4.6.1) is equal to

$$\int_{A^\infty_{M'}} \exp(-\langle\rho_P, H_P(a)\rangle) \int_{M'(\mathbb{A})\cap(\mathcal{P}'_n(F)N'_{\mathcal{P}_n}(\mathbb{A}))\backslash M'(\mathbb{A})} \int_{K'} W_{n-1}(amk,\theta_P)\Phi(e_{2n}ak)|\det(a)|^s\, dh. \tag{5.4.8.1}$$

By Lemma 5.4.6.1, we see that the expression $W_{n-1}(amk,\theta_P)$ is the sum over $w \in W(M)$ of

$$\int_{\kappa+i\mathfrak{a}_P^{G,*}} \exp(\langle w\lambda, H_P(a)\rangle) \int_{[N_{\mathcal{P}_{n-1}}]} (M(w,\lambda)I_P(\lambda, f)\varphi)(uamk)\psi(u)^{-1}\, du\beta(\lambda)d\lambda.$$

Writing $u = u_P u_M$ with $u_M \in (N_{\mathcal{P}_{n-1}} \cap M)$ and $u_P \in (N_{\mathcal{P}_{n-1}} \cap N_P)$, we see that

$$(M(w,\lambda)I_P(\lambda, f)\varphi)(uamk) = \exp(\langle\rho_P, H_P(a)\rangle)(M(w,\lambda)I_P(\lambda, f)\varphi)(u_M mk).$$

Let $A^\infty_{\mathcal{P}'_n}$ be the stabilizer of e_{2n} in $A^\infty_{M'}$. We have $|\det(a)|^s = \exp(s\langle\alpha, H_P(a)\rangle/2)$. The contribution in (5.4.8.1) corresponding to $w \in W(M)$ factors through the integral:

$$\int_{A^\infty_{\mathcal{P}'_n}} \exp(s\langle\alpha, H_P(a)\rangle/2) \int_{\kappa+i\mathfrak{a}_P^{G,*}} (M(w,\lambda)I_P(\lambda, f)\varphi)(umk)\exp(\langle w\lambda, H_P(a)\rangle)\beta(\lambda)\, d\lambda da \tag{5.4.8.2}$$

for some Haar measure on $A^\infty_{\mathcal{P}'}$. If $w = 1$, the expression (5.4.8.2) is simply

$$\int_{A^\infty_{\mathcal{P}'_n}} \exp(s\langle\alpha, H_P(a)\rangle/2) \int_{\kappa+i\mathfrak{a}_P^{G,*}} (I_P(\lambda, f)\varphi)(umk)\exp(\langle\lambda, H_P(a)\rangle)\beta(\lambda)\, d\lambda da.$$

By Fourier inversion, it is, up to a constant, $(I_P(-s\alpha/2, f)\varphi)(umk)\beta(-s\alpha/2)$ and thus vanishes.

If $w \neq 1$, then $w\lambda = -\lambda$ and the expression (5.4.8.2) can be written as

$$\int_{A^\infty_{\mathcal{P}'_n}} \exp(s\langle\alpha, H_P(a)\rangle/2) \int_{\kappa+i\mathfrak{a}_P^{G,*}} (M(w,\lambda)I_P(\lambda, f)\varphi)(umk)\exp(-\langle\lambda, H_P(a)\rangle)\beta(\lambda)\, d\lambda da.$$

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Assume that $\Re(s)$ is large enough so that $\Re(s)\langle\alpha,\alpha^\vee\rangle > 2\langle\kappa,\alpha^\vee\rangle$. One can check that there exists c such that the inner integral vanishes unless $\langle\alpha, H_P(a)\rangle \leqslant c$. Thus, one can restrict the outer integral to this 'half-line'. Then we can permute the two integrals. By Cauchy formula, we get that it is, up to a constant, $(M(w,s\alpha/2)I_P(s\alpha/2,f)\varphi)(umk)\beta(s\alpha/2)$, and thus, it also vanishes. This concludes the proof of Proposition 5.4.5.1.

5.5. *Final result*

5.5.1.
We keep the notations of previous sections. Let $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$. For $s \in \mathbb{C}$, let $L^S(s,\Pi_\lambda,\mathrm{As})$ be the Asai L-function 'outside S'. We have

$$L^S(s,\Pi_\lambda,\mathrm{As}) = L^S(s+\langle\lambda,\alpha^\vee\rangle,\sigma,\mathrm{As})L^S(s-\langle\lambda,\alpha^\vee\rangle,\sigma^*,\mathrm{As})L^S(s,\sigma\times\sigma^\vee).$$

The factor $L^S(s,\sigma\times\sigma^\vee)$ has a simple pole at $s=1$ and we set

$$L^{S,*}(1,\sigma\times\sigma^\vee) = \lim_{s\to 1}(s-1)L^S(s,\sigma\times\sigma^\vee).$$

Then we get an analytic function in λ by setting

$$L^{S,*}(1,\Pi_\lambda,\mathrm{As}) = L^S(1+\langle\lambda,\alpha^\vee\rangle,\sigma,\mathrm{As})L^S(1-\langle\lambda,\alpha^\vee\rangle,\sigma^*,\mathrm{As})L^{S,*}(1,\sigma\times\sigma^\vee).$$

Theorem 5.5.1.1. *We have the following equality of meromorphic functions on* $i\mathfrak{a}_{P,\mathbb{C}}^{G,*}$*:*

$$J(\xi_0,\varphi,\lambda) = (\Delta_{G'}^{S,*})^{-1}L^{S,*}(1,\Pi_\lambda,\mathrm{As})\int_{N'_{2n}(F_S)\backslash\mathcal{P}'(F_S)} \mathbb{W}(p_S,\varphi,\lambda)\,dp_S,$$

where the integral is absolutely convergent, and the left-hand side and the integrand in the right-hand side are respectively defined in (5.1.3.1) and (5.4.3.1). Moreover, both sides of the equality are holomorphic at $\lambda \in i\mathfrak{a}_P^{G,*}$ *in the following cases:*

- σ *is not* G'_n*-distinguished;*
- σ *is* G'_n*-distinguished and* $\lambda \neq 0$.

5.5.2. Proof of Theorem 5.5.1.1
By (5.4.4.2), we have for all $g_S \in G(F_S)$,

$$\mathbb{W}(g_S,\varphi,\lambda) = \frac{1}{L^S(1+\langle\lambda,\alpha^\vee\rangle,\sigma\times\sigma^c)}\mathbb{W}_S(g_S,\varphi_S,\lambda).$$

Let Φ_S a test function in the Schwartz space $\mathcal{S}(F_S^{2n})$. Let us consider the following integrals:

$$\int_{N'_{2n}(F_S)\backslash\mathcal{P}'(F_S)} \mathbb{W}_S(h_S,\varphi_S,\lambda)\,dh_S \tag{5.5.2.1}$$

and

$$\int_{N'_{2n}(F_S)\backslash G'(F_S)} \mathbb{W}_S(h_S,\varphi_S,\lambda)\Phi_S(e_{2n}h_S)|\det(h_S)|^s\,dh_S. \tag{5.5.2.2}$$

Lemma 5.5.2.1.

1. *There exists* $\eta > 0$ *(resp. and* $\varepsilon > 0$*) such that the integral (5.5.2.1), resp. (5.5.2.2), is absolutely convergent and holomorphic on the subset of* $\lambda \in \mathfrak{a}_P^{G,*}$ *such that* $|\langle\Re(\lambda),\alpha^\vee\rangle| < \eta$*, resp. and* $s \in \mathcal{H}_{1-\varepsilon}$.

2. *The integral (5.5.2.2) admits a meromorphic continuation to* $\mathbb{C} \times \mathfrak{a}_{P,\mathbb{C}}^{G,*}$ *denoted by* $Z_S(s, \lambda, \Phi_S)$.
3. *For any* $c > 0$, *there exists* $s_1 \in \mathbb{R}$ *such that for* $s \in \mathcal{H}_{s_1}$ *and* $|\langle \Re(\lambda), \alpha^\vee \rangle| < c$, *the integral (5.5.2.2) is absolutely convergent and coincides with* $Z_S(s, \lambda, \Phi_S)$.

Proof. All the results are slight variations on [BP18, lemma 3.3.1, lemma 3.3.2 and section 3.10]. To get the convergence of (5.5.2.1) or the precise lower bound $1 - \varepsilon$, we need to observe that φ_S belongs to the induced representation of the S-component of an irreducible automorphic cuspidal representation of $M(\mathbb{A})$. As such, it is an irreducible, generic and unitary representation. We can now use the classification of local irreductible generic and unitary representations as fully induced from essentially discrete series with exponents in $]-1/2; 1/2[$ (see, for example, [Tad09], [Zel80] and [BR10, Theorem 8.2.]). □

Let $\Phi = \Phi_S \otimes \Phi^S$, where Φ_S is the Schwartz space $\mathcal{S}(F_S^{2n})$ and Φ^S is the characteristic function of $(\mathcal{O}^S)^{2n}$. Let $c > \langle \kappa, \alpha^\vee \rangle$. By Lemma 5.5.2.1 assertion 3, there exists $s_1 \in \mathbb{R}$ such that for $s \in \mathcal{H}_{s_1}$ and $|\langle \Re(\lambda), \alpha^\vee \rangle| < c$, the integral (5.5.2.2) is absolutely convergent. We may and shall assume that s_1 is large enough so that Proposition 5.4.4.1 holds for s_1. Then by the factorization (5.4.4.2), some local computations [Fli88] and Lemma 5.5.2.1 assertion 2, we have

$$\int_{N'_{2n}(\mathbb{A})\backslash G'(\mathbb{A})} \mathbb{W}(h, \varphi, \lambda)\Phi(e_{2n}h)|\det(h)|^s\, dh = (\Delta_{G'}^{S,*})^{-1} \frac{L^S(s, \Pi_\lambda, \mathrm{As})}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma \times \sigma^c)} Z_S(s, \lambda, \Phi_S)$$

for any $\lambda \in \kappa + i\mathfrak{a}_P^{G,*}$ and $s \in \mathcal{H}_{>s_1}$. By analytic continuation of the equality of Corollary 5.4.5.2, we have

$$\sum_{\xi \in {}_PW_{P,2}, \xi \neq 1} J(\xi, \varphi, \lambda, \Phi, s) = (\Delta_{G'}^{S,*})^{-1} \frac{L^S(s, \Pi_\lambda, \mathrm{As})}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma \times \sigma^c)} Z_S(s, \lambda, \Phi_S)$$

for all $s \in \mathbb{C}$ and $\lambda \in \mathcal{D}_r$ given by Proposition 5.2.2.1. We can compute the residue of the left-hand side at $s = 1$ following Proposition 5.2.2.1. We get

$$\hat{\Phi}(0)J(\xi_0, \varphi, \lambda) = (\Delta_{G'}^{S,*})^{-1} Z_S(1, \lambda, \Phi_S) \frac{L^{S,*}(1, \Pi_\lambda, \mathrm{As})}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma \times \sigma^c)}. \tag{5.5.2.3}$$

Both sides are analytic in λ. Thus, the equality for $\lambda \in i\mathfrak{a}_P^{G,*}$. But by Lemma 5.5.2.1 assertion 1 and assertion 2, one has

$$Z_S(1, \lambda, \Phi_S) = \int_{N'_{2n}(F_S)\backslash G'(F_S)} \mathbb{W}_S(h_S, \varphi_S, \lambda)\Phi_S(e_{2n}h_S)|\det(h_S)|\, dh_S.$$

Let $\bar{N}_1$ be the unipotent radical of the opposite of the parabolic subgroup P_1 defined in §5.4.2. The standard Levi factor of P_1 decomposes as $G_{2n-1} \times G_1$. Thus, we have $N'_{2n}\backslash P'_1 \simeq N'_{2n}\backslash \mathcal{P}' \times G'_1$. By a usual decomposition of measures, we get

$$Z_S(1, \lambda, \Phi_S) = \int_{\bar{N}_1(F_S)} \int_{G'_1(F_S)} \left(\int_{N'_{2n}(F_S)\backslash \mathcal{P}'(F_S)} \mathbb{W}_S(htn, \varphi_S, \lambda)\, dh \right) \Phi_S(e_{2n}tn)|t|^{2n}\, dtdn.$$

However, we have also

$$\begin{aligned} \hat{\Phi}_S(0) &= \int_{F_S^{2n}} \Phi_S(X)\, dX \\ &= \int_{\bar{N}_1(F_S)} \int_{G'_1(F_S)} \Phi_S(e_{2n}tn)|t|^{2n}\, dtdn. \end{aligned}$$

Since (5.5.2.3) holds for any Schwartz function Φ_S, we get that $J(\xi_0, \varphi, \lambda)$ is equal to

$$\frac{L^{S,*}(1, \Pi_\lambda, \mathrm{As})}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma \times \sigma^c)} \int_{N'_{2n}(F_S)\backslash \mathcal{P}'(F_S)} \mathbb{W}_S(g_S, \varphi_S, \lambda)$$
$$= L^{S,*}(1, \Pi_\lambda, \mathrm{As}) \int_{N'_{2n}(F_S)\backslash \mathcal{P}'(F_S)} \mathbb{W}(p_S, \varphi, \lambda)\, dp_S,$$

where we have used the factorization (5.4.4.2). In the first expression above, the integral is holomorphic on $i\mathfrak{a}_P^{G,*}$; see Lemma 5.5.2.1. Using the factorization of $L^S(s, \sigma \times \sigma^c)$ in terms of Asai L-functions $L(s, \sigma, As^{\pm})$, we get

$$\frac{L^{S,*}(1, \Pi_\lambda, \mathrm{As})}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma \times \sigma^c)} = L^{S,*}(1, \sigma \times \sigma^\vee) \frac{L^S(1 - \langle \lambda, \alpha^\vee \rangle, \sigma^*, \mathrm{As})}{L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma, \mathrm{As}^-)}.$$

On $i\mathfrak{a}_P^{G,*}$, the L-function $L^S(1 + \langle \lambda, \alpha^\vee \rangle, \sigma, \mathrm{As}^-)$ does not vanish by [Sha81, theorem 5.1] and $L^S(1 - \langle \lambda, \alpha^\vee \rangle, \sigma^*, \mathrm{As})$ is holomorphic unless $\lambda = 0$ and σ^* (thus σ) is G'_n-distinguished (see [Fli88]). On the other hand, if σ is not G'_n-distinguished, then $J(\xi_0, \varphi, \lambda)$ is known to be holomorphic on $i\mathfrak{a}_P^{G,*}$; see [Lap06, lemma 8.1]. Otherwise, $J(\xi_0, \varphi, \lambda)$ is holomorphic on $i\mathfrak{a}_P^{G,*} \setminus \{0\}$, but it may have a simple pole at $\lambda = 0$.

6. The (G, H)-regular contribution to the Jacquet-Rallis trace formula: alternative proof

6.1. Statement

6.1.1.

The goal of this section is to provide an alternative proof of the following combination of Theorem 4.1.8.1 and Theorem 4.2.8.1.

Theorem 6.1.1.1. *Let $\chi \in \mathfrak{X}(G)$ be a (G, H)-regular cuspidal datum and let $f \in \mathcal{S}(G(\mathbb{A}))$. Then,*

1. *If χ is not Hermitian, we have $I_\chi(f) = 0$.*
2. *If χ is Hermitian, we have*

$$I_\chi(f) = 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} I_{\Pi_\lambda}(f) d\lambda, \qquad (6.1.1.1)$$

where we recall that (M, π) is a pair representing χ, $L \supset M$ is the Levi subgroup defined by (4.1.4.1), Π_λ stands for the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi_\lambda)$, for P a chosen parabolic subgroup with Levi factor M, and I_{Π_λ} is the relative character defined by (4.2.8.1).

More precisely, the proof will be very similar to that given in [BPCZ22, Section 8] and is based on two ingredients of independent interests. The first one is that the Rankin-Selberg period (over H) admits a continuous extension to the space $\mathcal{T}_\chi([G])$ of functions of uniform moderate growth supported on a H-regular cuspidal datum χ and that this extension can moreover be described in terms of the analytic continuation of Zeta integrals of Rankin Selberg type. This was already established in [BPCZ22, Section 7]. The second ingredient is an explicit spectral decomposition of the Flicker-Rallis period (over G') restricted to $\mathcal{S}_\chi([G'])$ when the cuspidal datum χ is (G-)regular. This was already done in [BPCZ22, Section 6] under the stronger assumption that χ is $*$-regular. The aim of the subsection 6.2 is to state and prove the extension of this result to the regular case. Once established, we will be able to give a proof of Theorem 6.1.1.1 in subsection 6.3 in much the same lines as [BPCZ22, §8.2].

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6.2. Spectral decomposition of the Flicker-Rallis period for regular cuspidal data

6.2.1. Zeta integrals

Let $n \geqslant 0$ be an integer. We will freely use the notation introduced in Section 4. For every $f \in \mathcal{T}([G_n])$, we denote by

$$W_f(g) = \int_{[N_n]} f(ug)\psi_n(u)^{-1}du, \quad g \in G_n(\mathbb{A})$$

its Whittaker function and, for $\phi \in \mathcal{S}(\mathbb{A}^n)$, we set

$$Z_\psi^{\mathrm{FR}}(s, f, \phi) := \int_{N_n'(\mathbb{A})\backslash G_n'(\mathbb{A})} W_f(h)\phi(e_n h)|\det h|^s \eta_{G_n'}(h)dh.$$

This expression is absolutely convergent for $\Re(s)$ sufficiently large. More precisely, for every $N > 0$, there exists $c > 0$ such that $Z_\psi^{\mathrm{FR}}(s, f, \phi)$ converges for $s \in \mathcal{H}_{>c}$ (see [BPCZ22, Theorem 6.2.5.1]).

6.2.2. Flicker-Rallis period

Recall that for every $f \in \mathcal{C}([G_n])$, the period integral

$$P_{G_n'}(f) = \int_{[G_n']} f(h)\eta_{G_n'}(h)dh$$

is convergent [BPCZ22, Theorem 6.2.6.1].

6.2.3. Hermitian cuspidal data

Let $\chi \in \mathfrak{X}(G_n)$ be a cuspidal datum. Then, we can find a pair (M, π) representing χ with

$$M = \prod_{i\in I} G_{n_i}^{\times d_i} \times \prod_{j\in J} G_{n_j}^{\times d_j} \times \prod_{k\in K} G_{n_k}^{\times d_k}$$

and

$$\pi = \bigotimes_{i\in I} \pi_i^{\boxtimes d_i} \boxtimes \bigotimes_{j\in J} \pi_j^{\boxtimes d_j} \boxtimes \bigotimes_{k\in K} \pi_k^{\boxtimes d_k}$$

for some disjoint finite sets I, J, K, families of positive integers $(n_l)_{l\in I\cup J\cup K}$, $(d_l)_{l\in I\cup J\cup K}$ and a family of distinct cuspidal automorphic representations $(\pi_l)_{l\in I\cup J\cup K}$ satisfying

- For every $i \in I$, $\pi_i \not\simeq \pi_i^*$;
- For every $j \in J$, $\pi_j \simeq \pi_j^*$ and $L(s, \pi_j, \mathrm{As}^{(-1)^{n+1}})$ has no pole at $s = 1$;
- For every $k \in K$, $\pi_k \simeq \pi_k^*$ and $L(s, \pi_k, \mathrm{As}^{(-1)^{n+1}})$ has a pole at $s = 1$.

Fixing data as above (which are unique up to reordering), we recall that χ is said *Hermitian* (see §4.1.3) if the following condition is satisfied:

(6.2.3.1) For every $i \in I$, there exists $i^* \in I$ such that $\pi_{i^*} \simeq \pi_i^*$ and for every $j \in J$, d_j is even.

6.2.4.

Assume furthermore that χ is *regular* in the sense of §4.1.3 or [BPCZ22, §2.9.7] and fix a pair (M, π) representing χ together with data I, J, K, $(n_l)_{l\in I\cup J\cup K}$, $(d_l)_{l\in I\cup J\cup K}$, $(\pi_l)_{l\in I\cup J\cup K}$ as in the previous paragraph. By the regularity assumption, we have $d_l = 1$ for every $l \in I \cup J \cup K$. Moreover, χ is Hermitian if and only if $J = \emptyset$ and there exists an involution $i \mapsto i^*$ of I without fixed point such that $\pi_{i^*} = \pi_i^*$ for every $i \in I$. If this is the case, we choose a subset $I' \subset I$ such that I is the disjoint union of

I' and $(I')^* = \{i^* \mid i \in I'\}$, and we define a Levi subgroup $L \supset M$ by

$$L := \prod_{i \in I'} G_{n_i+n_{i^*}} \times \prod_{k \in K} G_{n_k}.$$

6.2.5.
Let P be a parabolic subgroup with Levi factor M. For every $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ and $f \in \mathcal{C}([G_n])$, we set

$$\Pi_\lambda := \mathrm{Ind}_{P(\mathbb{A})}^{G_n(\mathbb{A})}(\pi_\lambda)$$

and

$$W_{f,\Pi_\lambda} := W_{f_{\Pi_\lambda}},$$

where $f_{\Pi_\lambda} \in \mathcal{T}([G_n])$ is defined as in [BPCZ22, Eq. (2.9.8.14)].

Assuming that χ is Hermitian, we define for every $\lambda \in i\mathfrak{a}_M^{L,*}$ a linear form β_n on $\mathcal{W}(\Pi_\lambda, \psi_n)$ by setting

$$\beta_n(W) = (\Delta_{G_n'}^{S,*})^{-1} L^{S,*}(1, \Pi, \mathrm{As}^{(-1)^{n+1}}) \int_{N_n'(F_S)\backslash \mathcal{P}_n'(F_S)} W(p_S)\eta_{G_n'}(p_S)dp_S$$

for every $W \in \mathcal{W}(\Pi_\lambda, \psi_n)$, where S is a sufficiently large finite set of places of F (depending on W). That the above integral is convergent follows from [BP18, Proposition 2.6.1, Lemma 3.3.1], [JS81b], and moreover, the product stabilizes for S sufficiently large by the unramified computation of [Fli88, Proposition 3].

6.2.6.
Theorem 6.2.6.1. *Let $\chi \in \mathfrak{X}^{\mathrm{reg}}(G_n)$ for which we adopt the notation introduced in the previous three paragraphs. Then, for every $f \in \mathcal{C}_\chi([G_n])$ and $\phi \in \mathcal{S}(\mathbb{A}^n)$, we have*

1. *If χ is Hermitian, the function $\lambda \in i\mathfrak{a}_M^{L,*} \mapsto \beta_n(W_{f,\Pi_\lambda})$ is Schwartz and the resulting map*

$$\mathcal{C}_\chi([G_n]) \to \mathcal{S}(i\mathfrak{a}_M^{L,*}), \quad f \mapsto \left(\lambda \in i\mathfrak{a}_M^{L,*} \mapsto \beta_n(W_{f,\Pi_\lambda})\right) \tag{6.2.6.1}$$

is continuous.

2. *The function $s \mapsto (s-1)Z_\psi^{\mathrm{FR}}(s, {}^0 f, \phi)$ admits an analytic continuation to $\mathcal{H}_{>1}$ with a limit at $s = 1$, and setting $Z_\psi^{\mathrm{FR},*}(1, {}^0 f, \phi) := \lim_{s \to 1^+} (s-1)Z_\psi^{\mathrm{FR}}(s, {}^0 f, \phi)$, we have*

$$Z_\psi^{\mathrm{FR},*}(1, {}^0 f, \phi) = \begin{cases} 2^{1-\dim(\mathfrak{a}_L)}\widehat{\phi}(0) \int_{i\mathfrak{a}_M^{L,*}} \beta_n(W_{f,\Pi_\lambda})d\lambda \text{ if } \chi \text{ is Hermitian,} \\ 0 \text{ otherwise.} \end{cases} \tag{6.2.6.2}$$

3. *The equality*

$$\widehat{\phi}(0)P_{G_n'}(f) = \frac{1}{2} Z_\psi^{\mathrm{FR},*}(1, {}^0 f, \phi). \tag{6.2.6.3}$$

6.2.7.
We note the following immediate corollary of Theorem 6.2.6.1.

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Corollary 6.2.7.1. *Let $\chi \in \mathfrak{X}^{\mathrm{reg}}(G_n)$ for which we adopt the notation introduced in the paragraphs 6.2.4 and 6.2.5. Then, for every $f \in \mathcal{C}_\chi([G_n])$, we have*

$$P_{G'_n}(f) = \begin{cases} 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} \beta_n(W_{f,\Pi_\lambda})d\lambda \text{ if } \chi \text{ is Hermitian}, \\ 0 \text{ otherwise.} \end{cases} \quad (6.2.7.1)$$

6.2.8. Proof of Theorem 6.2.6.1 1. and 2.

Here, we prove parts 1 and 2 of Theorem 6.2.6.1. We will explain how to deduce the last part in the next subsection. The proof is actually along the same lines as that of [BPCZ22, Theorem 6.2.5.1]. Therefore, we will be brief on the parts that are similar.

Set

$$\mathcal{A} := (i\mathbb{R})^{I\cup J\cup K}$$

and let $\mathcal{A}_0$ be the subspace of vectors $\underline{x} = (x_\ell)_{\ell\in I\cup J\cup K} \in \mathcal{A}$ such that $\sum_{\ell\in I\cup J\cup K} x_\ell = 0$. We equip $\mathcal{A}$ with the product of Lebesgue measures and $\mathcal{A}_0$ with the unique measure inducing on $\mathcal{A}/\mathcal{A}_0 \simeq i\mathbb{R}$ the Lebesgue measure via the map $\underline{x} \mapsto \sum_{\ell\in I\cup J\cup K} x_\ell$. We identify $\mathcal{A}$ with $i\mathfrak{a}_M^*$ by sending $\underline{x} \in \mathcal{A}$ to the unramified character

$$g = (g_\ell)_{\ell\in I\cup J\cup K} \in M(\mathbb{A}) = \prod_{\ell\in I\cup J\cup K} G_{n_\ell}(\mathbb{A}) \mapsto \prod_{\ell\in I\cup J\cup K} |\det g_\ell|_{\mathbb{A}_E}^{x_\ell/n_\ell}.$$

We note that this isomorphism sends $\mathcal{A}_0$ onto $i\mathfrak{a}_M^{G_r,*}$. Furthermore, by our choice of measure on $i\mathfrak{a}_M^*$ (see §2.1.8 as well as [BPCZ22, Eq. (2.3.1.1)]), this isomorphism sends the Haar measure just described on $\mathcal{A}$ to $(2\pi)^r P$ times the Haar measure on $i\mathfrak{a}_M^*$, where we have set

$$r = \dim(A_M) = |I\cup J\cup K|, \quad P = \prod_{\ell\in I\cup J\cup K} n_\ell.$$

Set $f_{\underline{x}} = f_{\Pi_{\underline{x}}}$ for every $\underline{x} \in \mathcal{A}$. Then, by a computation completely similar to that leading to [BPCZ22, Eq. (6.3.0.3)] (using one of the main results of [Lap13]), we have

$$Z_\psi^{\mathrm{FR}}(s, {}^0f, \phi) = \frac{n}{P}(2\pi)^{-r+1}\int_{\mathcal{A}_0} Z_\psi^{\mathrm{FR}}(s, f_{\underline{x}}, \phi)d\underline{x}$$

for $\Re(s) \gg 1$.

Let S_0 be a finite set of places of F including the Archimedean ones and outside of which π is unramified and let $S_{0,f} \subset S_0$ be the subset of finite places. We fix, for every $l \in I\cup J\cup K$ and $v \in S_{0,f}$, polynomials $Q_l(T), Q_{l,v}(T) \in \mathbb{C}[T]$ with all their roots in $\mathcal{H}_{]0,1[}$ and $\mathcal{H}_{]q_v^{-1},1[}$, respectively, such that the products $s \mapsto Q_l(s)L_\infty(s,\pi_l,\mathrm{As}^{(-1)^{n+1}})$ and $s \mapsto Q_{l,v}(q_v^{-s})L_v(s,\pi_l,\mathrm{As}^{(-1)^{n+1}})$ have no pole in $\mathcal{H}_{]0,1[}$. Let I_0 be the set of $i \in I$ for which there exists $i^* \in I$ (necessarily unique by regularity of χ) satisfying $\pi_{i^*} = \pi_i^*$ and fix a subset $I' \subset I_0$ such that I_0 is the disjoint union of I' and $(I')^*$. Set

$$P(s,\underline{x}) = \prod_{i\in I'}\left(s + \frac{x_i + x_{i^*}}{n_i}\right)\left(s + \frac{x_i + x_{i^*}}{n_i} - 1\right)\prod_{k\in K}\left(s + \frac{2x_k}{n_k}\right)\left(s - 1 + \frac{2x_k}{n_k}\right)$$
$$\times \prod_{l\in I\cup J\cup K} Q_l\left(s + \frac{2x_l}{n_l}\right)\prod_{\substack{l\in I\cup J\cup K \\ v\in S_{0,f}}} Q_{l,v}\left(q_v^{-s-\frac{2x_l}{n_l}}\right).$$

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Then, we can prove exactly as in [BPCZ22, §6.3] that the two functions

$$(s,\underline{x}) \in \mathbb{C}\times\mathcal{A}_0 \mapsto P\left(s+\frac{1}{2},\underline{x}\right) Z^{\mathrm{FR}}_{\psi}\left(s+\frac{1}{2}, f_{\underline{x}}, \phi\right)$$

and

$$(s,\underline{x}) \in \mathbb{C}\times\mathcal{A}_0 \mapsto P\left(\frac{1}{2}-s,\underline{x}\right) Z^{\mathrm{FR}}_{\psi^{-1}}\left(s+\frac{1}{2}, \widetilde{f_{\underline{x}}}, \widehat{\phi}\right),$$

where $\widetilde{f_{\underline{x}}}(g) = f_{\underline{x}}({}^t g^{-1})$, satisfy the conditions of [BPCZ22, Corollary A.0.11.1]. Therefore, by the conclusion of this corollary, the map

$$s \mapsto F_s := \left(\underline{x} \mapsto \prod_{i\in I'}\left(s+\frac{x_i+x_{i^*}}{n_i}-1\right)\prod_{k\in K}\left(s+\frac{2x_k}{n_k}-1\right) Z^{\mathrm{FR}}_{\psi}(s, f_{\underline{x}}, \phi)\right)$$

induces a holomorphic map $\mathcal{H}_{1-\epsilon} \to \mathcal{S}(\mathcal{A}_0)$ for some $\epsilon > 0$. This already implies that

$$s \mapsto Z^{\mathrm{FR}}_{\psi}(s, {}^0 f, \phi) = \frac{n}{P}(2\pi)^{-r+1}\int_{\mathcal{A}_0}\prod_{i\in I'}\left(s+\frac{x_i+x_{i^*}}{n_i}-1\right)^{-1}\prod_{k\in K}\left(s+\frac{2x_k}{n_k}-1\right)^{-1} F_s(\underline{x})d\underline{x}$$

has an analytic continuation to $\mathcal{H}_{>1}$. Moreover, if χ is not Hermitian, the linear forms $\underline{x} \in \mathcal{A}_0 \mapsto x_i + x_{i^*}$, $i \in I'$, and $\underline{x} \in \mathcal{A}_0 \mapsto x_k$, $k \in K$, are linearly independent, and thus, by [BP21, Lemma 3.11], $Z^{\mathrm{FR}}_{\psi}(s, {}^0 f, \phi)$ has a limit as $s \to 1$.

We assume from now on that χ is Hermitian so that $I = I' \cup (I')^*$ and $J = \emptyset$. Set $\mathcal{A}^L = (i\mathbb{R})^{I'}$, $\mathcal{A}' = (i\mathbb{R})^{I'\cup K}$ and let $\mathcal{A}'_0$ be the subspace of $\underline{x} \in \mathcal{A}'$ such that $\sum_{i\in I'} x_i + \sum_{k\in K} x_k = 0$. We have a short exact sequence

$$0 \to \mathcal{A}^L \to \mathcal{A}_0 \to \mathcal{A}'_0 \to 0,$$

where the first map sends $\underline{x} \in \mathcal{A}^L$ to the vector $\underline{y} \in \mathcal{A}_0$ with coordinates $y_i = x_i$ for $i \in I'$, $y_i = -x_{i^*}$ for $i \in (I')^*$ and $y_k = 0$ for $k \in K$, whereas the second map is given by

$$\underline{x} \in \mathcal{A}_0 \mapsto ((x_i + x_{i^*})_{i\in I'}, (x_k)_{k\in K}).$$

We equip $\mathcal{A}^L$ and $\mathcal{A}'$ with the products of Lebesgue measures and $\mathcal{A}'_0$ as before with the unique measure inducing on $\mathcal{A}'/\mathcal{A}'_0 \simeq i\mathbb{R}$ the Lebesgue measure. Then, it is easy to see that the above exact sequence is compatible with the different Haar measures. In particular, we have

$$Z^{\mathrm{FR}}_{\psi}(s, {}^0 f, \phi) = \frac{n}{P}(2\pi)^{-r+1}\int_{\mathcal{A}'_0}\prod_{i\in I'}\left(s+\frac{y_i}{n_i}-1\right)^{-1}\prod_{k\in K}\left(s+\frac{2y_k}{n_k}-1\right)^{-1}\int_{\mathcal{A}^L} F_s(\underline{x}+\underline{y})d\underline{x}d\underline{y}.$$

From this and [BP21, Proposition 3.12], we obtain

$$\begin{aligned}\lim_{s\to 1^+}(s-1)Z^{\mathrm{FR}}_{\psi}(s, {}^0 f, \phi) &= \frac{n}{P}(2\pi)^{-r+1}\frac{\prod_{i\in I'} n_i \prod_{k\in K}\frac{n_k}{2}}{\sum_{i\in I'} n_i + \sum_{k\in K}\frac{n_k}{2}}(2\pi)^{|I'\cup K|-1}\int_{\mathcal{A}^L} F_1(\underline{x})d\underline{x} \qquad (6.2.8.1)\\ &= \frac{(2\pi)^{-|I'|}}{P'}2^{1-|K|}\int_{\mathcal{A}^L} F_1(\underline{x})d\underline{x},\end{aligned}$$

where we have set $P' = \prod_{i\in I'} n_i$. Furthermore, from the unramified computation of [Fli88, Proposition 3] and [BP21, Lemma 2.16.3], we have

$$F_1(\underline{x}) = \lim_{s\to 1^+}(s-1)^{|I'\cup K|} Z^{FR}_{\psi}(s, f_{\underline{x}}, \phi) = \widehat{\phi}(0)\beta_n(W_{f_{\underline{x}}}) \tag{6.2.8.2}$$

for $\underline{x} \in \mathcal{A}^L$. However, the isomorphism $\mathcal{A} \simeq i\mathfrak{a}^*_M$ sends $\mathcal{A}^L$ onto $i\mathfrak{a}^{L,*}_M$ and sends the measure on $\mathcal{A}^L$ to $\frac{(2\pi)^{|I'|}P'}{2^{|I'|}}$ times the measure on $i\mathfrak{a}^{L,*}_M$. As, by [BPCZ22, Corollary A.0.11.1], the function F_1 is Schwartz, so is the function $\lambda \in i\mathfrak{a}^{L,*}_M \mapsto \beta_n(W_{f,\Pi_\lambda})$. Moreover, that the map (6.2.6.1) is continuous follows from (6.2.8.2) together with [BPCZ22, Theorem 6.2.5.1 1., Eq. (A.0.4.3)] and the closed graph theorem. Finally, combining (6.2.8.1) with (6.2.8.2) and the above comparison of measures readily gives the identity (6.2.6.2) and ends the proof of Theorem 6.2.6.1 2.

6.2.9. Proof of Theorem 6.2.6.1 3.
The proof of [BPCZ22, Theorem 6.2.6.1] applies verbatim noting that in *loc. cit.* the condition that χ is $*$-regular is only used at the end of the proof to show that the family of bilinear forms denoted by

$$s \mapsto P_{G'_r}\widehat{\otimes}\mathcal{Z}^{FR}_{n-r}(s)$$

on $\mathcal{C}_\chi([M_r]) \times \mathcal{S}(\mathbb{A}^{n-r})$ extends holomorphically to $\mathcal{H}_{s>1}$. However, thanks to the proof of part 2 of Theorem 6.2.6.1 given in the previous subsection, we know that this property continues to hold for cuspidal data that are only assumed to be regular.

6.3. Proof of Theorem 6.1.1.1

Since χ is H-regular, by [BPCZ22, Theorem 7.1.3.1], the period integral

$$P_H : \phi \in \mathcal{S}_\chi([G]) \mapsto \int_{[H]} \phi(h)dh$$

extends continuously to a linear form on $\mathcal{T}_\chi([G])$ that we shall denote by P^*_H. Then, by the very same argument as for [BPCZ22, Eq. (8.2.3.5)], we have

$$I_\chi(f) = P^*_H\left(\int_{[G']} K_{f,\chi}(.,g')\eta_{G'}(g')dg'\right), \tag{6.3.0.1}$$

where the inner integral is taken in $\mathcal{T}_N([G])$ for N large enough. Furthermore, applying Corollary 6.2.7.1 instead of [BPCZ22, Corollary 6.2.7.1], the discussion of [BPCZ22, §8.2.4] shows that this inner integral is identically zero if χ is not Hermitian, whereas otherwise, it leads to the following expansion:

$$\int_{[G']} K_{f,\chi}(.,g')\eta_{G'}(g')dg' = 2^{-\dim(\mathfrak{a}_L)}\int_{i\mathfrak{a}^{L,*}_M}\sum_{\phi\in\mathcal{B}_{P,\pi}} E(.,I_P(\lambda,f)\phi,\lambda)\overline{\beta_\eta(W(\phi,\lambda))}d\lambda \tag{6.3.0.2}$$

for $\mathcal{B}_{P,\pi}$ a K-basis of $I^{G(\mathbb{A})}_{P(\mathbb{A})}(\pi)$. We assume from now on that χ is Hermitian. We claim the following:

(6.3.0.3) There exists $N > 0$ such that for every $\lambda \in i\mathfrak{a}^{L,*}_M$, the series

$$\sum_{\phi\in\mathcal{B}_{P,\pi}} E(.,I_P(\lambda,f)\phi,\lambda)\overline{\beta_\eta(W(\phi,\lambda))} \tag{6.3.0.4}$$

converges absolutely in $\mathcal{T}_N([G])$, and furthermore, the integral over $i\mathfrak{a}^{L,*}_M$ in the right-hand side of (6.3.0.2) converges absolutely in $\mathcal{T}_N([G])$.

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Indeed, that the series (6.3.0.4) converges absolutely in $\mathcal{T}_N([G])$ for large enough N (independent of λ) follows from [BPCZ22, Theorem 2.9.8.1]. Note that for every $\lambda \in i\mathfrak{a}_M^{L,*}$ and $g \in [G]$, we have

$$\sum_{\phi \in \mathcal{B}_{P,\pi}} E(g, I_P(\lambda, f)\phi, \lambda)\overline{\beta_\eta(W(\phi, \lambda))} = \beta_\eta(K_f(g,.)_{\Pi_\lambda}),$$

where $K_f(g,.)_{\Pi_\lambda}$ denotes the projection of the function $K_f(g,.) \in \mathcal{S}([G])$ to Π_λ as defined in [BPCZ22, Eq. (2.9.8.14)]. Moreover, by [BPCZ22, Lemma 2.10.1.1], for every continuous semi-norm μ on $\mathcal{S}([G])$, we can find $N_\mu > 0$ such that for every $f' \in \mathcal{S}(G(\mathbb{A}))$, we have

$$\mu(K_{f'}(g,.)) \ll_{\mu,f'} \|g\|_G^{N_\mu}, \text{ for } g \in [G].$$

In particular, combining this estimate with the first part of Theorem 6.2.6.1, we deduce that for each $d > 0$, we can find $N_d > 0$ such that, for $f' \in \mathcal{S}(G(\mathbb{A}))$, we have

$$\left| \sum_{\phi \in \mathcal{B}_{P,\pi}} E(g, I_P(\lambda, f')\phi, \lambda)\overline{\beta_\eta(W(\phi, \lambda))} \right| = |\beta_\eta(K_{f'}(g,.)_{\Pi_\lambda})| \ll_{d,f'} \|g\|_G^{N_d}(1 + \|\lambda\|)^{-d}$$

for $(\lambda, g) \in i\mathfrak{a}_M^{L,*} \times [G]$. Choosing d such that $\lambda \mapsto (1 + \|\lambda\|)^{-d}$ is integrable on $i\mathfrak{a}_M^{L,*}$ and applying the above inequality to $f' = R(X)f$ for $X \in \mathcal{U}(\mathfrak{g})$ gives the second part of (6.3.0.3).

We now need to recall how the extension P_H^* is defined in [BPCZ22, Theorem 7.1.3.1]: for $\Phi \in \mathcal{T}_\chi([G])$, the integral

$$Z_\psi^{RS}(s, \Phi) = \int_{N_H(\mathbb{A})\backslash H(\mathbb{A})} W_\Phi(h)|\det h|_{\mathbb{A}_E}^s dh,$$

where $W_\Phi(h) := \int_{[N]} \Phi(uh)\psi_N(u)^{-1}du$, a priori only defined when $\Re(s)$ is sufficiently large, admits an analytic continuation to $\mathbb{C}$ and we have

$$P_H^*(\Phi) = Z_\psi^{RS}(0, \Phi).$$

Therefore, by continuity of P_H^*, (6.3.0.2) and (6.3.0.3), we obtain

$$\begin{aligned} P_H^*\left(\int_{[G']} K_{f,\chi}(.,g')\eta_{G'}(g')dg'\right) &= 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} \sum_{\phi \in \mathcal{B}_{P,\pi}} P_H^*(E(., I_P(\lambda, f)\phi, \lambda))\overline{\beta_\eta(W(\phi, \lambda))}d\lambda \\ &= 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} \sum_{\phi \in \mathcal{B}_{P,\pi}} Z_\psi^{RS}(0, W(I_P(\lambda, f)\phi, \lambda))\overline{\beta_\eta(W(\phi, \lambda))}d\lambda \\ &= 2^{-\dim(\mathfrak{a}_L)} \int_{i\mathfrak{a}_M^{L,*}} I_{\Pi_\lambda}(f)d\lambda. \end{aligned}$$

Together with (6.3.0.1), this ends the proof of the theorem.

7. Proof of Gan-Gross-Prasad conjecture: Eisenstein case

7.1. Global comparison of relative characters

7.1.1. Notations

We follow the notations of Sections 3 and 4. In particular, we fix an integer $n \geqslant 1$. We consider the group $G = G_n \times G_{n+1}$ and, for $h \in \mathcal{H}_n$, the groups $U_h' \subset U_h$. Since n is fixed, we drop it from the notation: $\mathcal{H} = \mathcal{H}_n$. The various Haar measures are those considered in §2.1.8.

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7.1.2.
Let $V_{F,\infty} \subset S_0 \subset V_F$ be a finite set of places containing all the places that are ramified in E. For every $v \in V_F$, we set $E_v = E \otimes_F F_v$, and when $v \notin V_{F,\infty}$, we denote by $\mathcal{O}_{E_v} \subset E_v$ its ring of integers. Let $\mathcal{H}^\circ \subset \mathcal{H}$ be the (finite) subset of Hermitian spaces of rank n over E that admits a selfdual $\mathcal{O}_{E_v}$-lattice for every $v \notin S_0$. For each $h \in \mathcal{H}^\circ$, the group U_h is then defined over $\mathcal{O}_F^{S_0}$, and we fix a choice of such a model. For $v \notin S_0$, we define the open compact subgroups $K_{h,v} = U_h(\mathcal{O}_v)$ and $K_v = G(\mathcal{O}_v)$, respectively, of $U_h(F_v)$ and $G(F_v)$. We set

$$K_h^\circ = \prod_{v\notin S_0} K_{h,v} \text{ and } K^\circ = \prod_{v\notin S_0} K_v.$$

We choose also for each $v \in S_0$ some maximal compact subgroup $K_{h,v} \subset U_h(F_v)$ and $K_v \subset G(F_v)$ (see §3.1.3).

Let $v \notin S_0$. We denote by $\mathcal{S}^\circ(U_h(F_v))$, resp. $\mathcal{S}^\circ(G(F_v))$, the corresponding spherical Hecke algebra. We have the base change homomorphism

$$BC_{h,v} : \mathcal{S}^\circ(G(F_v)) \to \mathcal{S}^\circ(U_h(F_v)).$$

We denote by $\mathcal{S}^\circ(U_h(\mathbb{A}^{S_0}))$, resp. $\mathcal{S}^\circ(G(\mathbb{A}^{S_0}))$, the restricted tensor product of $\mathcal{S}^\circ(U_h(F_v))$, resp. $\mathcal{S}^\circ(G(F_v))$, for $v \notin S_0$. We set $BC_h^{S_0} = \otimes_{v\notin S_0} BC_{h,v}$.

We also denote by $\mathcal{S}^\circ(G(\mathbb{A})) \subset \mathcal{S}(G(\mathbb{A}))$ and $\mathcal{S}^\circ(U_h(\mathbb{A})) \subset \mathcal{S}(U_h(\mathbb{A}))$, for $h \in \mathcal{H}^\circ$, the subspaces of functions that are, respectively, bi-K°-invariant and bi-K_h°-invariant.

7.1.3. Transfer
Let $h \in \mathcal{H}^\circ$. We shall say that $f_{S_0} \in \mathcal{S}(G(F_{S_0}))$ and $f_{S_0}^h \in \mathcal{S}(U_h(F_{S_0}))$ are transfers if the functions f_{S_0} and $f_{S_0}^h$ have matching regular orbital integrals in the sense of [BLZZ21, Definition 4.4].

7.1.4. Cuspidal datum χ_0 and H-regular Hermitian Arthur parameter Π
Let P be a standard parabolic subgroup of G and π be a cuspidal automorphic representation of $M = M_P$. Let $\chi_0 \in \mathfrak{X}(G)$ be the class of the pair (M_P, π). We assume henceforth that χ_0 is a Hermitian (G,H)-regular cuspidal datum in the sense of §4.1.3. We assume also that (M,π) satisfies the conditions of §4.1.4 and that L is the standard Levi subgroup containing M defined in this §. Set $\Pi = \mathrm{Ind}_P^G(\pi)$: this is a H-regular Hermitian Arthur parameter in the sense of §1.2.1. We assume that S_0 is large enough so that Π admits K°-fixed vectors. Let Π_0 be the discrete component of Π (see §1.1.2).

The group $S_\Pi = S_{\Pi_0}$ is defined in §§1.1.2 and 1.1.4. Its order can be computed as follows:

$$|S_\Pi| = 2^{\dim(\mathfrak{a}_L)-\dim(\mathfrak{a}_M^L)}. \tag{7.1.4.1}$$

For any $\lambda \in i\mathfrak{a}_M^{L,*}$, we shall use the distribution $I_{P,\pi}(\lambda,\cdot)$ on $\mathcal{S}(G(\mathbb{A}))$ defined in §4.1.7.

7.1.5.
Let S_0' be the union of $S_0 \setminus V_{F,\infty}$ and the set of all finite places of F that are inert in E. Let $h \in \mathcal{H}^\circ$ and let $\mathfrak{X}_\Pi^h \subset \mathfrak{X}(U_h)$ be the set of cuspidal data represented by pairs (M_h, σ) such that M_h is the standard Levi factor of a standard parabolic sugroup P_h of U_h and σ is a cuspidal automorphic representation of $M_h(\mathbb{A})$ such that

- Π is a weak base change of (P_h, σ) (in the sense of §§1.1.3 and 1.1.4);
- σ is $\prod_{v\notin S_0}(M_h(F_v) \cap K_{h,v})$-unramified;

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- with the identifications $M_h = G^\sharp \times U$ where $G^\sharp$ is a product of linear groups and $U = U(h_{n_0}) \times U(h_{n'_0})$ for some $h_{n'_0} \in \mathcal{H}_{n_0}$ and $h_{n_0} \in \mathcal{H}_{n'_0}$ and $\sigma = \sigma^\sharp \boxtimes \sigma_0$ accordingly (where σ_0 is a cuspidal automorphic representation of $U(\mathbb{A})$), for all $v \notin S'_0 \cup V_{F,\infty}$, the representation $\Pi_{0,v}$ is the split base change of the representation $\sigma_{0,v}$.

Since Π is a H-regular Hermitian Arthur parameter, the class of (M_h, σ) is (U, U')-regular in the sense of §3.5.2. Moreover, there is a natural isomorphism

$$bc = bc_{(M_h,\sigma)} : \mathfrak{a}_M^{L,*} \to \mathfrak{a}_{M_h}^*. \tag{7.1.5.1}$$

More precisely, we wrote $M_h = G^\sharp \times U$, where $G^\sharp$ is a product of linear groups G_{m_i} for some integers m_i and $1 \leqslant i \leqslant s$ so that $\mathfrak{a}_{M_h}^* = \oplus_{i=1}^s \mathfrak{a}_{G_{m_i}}^*$. But the product $\prod_{i=1}^s (G_{m_i} \times G_{m_i})$ is also a factor in the decomposition of M into a product of linear groups. So the space $\mathfrak{a}_{M_h}^* \oplus \mathfrak{a}_{M_h}^* = \oplus_{i=1}^s (\mathfrak{a}_{G_{m_i}}^* \oplus \mathfrak{a}_{G_{m_i}}^*)$ is a subspace of $\mathfrak{a}_M^*$. The antidiagonal map $x \mapsto (x, -x)$ then identifies $\mathfrak{a}_{M_h}^*$ with a subspace of $\mathfrak{a}_{M_h}^* \oplus \mathfrak{a}_{M_h}^*$ and thus of $\mathfrak{a}_M^*$, and this subspace is precisely $\mathfrak{a}_M^{L,*}$.

Remarks 7.1.5.1. The base change map bc depends implicitly on the choice of the pair (M_h, σ). Indeed, if $(M_{1,h}, \sigma_1)$ is a pair equivalent to (M_h, σ) and if $w \in W(M_h, M_{1,h})$ is such that $\sigma_1 = w \cdot \sigma$, then $bc_{(M_{1,h},\sigma_1)} = w \circ bc_{(M_h,\sigma)}$ (where we view w as a map $\mathfrak{a}_{M_h}^* \to \mathfrak{a}_{M_{1,h}}^*$). In order to not burden the notation, we shall omit the subscript (M_h, σ) from the notation bc in (7.1.5.1).

We warn also the reader that the map (7.1.5.1) does not preserve the various choices of Haar measures. In fact, the pullback of the measure on $\mathfrak{a}_{M_h}^*$ is $2^{\dim(\mathfrak{a}_M^L)}$ times the measure on $\mathfrak{a}_M^{L,*}$.

For any $\lambda \in i\mathfrak{a}_M^{L,*}$, we set

$$J_\Pi^h(\lambda, f) = \sum_{(M_h,\sigma)} J_{P_h,\sigma}^h(bc(\lambda), f), \tag{7.1.5.2}$$

where the sum is over a set of representatives (M_h, σ) of classes in $\mathfrak{X}_\Pi^h$ and $J_{P_h,\sigma}^h(bc(\lambda), f) = J_{P_h,\sigma}^{U_h}(bc(\lambda), f)$ is the distribution introduced in §3.5.5. By Proposition 3.5.5.1 and remarks 7.1.5.1, the distribution does not depend on the choice of the representative.

The sum above is *a priori* absolutely convergent and the convergence is uniform on compact subset of $i\mathfrak{a}_M^{L,*}$ (as follows from from a reinforcement of [BPCZ22, proposition 2.8.4.1] based on results of Müller; see [Mül98, corollary 0.3]). In particular, the expression $J_\Pi^h(\lambda, f)$ is holomorphic for $\lambda \in i\mathfrak{a}_M^{L,*}$.

7.1.6. A global relative characters identity

Theorem 7.1.6.1. *Let $f \in \mathcal{S}^\circ(G(\mathbb{A}))$ and $f^h \in \mathcal{S}^\circ(U_h(\mathbb{A}))$ for every $h \in \mathcal{H}^\circ$. Assume that the following properties are satisfied for every $h \in \mathcal{H}^\circ$:*

1. $f = (\Delta_H^{S_0,*}\Delta_{G'}^{S_0,*}) f_{S_0} \otimes f^{S_0}$ *with* $f_{S_0} \in \mathcal{S}(G(F_{S_0}))$ *and* $f^{S_0} \in \mathcal{S}^\circ(G(\mathbb{A}^{S_0}))$.
2. $f^h = (\Delta_{U'_h}^{S_0})^2 f_{S_0}^h \otimes f^{h,S_0}$ *with* $f_{S_0}^h \in \mathcal{S}(U_h(F_{S_0}))$ *and* $f^{h,S_0} \in \mathcal{S}^\circ(U_h(\mathbb{A}^{S_0}))$.
3. *The functions* f_{S_0} *and* $f_{S_0}^h$ *are transfers.*
4. $f^{h,S_0} = BC_h^{S_0}(f^{S_0})$
5. *The function f^{S_0} is a product of a smooth compactly supported function on the restricted product $\prod'_{v \notin S'_0 \cup V_{F,\infty}} G(F_v)$ by the characteristic function of $\prod_{v \in S'_0 \setminus S_0} G(\mathcal{O}_v)$.*

Then for any $\lambda \in i\mathfrak{a}_M^{L,*}$, we have

$$\sum_{h \in \mathcal{H}^\circ} J_\Pi^h(\lambda, f^h) = |S_\Pi|^{-1} I_{P,\pi}(\lambda, f). \tag{7.1.6.1}$$

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Remark 7.1.6.2. If Π is a discrete Arthur parameter (that is, $L = M$), the statement reduces to [BPCZ22, proposition 10.1.6.1]. As we observed in [BPCZ22], if the assumptions hold for the set S_0, they also hold for any large enough finite set containing S_0.

7.1.7. Proof of Theorem 7.1.6.1

As in [BLZZ21] and [BPCZ22], our proof is based on the global comparison of Jacquet-Rallis trace formulas and the use of multipliers to isolate some spectral contributions. However, the spectral contributions we consider here are continuous in nature, and we need further considerations.

In [BPCZ22, Theorem 3.2.4.1], we defined a distribution I on $\mathcal{S}(G(\mathbb{A}))$: this is the 'Jacquet-Rallis trace formula' for G. For unitary groups, we have an analogous distribution $J^h = J^{U_h}$ on $\mathcal{S}(U_h(\mathbb{A}))$ for each $h \in \mathcal{H}$ (see Theorem 3.2.3.1). By [CZ21, théorème 1.6.1.1], we have for functions f and f^h as in the statement

$$I(f) = \sum_{h\in\mathcal{H}^\circ} J^h(f^h). \tag{7.1.7.1}$$

In the following, for each finite place outside S_0, we fix open compact subgroups $K'_v \subset K_v$ and $K'_{h,v} \subset K_{h,v}$ of finite index. We set

$$K_1^\infty = \prod_{v\in S_0\setminus V_{F,\infty}} K'_v \prod_{v\notin S_0} K_v \text{ and } K_{h,1}^\infty = \prod_{v\in S_0\setminus V_{F,\infty}} K'_{h,v} \prod_{v\notin S_0} K_{h,v}.$$

We denote by $\mathcal{S}(G(\mathbb{A}), K_1^\infty) \subset \mathcal{S}^\circ(G(\mathbb{A}))$, resp. $\mathcal{S}(U_h(\mathbb{A}), K_{h,1}^\infty) \subset \mathcal{S}^\circ(U_h(\mathbb{A}))$, the subalgebras of bi-$K_1^\infty$ (resp. bi-$K_{h,1}^\infty$) invariant functions. Since we can shrink K_1^∞ and $K_{h,1}^\infty$ if necessary, it suffices to prove the theorem for functions in these subalgebras.

We denote by $\mathcal{M}^{S'_0}(G(\mathbb{A}))$, resp. $\mathcal{M}^{S'_0}(U_h(\mathbb{A}))$, the algebra of S'_0-multipliers defined in [BLZZ21, definition 3.5] relatively to the subgroup $\prod_{v\notin S'_0} K_v$, resp. $\prod_{v\notin S'_0} K_{h,v}$. Any multiplier $\mu \in \mathcal{M}^{S'_0}(G(\mathbb{A}))$, resp. $\mu \in \mathcal{M}^{S'_0}(U_h(\mathbb{A}))$, gives rise to a linear operator $\mu*$ of the algebra $\mathcal{S}(G(\mathbb{A}), K_1^\infty)$, resp. $\mathcal{S}(U_h(\mathbb{A}), K_{h,1}^\infty)$. For every admissible irreducible representation ρ of $G(\mathbb{A})$, resp. of $U_h(\mathbb{A})$, there exists a complex number $\mu(\rho)$ such that $\rho(\mu * f) = \mu(\rho)\rho(f)$ for all $f \in \mathcal{S}^\circ(G(\mathbb{A}), K_1^\infty)$, resp. $f \in \mathcal{S}^\circ(U_h(\mathbb{A}), K_{h,1}^\infty)$. Note that $\mu(\rho)$ depends only on the infinitesimal character of the archimedean component of ρ and on the components outside S'_0. Moreover, if Q is a standard parabolic subgroup of G, resp. U_h, and if ρ is an admissible irreducible representation of the Levi component $M_Q(\mathbb{A})$, then we have for any $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*$ and $R_\lambda = \mathrm{Ind}_Q^G(\rho_\lambda)$, resp. $\mathrm{Ind}_Q^{U_h}(\rho_\lambda)$,

$$R_\lambda(\mu * f) = \mu(R_\lambda)R_\lambda(f),$$

where $\mu(R_\lambda) \in \mathbb{C}$ and the map $\lambda \mapsto \mu(R_\lambda)$ is holomorphic.

The following two lemmas are based on [BLZZ21, theorem 3.17] and a strong multiplicity one theorem of Ramakrishnan; see [Ram18].

Lemma 7.1.7.1. *Let $h \in \mathcal{H}^\circ$ and $\lambda \in i\mathfrak{a}_M^{L,*}$ in general position. Then there exists a multiplier $\mu_h \in \mathcal{M}^{S'_0}(U_h(\mathbb{A}))$ such that*

1. *For all $f^h \in \mathcal{S}(U_h(\mathbb{A}), K_{h,1}^\infty)$, the right convolution by $\mu_h * f^h$ sends $L^2([U_h])$ into*

$$\widehat{\bigoplus}_{\chi\in\mathfrak{X}_\Pi^h} L^2_\chi([U_h]).$$

2. *For $(M_h, \sigma) \in \mathfrak{X}_\Pi^h$, we have $\mu_h(\mathrm{Ind}_{P_h}^{U_h}(\sigma_{bc(\lambda)})) = 1$.*

Proof. By [GRS11], there exists at least one $h^\flat \in \mathcal{H}^\circ$ such that the set $\mathfrak{X}_\Pi^{h^\flat}$ is nonempty. We fix such a form $h^\flat$ and a pair $(M_{h^\flat}, \sigma^\flat)$. Let $\lambda \in i\mathfrak{a}_M^{L,*}$ in general position. Let $h \in \mathcal{H}^\circ$ and let $\mathfrak{X}_\lambda^h$ be the set of

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cuspidal data of U_h represented by pairs (M_Q,ρ) with $Q\subset U_h$ a standard parabolic subgroup for which there exists $\lambda'\in\mathfrak{a}^*_{Q,\mathbb{C}}$ such that

(A) the Archimedean infinitesimal characters of $\mathrm{Ind}_{P_{h^\flat}}^{U_{h^\flat}}(\sigma^\flat_{bc(\lambda)})$ and $\mathrm{Ind}_Q^{U_h}(\rho_{\lambda'})$ are the same;

(B) for all finite places v outside S_0', the irreducible representation $\mathrm{Ind}_{P_{h^\flat}(F_v)}^{U_{h^\flat}(F_v)}(\sigma^\flat_{bc(\lambda),v})$ is a constituent of $\mathrm{Ind}_{Q(F_v)}^{U_h(F_v)}(\rho_{\lambda',v})$.

By [BLZZ21, theorem 3.19], there exists a S_0'-multiplier μ_h such that

(C) For all $f^h\in\mathcal{S}(U_h(\mathbb{A}),K^\infty_{h,1})$, the right convolution by $\mu_h * f^h$ sends $L^2([U_h])$ into

$$\widehat{\oplus}_{\chi\in\mathfrak{X}^h_\lambda}L^2_\chi([U_h]).$$

(D) $\mu_h(\mathrm{Ind}_{P_{h^\flat}}^{U_{h^\flat}}(\sigma^\flat_{bc(\lambda)}))=1$.

Let $\lambda'\in\mathfrak{a}^*_{Q,\mathbb{C}}$ and (M_Q,ρ) be a representative of an element $\mathfrak{X}^h_\lambda$ such that conditions (A) and (B) hold. We can write the Levi factor M_Q of Q as a product $G^\sharp\times U_0$, where $G^\sharp$ is a product of linear groups and U_0 is a product of two unitary groups. Accordingly, we write $\rho=\rho^\sharp\times\rho_0$. Let $Q_E=\mathrm{Res}_{E/F}(Q\times_F E)$ and let $\lambda'_E\in\mathfrak{a}^*_{Q_E}$ be the base change of λ'. Observe that Π_v is generic for all places v and is the split base of the representation $\mathrm{Ind}_{P_{h^\flat}(F_v)}^{U_{h^\flat}(F_v)}(\sigma^\flat_{bc(\lambda),v})$ for finite $v\notin S_0'$. So the representation $\mathrm{Ind}_{P_{h^\flat}(F_v)}^{U_{h^\flat}(F_v)}(\sigma^\flat_{bc(\lambda),v})$ is also generic, and thus, $\mathrm{Ind}_{Q(F_v)}^{U_h(F_v)}(\rho_{\lambda',v})$ and $\rho_{\lambda',v}$ are also generic for finite $v\notin S_0'$. By [BLZZ21, Theorem 4.14 (1)], there exists an isobaric automorphic representation ρ_E of M_{Q_E} such that the split base change of ρ is ρ_E at almost all split places. By the strong multiplicity one theorem of Ramakrishnan (see [Ram18]), we deduce that the isobaric automorphic representations of $G(\mathbb{A})$ associated to $(P,\pi\otimes\lambda)$ and $(Q_E,\rho_E\otimes\lambda'_E)$ are isomorphic. Using [JS81a, theorem 4.4] and the fact that λ is in general position, we conclude that, up to a change of representative, we have $P\subset Q_E$, the inclusion given by base change $\mathfrak{a}^*_Q\subset\mathfrak{a}^*_{Q_E}$ induces an isomorphism of $\mathfrak{a}^*_Q$ onto $\mathfrak{a}^{L,*}_M$ which identifies λ' to $bc(\lambda)$, the representation Π_0 is the weak base change of ρ_0 and $\mathrm{Ind}_{Q_E}^G(\rho^\sharp\boxtimes\Pi_0\boxtimes\rho^{\sharp,*})=\Pi$. Using condition (B), we deduce that $\Pi_{0,v}$ is also the split base change of $\rho_{0,v}$ for all finite places v outside S_0'. We get that $\mathfrak{X}^h_\lambda\subset\mathfrak{X}^h_\Pi$, and so (C) implies assertion 1.

Still, we have to check assertion 2. Let (M_Q,ρ) be a representative of an element in $\mathfrak{X}^h_\Pi$. We claim that $\mathrm{Ind}_Q^{U_h}(\rho)$ and $\mathrm{Ind}_{P_{h^\flat}}^{U_{h^\flat}}(\sigma^\flat_{bc(\lambda)})$ have the same Archimedean infinitesimal character and have the same local component for all finite places v outside S_0'. The latter condition follows directly from the definition of the set $\mathfrak{X}^h_\Pi$ and the fact that the split base is injective. The former condition follows from [BLZZ21, Theorem 4.14 (4)] (applied to ρ) and the fact that the base change map is injective at the level of archimedean infinitesimal characters. Since condition 2 depends only on the components of $\mathrm{Ind}_Q^{U_h}(\rho)$ on $V_F\setminus S_0'$, we see that (D) implies assertion 2. □

Lemma 7.1.7.2. *Let $\lambda\in i\mathfrak{a}^{L,*}_M$. Then there exists a multiplier $\mu\in\mathcal{M}^{S_0'}(G(\mathbb{A}))$ such that*

1. *For all $f\in\mathcal{S}(G(\mathbb{A}),K^\infty_{h,1})$, the right convolution by $\mu * f$ sends $L^2(G(F)A_G(\mathbb{A})\backslash G(\mathbb{A}))$ into $L^2_{\chi_0}([G])$.*
2. *We have $\mu(\Pi_\lambda)=1$.*

Proof. The proof is similar to (but simpler than) that of Lemma 7.1.7.1 and is based on [BLZZ21, Theorem 1.3]). □

Let λ_0 be an element of $i\mathfrak{a}^{L,*}_M$ in general position. Let $\mu\in\mathcal{M}^{S_0'}(G(\mathbb{A}))$ and $\mu_h\in\mathcal{M}^{S_0'}(U_h(\mathbb{A}))$ for $h\in\mathcal{H}^\circ$ satisfying assertions of Lemmas 7.1.7.1 and 7.1.7.2 for $\lambda=\lambda_0$. We may and we shall also assume that μ^h is the 'base change' of μ (see [BLZZ21, Lemma 4.12]). Let $f\in\mathcal{S}(G(\mathbb{A}),K^\infty_1)$, and for $h\in\mathcal{H}^\circ$,

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let $f^h \in \mathcal{S}(U_h(\mathbb{A}), K^\infty_{h,1})$ that satisfy the hypotheses of Theorem 7.1.6.1. Then $\mu * f$ and $\mu^h * f^h$, for $h \in \mathcal{H}^\circ$ still satisfy the hypotheses (see [BLZZ21, Proposition 4.8]). Hence, by (7.1.7.1), we have

$$I(\mu * f) = \sum_{h\in\mathcal{H}^\circ} J^h(\mu^h * f^h). \tag{7.1.7.2}$$

It follows from Lemma 7.1.7.2 and [BPCZ22, Proposition 3.3.3.1 and Theorem 3.3.9.1] that we have $I(\mu * f) = I_{\chi_0}(\mu * f)$. In the same way, Lemma 7.1.7.1 and Corollary 3.3.5.2 show that the right-hand side of (7.1.7.2) reduces to

$$\sum_{h\in\mathcal{H}^\circ}\sum_{\chi\in\mathfrak{X}^h_\Pi} J^h_\chi(\mu_h * f^h).$$

By Theorems 3.5.7.1 and 4.1.8.1 and by an elementary change of variables, we get

$$\begin{aligned}2^{-\dim(\mathfrak{a}_L)}\int_{i\mathfrak{a}^{L,*}_M} I_{P,\pi}(\lambda,\mu * f)\,d\lambda &= \sum_{h\in\mathcal{H}^\circ}\sum_{(M_h,\sigma)\in\mathfrak{X}^h_\Pi}\int_{i\mathfrak{a}^*_{M_h}} J^h_{P_h,\sigma}(bc(\lambda),\mu_h * f^h)\,d\lambda\\ &= 2^{-\dim(\mathfrak{a}^L_M)}\sum_{h\in\mathcal{H}^\circ}\int_{i\mathfrak{a}^{L,*}_M} J^h_\Pi(\lambda,\mu_h * f^h)\,d\lambda. \end{aligned}\tag{7.1.7.3}$$

Let v_1, v_2 two finite places outside S'_0 with distinct residual characteristics. Let $S_1 \subset V_F \setminus S'_0$ be a finite set of finite places. We assume that S_1 contains v_1 and v_2. Let $\mathcal{A}_{S_1}$ be the spherical algebra $\otimes_{v\in S_1}\mathcal{S}^\circ(G(F_v))$. Let $g \in \mathcal{A}_{S_1}$. For all $h \in \mathcal{H}^\circ$, let $g^h = (\otimes_{v\in S_1} BC_{h,v})(g)$ be its base change to $\otimes_{v\in S_1}\mathcal{S}^\circ(U_h(F_v))$. The assumptions of Theorem 7.1.6.1 still hold for the convolutions $f * g$ and $f^h * g^h$. Note that we have for any $\lambda \in i\mathfrak{a}^{L,*}_M$,

$$J^h_\Pi(\lambda,\mu_h * (f^h * g^h)) = J^h_\Pi(\lambda,\mu_h * f^h)\hat{g}(\Pi_{\lambda,S_1}),$$

where $\hat{g}$ is the Satake transform and $\hat{g}(\Pi_{\lambda,S_1})$ is the scalar by which g acts on $\Pi_{\lambda,S_1} = \otimes_{v\in S_1}\Pi_{\lambda,v}$. For all $\lambda \in i\mathfrak{a}^{L,*}_M$, we set

$$h(\lambda) = |S_\Pi|^{-1} I_{P,\pi}(\lambda,\mu * f) - \sum_{h\in\mathcal{H}^\circ} J^h_\Pi(\lambda,\mu_h * f^h).$$

The equality (7.1.7.3) implies that for all $g \in \mathcal{A}_{S_1}$, we have

$$\int_{i\mathfrak{a}^{L,*}_M} \hat{g}(\Pi_{\lambda,S_1})h(\lambda)\,d\lambda = 0.$$

Let $\mathcal{T}_{S_1} : \lambda \in i\mathfrak{a}^{L,*}_M \mapsto \Pi_{\lambda,S_1}$. Because v_1 and v_2 have distinct residual characteristics, the morphism from $i\mathfrak{a}^*_M$ to the group of unramified characters of $M(F_{v_1}\times F_{v_2})$ given by $\lambda \in i\mathfrak{a}^*_M \mapsto (x \in M(F_{v_1}\times F_{v_2}) \mapsto \exp(\langle\lambda, H_P(x)\rangle)$ is injective. We deduce that the map $\mathcal{T}_{\{v_1,v_2\}}$ has finite fibers of uniformly bounded cardinality. In particular, the map $\mathcal{T}_{S_1}$ has the same property with the same bound. By Stone-Weierstrass theorem, the set $\{\hat{g} \mid g \in \mathcal{A}_{S_1}\}$ is dense in the set of continuous fonctions on the unramified unitary dual of $\prod_{v\in S_1} G(F_v)$.The push-forward of the measure $h(\lambda)\,d\lambda$ by the map $\mathcal{T}_{S_1}$ is thus zero. In this way, we get that for almost all $\lambda \in i\mathfrak{a}^{L,*}_M$,

$$\sum_{\lambda_1\in\mathcal{T}^{-1}_{S_1}(\Pi_{\lambda,S_1})} h(\lambda_1) = 0 \tag{7.1.7.4}$$

for all finite sets S_1, as above. By continuity of h, this equality holds for all $\lambda \in i\mathfrak{a}^{L,*}_M$ and in particular for $\lambda = \lambda_0$.

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Let us write $\mathcal{T}_{S_1}^{-1}(\Pi_{\lambda_0,S_1}) = \{\lambda_0, \lambda_1, \ldots, \lambda_j\}$, where $j+1$ is the cardinality of this set. Let $1 \leqslant i \leqslant j$. If $\Pi_{\lambda_i} \neq \Pi_{\lambda_0}$, by the strong multiplicity one theorem of Ramakrishnan (see [Ram18]), there exists a finite place $v_i \notin S_0' \cup S_1$ such that $\Pi_{\lambda_i,v_i} \neq \Pi_{\lambda_0,v_i}$. Since we may enlarge S_1 by adding such a place v_i, we will assume without loss of generality that for any $\lambda \in \mathcal{T}_{S_1}^{-1}(\Pi_{\lambda_0,S_1})$, we have $\Pi_\lambda = \Pi_{\lambda_0}$. By [JS81a, theorem 4.4] and the fact that the cuspidal datum (M,π) is G-regular, we have even $\pi_\lambda = \pi_{\lambda_0}$, and thus, $\lambda = \lambda_0$. So (7.1.7.4) reduces to $h(\lambda_0) = 0$. Because λ_0 is in general position, we may use Condition 2 of Lemmas 7.1.7.1 and 7.1.7.2: we get

$$h(\lambda_0) = |S_\Pi|^{-1} I_{P,\pi}(\lambda_0, f) - \sum_{h \in \mathcal{H}^\circ} J_\Pi^h(\lambda_0, f^h).$$

So we get (7.1.6.1) for λ in general position, and so for all $\lambda \in i\mathfrak{a}_M^{L,*}$, since both members of (7.1.6.1) are continuous (and even analytic).

7.2. *Proof of Theorem 1.2.3.1*

7.2.1.
Once we have Theorem 7.1.6.1, the proof of Theorem 1.2.3.1 is very similar to the proof of [BPCZ22, theorem 1.1.5.1]. For the reader's convenience, we recall some steps. We use notations of Section 7.1. Let $\Pi = \mathrm{Ind}_P^G(\pi)$ be a H-regular Hermitian Arthur parameter of G and let $\lambda \in i\mathfrak{a}_M^{L,*}$. The relative character I_{Π_λ} defined in (4.2.8.1) is built upon two linear forms – namely, $Z^{RS}(0)$ and β_η. The linear form β_η is not identically zero (as a consequence of [GK72], [Jac10, Proposition 5] and [Kem15]) and $Z^{RS}(0)$ is nonzero if and only if $L(\frac{1}{2}, \Pi_\lambda) \neq 0$ (as follows from the work Jacquet, Piatetski-Shapiro and Shalika [JPSS83], [Jac04]). Since by Theorem 4.2.8.1, we have $I_{\Pi_\lambda} = I_{P,\pi}(\lambda)$, we deduce that $I_{P,\pi}(\lambda)$ is nonzero if and only if $L(\frac{1}{2}, \Pi_\lambda) \neq 0$.

Let us consider $h \in \mathcal{H}$, a parabolic subgroup $P_h = M_h N_{P_h}$ of U_h and a cuspidal *subrepresentation* σ of M_h. Let $\mathcal{A}_{P_h,\sigma_h}(U_h) \subset \mathcal{A}_{P_h}(U_h)$ be the space of forms $\varphi \in \mathcal{A}_P(G)$ such that

$$m \in [M_P] \mapsto \exp(-\langle \rho_P, H_P(m) \rangle)\varphi(mg)$$

belongs to the space of σ for every $g \in G(\mathbb{A})$. By a variation on §3.5.5, we define for $\mu \in \mathfrak{a}_{M_h}^*$ the relative character

$$J_{P_h,\sigma}^{U_h}(\mu, f) = \sum_{\varphi \in \mathcal{B}_{P_h,\sigma}} \mathcal{P}_{U_h'}(I_{P_h}(\mu, f)\varphi, \mu)\overline{\mathcal{P}_{U_h'}(\varphi, \mu)},$$

where $\mathcal{B}_{P_h,\sigma}$ is a K-basis of $\mathcal{A}_{P_h,\sigma_h}(U_h)$; that is, it is the union over of $\tau \in \hat{K}_h$ of orthonormal bases $\mathcal{B}_{P_h,\sigma,\tau}$ for the Petersson inner product of the finite dimensional subspaces $\mathcal{A}_{P_h,\sigma}(U_h, \tau)$ of functions in $\mathcal{A}_{P_h,\sigma}(U_h)$ which transform under K_h according to τ.

Let us assume that Π is the weak base change of (P_h, σ). Then we have the map $bc : a_M^{L,*} \to \mathfrak{a}_{M_h}^*$; see (7.1.5.1). It is clear that the distribution $J_{P_h,\sigma}^h(bc(\lambda))$ is nonzero if and only if the period integral $\mathcal{P}_{U_h'}(\cdot, bc(\lambda))$ induces a nonzero linear form on the space of σ. Then Theorem 1.2.3.1 reduces to the equivalence between the two assertions:

(A) The distribution $I_{P,\pi}(\lambda)$ is nonzero.
(B) There exist $h \in \mathcal{H}$, a parabolic subgroup $P_h = M_h N_{P_h}$ of U_h and a cuspidal subrepresentation σ of M_h such that Π is the weak base change of (P_h, σ) and $J_{P_h,\sigma}^h(bc(\lambda), f) \neq 0$.

7.2.2. Proof of $(A) \Rightarrow (B)$
Let S_0 be the finite set of §7.1.2 such that I_Π is not identically zero on $\mathcal{S}^\circ(G(\mathbb{A}))$. Then by results of [Xue19] towards the archimedean transfert and the existence of p-adic transfer of [Zha14b], we see that

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there exist functions f and f^h for $h \in \mathcal{H}^\circ$ satisfying the hypotheses of Theorem 7.1.6.1 and such that $I_{P,\pi}(\lambda, f) \neq 0$. Then (B) follows from (7.1.6.1).

7.2.3. Proof of $(B) \Rightarrow (A)$

We may choose the set S_0 so that there exist $h_0 \in \mathcal{H}^\circ$, a parabolic subgroup $P_{h_0} = M_{h_0}N_{P_{h_0}} \subset U_{h_0}$, a cuspidal subrepresentation σ_0 of $M_{h_0}(\mathbb{A})$ and a function $\xi \in \mathcal{S}^\circ(U_{h_0}(\mathbb{A}))$ such that

- the class of (M_{h_0}, σ_0) belongs to $\mathfrak{X}_\Pi^{h_0}$;
- $J^{h_0}_{P_{h_0},\sigma_0}(bc(\lambda), \xi) \neq 0$.

We set $f_0^{h_0} = \xi * \xi^\vee$, where $\xi^\vee(g) = \overline{\xi(g^{-1})}$. Then (see [Zha14b, p. 993]) we have $J^{h_0}_{P,\sigma}(bc(\lambda), f_0^{h_0}) \geqslant 0$ for all pairs (P, σ) whose class belongs to $\mathfrak{X}_\Pi^{h_0}$. Moreover, $J^{h_0}_{P_{h_0},\sigma_0}(bc_{h_0}(\lambda), f_0^{h_0}) > 0$. For any $h \in \mathcal{H}^\circ$ such that $h \neq h_0$, we set $f_0^h = 0$. Up to enlarging S_0, we may and shall assume that the family $(f_0^h)_{h\in\mathcal{H}^\circ}$ satisfies conditions 2 and 5 of Theorem 7.1.6.1. The left-hand side of (7.1.6.1) for the family $(f_0^h)_{h\in\mathcal{H}^\circ}$ is nonzero. By the existence of transfer in [Zha14b] and the results towards archimedean transfer in [Xue19], we can find test functions $f \in \mathcal{S}^\circ(G(\mathbb{A}))$ and $f^h \in \mathcal{S}^\circ(U_h(\mathbb{A}))$, for $h \in \mathcal{H}^\circ$, satisfying all the conditions of Theorem 7.1.6.1 and such that the left-hand side of (7.1.6.1) is nonzero. Assertion (A) is then clear.

7.3. *Proof of Theorem 1.2.4.1*

7.3.1.

Let $h \in \mathcal{H}$. Let $P = M_P N_P$ be a parabolic subgroup of U_h and σ be a cuspidal automorphic subrepresentation of $M_P(\mathbb{A})$ which is tempered everywhere. Then the group $\mathrm{Res}_{E/F}(P \times_F E)$ obtained by extension to E and restriction of scalars to F can be identified to a parabolic subgroup $Q = M_Q N_Q$ of G. Then by [Mok15], [KMSW], σ admits a strong base-change π to M_Q; namely, for every place v of F, the local base-change of σ_v (defined in [Mok15] and [KMSW]) coincides with π_v. Let $\Pi = \mathrm{Ind}_Q^G(\pi)$. It follows that π and Π are also tempered everywhere. We assume that Π is a H-regular hermitian Arthur parameter. As in §1.1.3, we shall not distinguish in the notation the spaces $\mathfrak{a}_\Pi^*$ and $\mathfrak{a}_P^*$. Let $\lambda \in i\mathfrak{a}_\Pi^*$.

We choose a finite set of places S_0 as in §7.1.2 such that $h \in \mathcal{H}^\circ$ and σ as well as the additive character ψ' used to normalize local Haar measures in §2.1.8 are unramified outside of S_0.

We have a decomposition $\sigma = \otimes_{v\in V_F}\sigma_v$. Let $\Sigma_{\lambda,v}$ be the full induced representation $\mathrm{Ind}_{P(F_v)}^{U_h(F_v)}(\sigma_v \otimes \lambda)$. Let $v \in V_F$. We define a distribution $J_{\Sigma_{\lambda,v}}$ on $\mathcal{S}(U_h(F_v))$ by

$$J_{\Sigma_{\lambda,v}}(f_v^h) = \int_{U'_h(F_v)} \mathrm{Trace}(\Sigma_{\lambda,v}(h_v)\Sigma_{\lambda,v}(f_v^h))dh_v, \quad f_v^h \in \mathcal{S}(U_h(F_v)),$$

where

$$\Sigma_{\lambda,v}(f_v^h) = \int_{U_h(F_v)} f_v^h(g_v)\Sigma_{\lambda,v}(g_v)dg_v.$$

By [Har14], since the representations $\Sigma_{\lambda,v}$ are all tempered, the expression defining $J_{\Sigma_{\lambda,v}}$ is absolutely convergent, and for every $v \notin S_0$, we have

$$J_{\Sigma_{\lambda,v}}(\mathbf{1}_{U_h(\mathcal{O}_v)}) = \Delta^{-2}_{U'_h,v}\frac{L\left(\frac{1}{2}, \Pi_{\lambda,v}\right)}{L(1, \Pi_{\lambda,v}, \mathrm{As}')}.$$

If there exists a place $v \in S_0$ such that σ_v does not support any nonzero continuous $U'_h(F_v)$-invariant functional, both sides of (1.2.4.1) in Theorem 1.2.4.1 are clearly automatically zero. So we shall assume that for every $v \in S_0$, the local representation σ_v supports a nonzero continuous

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$U'_h(F_v)$-invariant functional. Then the semi-local distribution $\prod_{v\in S_0} J_{\Sigma_{\lambda,v}}(f_v^h)$ does not vanish identically by [BP20, theorem 8.2.1]. According to our choice of local measure, Theorem 1.2.4.1 is then equivalent to the following assertion: for all factorizable test function $f^h \in \mathcal{S}(U_{h_0}(\mathbb{A}))$ of the form $f^h = (\Delta_{U'_h}^{S_0})^2 \prod_{v\in S_0} f_v^h \times \prod_{v\notin S_0} \mathbf{1}_{U_h(\mathcal{O}_v)}$, we have

$$J_{P,\sigma}^h(\lambda, f^h) = |S_\Pi|^{-1} \frac{L^{S_0}\left(\frac{1}{2}, \Pi_\lambda\right)}{L^{S_0}(1, \Pi_\lambda, \mathrm{As}')} \prod_{v\in S_0} J_{\Sigma_{\lambda,v}}(f_v^h). \tag{7.3.1.1}$$

7.3.2.
By Theorem 4.2.8.1 and the definition of I_{Π_λ} there, for every factorizable test function $f \in \mathcal{S}(G(\mathbb{A}))$ of the form $f = \Delta_H^{S_0,*}\Delta_{G'}^{S_0,*} \prod_{v\in S_0} f_v \times \prod_{v\notin S_0} \mathbf{1}_{G(\mathcal{O}_v)}$, we have

$$I_{Q,\pi}(\lambda, f) = \frac{L^{S_0}\left(\frac{1}{2}, \Pi_\lambda\right)}{L^{S_0,*}(1, \Pi_\lambda, \mathrm{As}')} \prod_{v\in S_0} I_{\Pi_{\lambda,v}}(f_v), \tag{7.3.2.1}$$

where for for every place $v \in S_0$, we introduce the local relative character $I_{\Pi_{\lambda,v}}$ defined by

$$I_{\Pi_{\lambda,v}}(f_v) = \sum_{W_v} \frac{\alpha_v(\Pi_v(f_v)W_v)\overline{\beta_{\eta,v}(W_v)}}{\langle W_v, W_v\rangle_{\mathrm{Whitt},v}}, \quad f_v \in \mathcal{S}(G(F_v)).$$

Here, the sum runs over a K_v-basis of the Whittaker model $\mathcal{W}(\Pi_{\lambda,v}, \psi_{N,v})$, and λ_v, $\beta_{\eta,v}$, $\langle .,. \rangle_{\mathrm{Whitt},v}$ are given by

$$\alpha_v(W_v) = \int_{N_H(F_v)\backslash H(F_v)} W_v(h_v)dh_v$$
$$\beta_{\eta,v}(W_v) = \int_{N'(F_v)\backslash \mathcal{P}'(F_v)} W_v(p_v)\eta_{G',v}(p_v)dp_v,$$
$$\langle W_v, W_v\rangle_{\mathrm{Whitt},v} = \int_{N(F_v)\backslash \mathcal{P}(F_v)} |W_v(p_v)|^2 dp_v.$$

The above expressions, especially $\alpha_v(W_v)$, are all absolutely convergent due to the fact that $\Pi_{\lambda,v}$ is tempered (see [JPSS83, Proposition 8.4]). The above definition implicitly depends on the choice of an additive character ψ of $\mathbb{A}_E/E$ trivial on $\mathbb{A}$ which, up to enlarging S_0, we may assume to be unramified outside of S_0.

7.3.3.
Let f^h be a test function as in (7.3.1.1). Since both sides of (7.3.1.1) are continuous functionals in f_v^h for $v \in S_0$, we may assume that the function f_v^h admits a transfer $f_v \in \mathcal{S}(G(F_v))$ for every $v \in S_0$ using results of [Xue19] and [Zha14b]. Moreover, by the results of those references, we may also assume that for every $h' \in \mathcal{H}^\circ$ with $h' \neq h$, the zero function on $U_{h'}(F_{S_0})$ is a transfer of $f_{S_0} = \prod_{v\in S_0} f_v$. We set $f = \Delta_H^{S_0,*}\Delta_{G'}^{S_0,*} f_{S_0} \times \prod_{v\notin S_0} \mathbf{1}_{G(\mathcal{O}_v)}$. Then, setting $f^{h'} = 0$ for every $h' \in \mathcal{H}^\circ \setminus \{h\}$, the functions f and $(f^{h'})_{h'\in\mathcal{H}^\circ}$ satisfy the assumptions of Theorem 7.1.6.1. Therefore, we have

$$J_\Pi^h(\lambda, f^h) = |S_\Pi|^{-1} I_{Q,\pi}(\lambda, f). \tag{7.3.3.1}$$

7.3.4.
By the local Gan-Gross-Prasad Conjecture [BP20], the classification of cuspidal automorphic representations of U_h in terms of local L-packets [Mok15], [KMSW], all the terms in the definition of $J_\Pi^h(\lambda, f^h)$

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(see (7.1.5.2)) vanish except possibly $J^h_{P,\sigma}(\lambda, f^h)$. So (7.3.3.1) reduces to

$$J^h_{P,\sigma}(\lambda, f^h) = |S_\Pi|^{-1} I_{Q,\pi}(\lambda, f). \tag{7.3.4.1}$$

By [BP21, Theorem 5.4.1] and since Π_v is the local base-change of σ_v, there are explicit constants $\kappa_v \in \mathbb{C}^\times$ for $v \in S_0$ satisfying $\prod_{v\in S_0} \kappa_v = 1$ and such that

$$I_{\Pi_{\lambda,v}}(f_v) = \kappa_v J_{\Sigma_{\lambda,v}}(f^h_v) \tag{7.3.4.2}$$

for every $v \in S_0$. Now (7.3.1.1) results from the combination of (7.3.4.1), the factorization (7.3.2.1) and the local comparison (7.3.4.2).

8. Application to the Gan-Gross-Prasad and Ichino-Ikeda conjectures for Bessel periods

8.1. Groups

8.1.1. Notations

They are as in Sections 2, 3 and 4. In particular, E/F is a quadratic extension of number fields. For every place v of F, we set $E_v = F_v \otimes_F E$ and we let $\mathcal{O}_{E_v}$ be the 'ring of integers' in the quadratic étale extension E_v of F_v. We denote by $\mathbb{A}_E = \prod'_v E_v$ the ring of adeles of E. We fix a nontrivial character $\psi : \mathbb{A}_E/E \to \mathbb{C}^\times$ that is trivial on $\mathbb{A}$. We also set (see §2.1.7)

$$\|x\|_G = \inf_{\gamma\in G(F)} \|\gamma x\|,$$

$$\sigma(x) = 1 + \log\|x\|, \quad \text{for } x \in G(\mathbb{A}).$$

If V is a finite dimensional vector space over F, we also fix a height $\|.\|_{V_\mathbb{A}} : V_\mathbb{A} = V \otimes_F \mathbb{A} \to \mathbb{R}_{\geqslant 1}$ as in [BPCZ22, §2.4.2], and for every place v of F, we denote by $\|.\|_{V_v}$ the restriction of $\|.\|_{V_\mathbb{A}}$ to $V_v := V \otimes_F F_v$. We will also write $\mathcal{S}(V_v)$ for the usual Schwartz-Bruhat space on the vector space V_v (i.e., the usual Schwartz space if v is Archimedean or the space of compactly supported locally constant functions when v is non-Archimedean).

Recall also, that for every integer $n \geqslant 0$, $\mathcal{H}_n$ stands for the set of isomorphism classes of Hermitian forms of rank n for the extension E/F. For each $h \in \mathcal{H}_n$, we denote by $U(h)$ the corresponding unitary group. We also fix, as in 3.1, some Hermitian form of rank one $h_0 \in \mathcal{H}_1$.

8.1.2. Linear groups

For every $k \geqslant 0$, we set $G_k = \mathrm{Res}_{E/F}\,\mathrm{GL}_{k,E}$ equipped with the pair (B_k, T_k) as in §4.1.1. Let $N_k \subset B_k$ be the unipotent radical of B_k. We define two generic characters $\psi_k, \psi_{-k} : [N_k] \to \mathbb{C}^\times$ by

$$\psi_k(u) = \psi\left(\sum_{i=1}^{k-1} u_{i,i+1}\right).$$

$$\psi_{-k}(u) = \psi\left(-\sum_{i=1}^{k-1} u_{i,i+1}\right), \quad u \in [N_k].$$

We also let $P_k \subset G_k$ be the *mirabolic* subgroup consisting of matrices with last row $(0, \dots, 0, 1)$ and $K_k = \prod_v K_{k,v}$ be the standard maximal compact subgroup of $G_k(\mathbb{A})$.

8.1.3. Unitary groups

We define the Hermitian form $h_s \in \mathcal{H}_2$ by $h_s = h_0 \oplus -h_0$. We also fix a basis (x, y) of E^2 consisting of isotropic vectors for h_s (i.e. $h_s(x, x) = h_s(y, y) = 0$) such that $h_s(x, y) = 1$, and we set $X = Ex, Y = Ey$.

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We fix once and for all two positive integers $n \geqslant m$ of the same parity. Thus, $n = m + 2r$ for some $r \geqslant 0$. Let $h_m \in \mathcal{H}_m$. We set $h_{m+1} = h_m \oplus h_0$, $h_n = h_s^{\oplus r} \oplus h_m$, $h_{n+1} = h_n \oplus h_0 = h_s^{\oplus r} \oplus h_{m+1}$ and

$$U_m = U(h_m),\ U_{m+1} = U(h_{m+1}),\ U_n = U(h_n),\ U_{n+1} = U(h_{n+1}).$$

Note that we have natural inclusions $U_m \hookrightarrow U_{m+1} \hookrightarrow U_n \hookrightarrow U_{n+1}$. We also define the following products:

$$\mathcal{G} = U_m \times U_{n+1},\quad U = U_n \times U_{n+1},\quad \widetilde{U} = U_m \times U_{m+1}. \tag{8.1.3.1}$$

For every $0 \leqslant i \leqslant r$, we set

$$x_i = (\underbrace{0,\dots,0}_{2i-2}, x, \underbrace{0,\dots,0}_{1+m+2r-2i}),\ y_i = (\underbrace{0,\dots,0}_{2i-2}, y, \underbrace{0,\dots,0}_{1+m+2r-2i}),\ v_0 = (\underbrace{0,\dots,0}_{2r+m}, 1) \in E^{n+1},$$

and we define the subvector spaces $X_i = \langle x_1,\dots,x_i\rangle$ and $Y_i = \langle y_1,\dots,y_i\rangle$ of E^{n+1} spanned by $(x_1,\dots,x_i)$ and $(y_1,\dots,y_i)$, respectively. Then, $0 = X_0 \subsetneq X_1 \dots \subsetneq X_r$ is a flag of h_{n+1}-isotropic subspaces, and we let $P \subset U_{n+1}$ be the parabolic subgroup stabilizing it. Let N be the unipotent radical of P. We define a character $\psi_N : [N] \to \mathbb{C}^\times$ by

$$\psi_N(u) = \psi\left(\sum_{i=1}^{r-1} h_{n+1}(ux_{i+1}, y_i) + h_{n+1}(uv_0, y_r)\right),\quad u \in [N].$$

The subgroup $U_m \subset U_{n+1}$ normalizes N, and ψ_N is invariant by $U_m(\mathbb{A})$-conjugation. We define the following three subgroups of the respective groups in (8.1.3.1):

$$\mathcal{B} = U_m \ltimes N,\quad U' = U_n,\quad ,\widetilde{U}' = U_m, \tag{8.1.3.2}$$

where the embedding $\mathcal{B} \subset \mathcal{G}$ is the product of the inclusion $\mathcal{B} \subset U_{n+1}$ and the natural projection $\mathcal{B} \to U_m$, whereas the embeddings $U' \subset U$, $\widetilde{U}' \subset \widetilde{U}$ are the diagonal ones. We also let $\psi_{\mathcal{B}} : [\mathcal{B}] \to \mathbb{C}^\times$ be the character that coincides with ψ_N on $[N]$ and is trivial on $[U_m]$.

We will also need some auxiliary parabolic subgroups. First, we let $P' = P \cap U_n$ be the stabilizer of the flag $X_0 \subsetneq \dots \subsetneq X_r$ in U_n. We also let $Q_{n+1} \subset U_{n+1}$ (resp. $Q_n \subset U_n$) be the parabolic subgroup stabilizing the h_{n+1}-isotropic (resp. h_n-isotropic) subspace X_r and we denote by V_{n+1} (resp. V_n) its unipotent radical.

Using the basis $(x_1,\dots,x_r)$ of X_r, we will identify G_r as the subgroup of elements in both U_n and U_{n+1} that stabilize the two isotropic subspaces X_r, Y_r and act trivially on the orthogonal complement $(X_r \oplus Y_r)^\perp$. We then have a semi-direct decomposition

$$N = N_r \ltimes V_{n+1}.$$

We also let $L_n \subset Q_n$ be the Levi factor of Q_n stabilizing Y_r. We have

$$L_n = R_{E/F}GL(X_r) \times U(h_m) = G_r \times U_m. \tag{8.1.3.3}$$

We will also use the natural identification $a_{Q_n}^* \simeq a_{G_r}^* \simeq \mathbb{C}$ sending $s \in \mathbb{C}$ to the unramified character $(g_r, h) \in L_n(\mathbb{A}) \mapsto |\det(g_r)|_{\mathbb{A}_E}^s$. Similarly, for any representation τ of $L_n(\mathbb{A})$, we will denote by τ_s the twist of τ by this character. Also, once we have fixed suitable maximal compact subgroups below giving rise to a Harish-Chandra map $H_{Q_n} : U_n(\mathbb{A}) \to a_{Q_n}$, for each $\phi_n \in I_{Q_n}^{U_n}(\tau)$ and $s \in \mathbb{C}$, we will denote by $\phi_{n,s} \in I_{Q_n}^{U_n}(\tau_s)$ the section given by $g \mapsto \exp(s\langle \det, H_{Q_n}(g)\rangle)\phi_n(g)$. This applies in particular to

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$\phi_n \in \mathcal{A}_{Q_n}(U_n)$, and we will write $E_{Q_n}^{U_n}(\phi_n, s)$ for the Eisenstein series

$$E_{Q_n}^{U_n}(g, \phi_n, s) := \sum_{\gamma \in Q_n(F)\backslash U_n(F)} \phi_{n,s}(\gamma g),$$

which converges absolutely when $\Re(s)$ is large enough.

We also set $Q = Q_n \times U_{n+1}$ (a maximal parabolic subgroup of U) and

$$L = L_n \times U_{n+1} = G_r \times \mathcal{G}$$

(a Levi factor of Q). We define the following subgroup of L:

$$\mathcal{B}^L = N_r \times \mathcal{B}$$

as well as the character $\psi_{\mathcal{B}}^L = \psi_{-r} \boxtimes \psi_{\mathcal{B}}$ of $[\mathcal{B}^L]$. We also set

$$\mathcal{B}' = U' \cap \mathcal{B} = (N_r \times U_m) \ltimes V_n.$$

We fix a finite set S of places of F containing the Archimedean places as well a the places dividing 2 and such that the character ψ and the Hermitian form h_m are both unramified outside S (i.e., there exists a lattice $\Lambda_m \subset E^m$ such that $\Lambda_{m,v} := \Lambda_m \otimes_{\mathcal{O}_E} \mathcal{O}_{E_v}$ is self-dual with respect to h_m for every $v \notin S$). Then, the same holds for the lattices $\Lambda_{m+1} = \Lambda_m \oplus \mathcal{O}_E$, $\Lambda_n = \mathcal{O}_E^{\oplus 2r} \oplus \Lambda_m$ and $\Lambda_{n+1} = \Lambda_n \oplus \mathcal{O}_E$ with respect to the Hermitian forms h_{m+1}, h_n and h_{n+1}, respectively.

For $\ell \in \{m, m+1, n, n+1\}$, we fix a maximal compact subgroup $K_\ell^U = \prod_v K_{\ell,v}^U \subset U_\ell(\mathbb{A})$ such that for every $v \notin S$, $K_{\ell,v}^U$ is the stabilizer of the lattice $\Lambda_\ell \otimes_{\mathcal{O}_E} \mathcal{O}_{E_v}$.

8.2. Measures

8.2.1.

For every linear algebraic group $\mathbb{G}$ defined over F, we have equipped $\mathbb{G}(\mathbb{A})$ with its (left) Tamagawa measure dg; see §2.1.8. Also, for each $\mathbb{G} \in \{U_m, U_n, U_{n+1}, U_{n+2}, G_r, V_n\}$, we fix a factorization $dg = \prod_v dg_v$ into local Haar measures giving $\mathbb{G}(\mathcal{O}_v)$ measure one for almost all v. In the case where $\mathbb{G} = N_k$, it will be convenient to fix more precisely the local measures as follows: for every place v of F, let $d_{\psi_v} x_v$ be the Haar measure on E_v that is self-dual with respect to ψ_v. Then we equip $N_k(F_v)$ with the product measure

$$du_v = \prod_{1 \leqslant i < j \leqslant k} d_{\psi_v} u_{v,i,j}.$$

It is well known, and easy to check, that $du = \prod_v du_v$ is indeed the global Tamagawa measure on $N_k(\mathbb{A})$ (i.e., it gives $[N_k]$ volume one).

Similarly, for every v, there is another natural measure on $G_k(F_v) = \mathrm{GL}_k(E_v)$ defined by

$$d_{\psi_v} g_v = \frac{\prod_{1 \leqslant i,j \leqslant k} d_{\psi_v} g_{v,i,j}}{|\det(g_v)|_{E_v}^k}.$$

We will denote by $\nu(G_{k,v}) \in \mathbb{R}_{>0}$ the quotient $dg_v (d_{\psi_v} g_v)^{-1}$ between the Haar measure we have fixed on $G_r(F_v)$ and the above one. Set

$$\Delta_{G_k}^* = \zeta_E^*(1) \prod_{i=2}^{k} \zeta_E(i),$$

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where $\zeta_E(s)$ denotes the completed Dedekind Zeta function of E and $\zeta_E^*(1)$ its residue at $s = 1$. Similarly for every place v (resp. every finite set T of places), we set

$$\Delta_{G_k,v} = \prod_{i=1}^{k} \zeta_{E_v}(i) \quad (\text{resp. } \Delta_{G_k}^{T,*} = \zeta_E^{T,*}(1) \prod_{i=2}^{k} \zeta_E^T(i)),$$

where ζ_{E_v} denotes the local Eulerian factor[1] of ζ_E at v (resp. $\zeta_E^T(s)$ denotes the corresponding partial Dedekind zeta function and $\zeta_E^{T,*}(1)$ is its residue at $s = 1$). Then, for every non-Archimedean v where ψ_v is unramified, we have

$$\mathrm{vol}(K_{k,v}, d_{\psi_v} g_v) = \Delta_{G_k,v}^{-1},$$

and the global Tamagawa measure on $G_k(\mathbb{A})$ is, by definition,

$$dg = (\Delta_{G_k}^*)^{-1} \prod_v \Delta_{G_k,v} d_{\psi_v} g_v.$$

In particular, it follows that if T is a sufficiently large finite set of places, we have $\nu(G_{k,v}) = \Delta_{G_k,v}$ for $v \notin T$ and

$$\prod_{v \in T} \nu(G_{k,v}) = (\Delta_{G_k}^{T,*})^{-1}. \tag{8.2.1.1}$$

8.2.2.
Finally, we record the following Fourier inversion formula: for every $f \in C_c^\infty(P_{k+1}(F_v))$ setting

$$W_f(g_1, g_2) = \int_{N_{k+1}(F_v)} f(g_1^{-1} u_v g_2) \psi_{k+1}(u_v) du_v, \quad g_1, g_2 \in P_{k+1}(F_v),$$

we have

$$f(p) = \int_{N_k(F_v)\backslash G_k(F_v)} W_f(\gamma, \gamma p) d_{\psi_v} \gamma$$

for every $p \in P_{k+1}(F_v)$, where $d_{\psi_v}\gamma$ denotes the quotient of the Haar measure $d_{\psi_v} g_v$ on $G_k(F_v)$ by the Haar measure $du_v = d_{\psi_v} u_v$ on $N_k(F_v)$. In particular, replacing $d_{\psi_v}\gamma$ by the quotient measure $d\gamma$ of dg_v by du_v, we obtain the following renormalized inversion formula:

$$f(p) = \nu(G_{k,v})^{-1} \int_{N_k(F_v)\backslash G_k(F_v)} W_f(\gamma, \gamma p) d\gamma. \tag{8.2.2.1}$$

8.3. Global periods

8.3.1.
We define the *Whittaker period* on G_r as the linear form

$$\mathcal{P}_{N_r, \psi_{-r}} : \mathcal{A}([G_r]) \ni \phi \mapsto \int_{[N_r]} \phi(u) \psi_{-r}(u) du.$$

(Note the minus sign.)

[1] More precisely, this is really a product of two such factors when v splits in E.

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8.3.2.
Under extra assumptions ensuring absolute convergence (e.g., cuspidality), we will consider the following *global Bessel period* for $\phi \in \mathcal{A}([\mathcal{G}])$ (resp. $\phi \in \mathcal{A}([U])$):

$$\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}}}(\phi) = \int_{[\mathcal{B}]} \phi(s)\psi_{\mathcal{B}}(s)ds.$$

$$(\text{resp. } \mathcal{P}_{U'}(\phi) = \int_{[U']} \phi(h)dh.)$$

For example, it is readily seen that the period $\mathcal{P}_{U'}(\phi)$ is absolutely convergent for $\phi = \phi_n \otimes \phi_{n+1}$, where $\phi_n \in \mathcal{A}([U_n])$ and $\phi_{n+1} \in \mathcal{A}_{cusp}([U_{n+1}])$.

8.3.3.
Again provided it is convergent, for $\phi \in \mathcal{A}([L])$, we define the *mixed Whittaker-Bessel period*

$$\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}}}(\phi) = \int_{[\mathcal{B}^L]} \phi(s)\psi^L_{\mathcal{B}}(s)ds.$$

For example, this period is absolutely convergent when $\phi = \phi_r \otimes \phi'$ where $\phi_r \in \mathcal{A}([G_r])$ and $\phi' \in \mathcal{A}_{cusp}([\mathcal{G}])$, in which case we have

$$\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}}}(\phi) = \mathcal{P}_{N_r,\psi_{-r}}(\phi_r)\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}}}(\phi').$$

8.4. Local Bessel periods

8.4.1.
Let v be a place of F. For every connected reductive group $\mathbb{G}$ defined over F_v, we let $\mathcal{C}^w(\mathbb{G}(F_v))$ be the space of *tempered functions* on $\mathbb{G}(F_v)$ as defined in [BP20, §1.5] (where it is called the *weak Harish-Chandra Schwartz space*). Since we will need it, let us recall quickly its definition. Let $\Xi^{\mathbb{G}}$ Harish-Chandra special spherical function on $\mathbb{G}(F_v)$, which, strictly speaking, depends on the choice of a maximal compact subgroup $\mathbb{K}_v \subset \mathbb{G}(F_v)$ (such a choice has already been made for all the groups we will have to consider). For each $d > 0$, we let $\mathcal{C}^w_d(\mathbb{G}(F_v))$ be the space of functions $f : \mathbb{G}(F_v) \to \mathbb{C}$ such that:

- If v is non-Archimedean, f is biinvariant by a compact-open subgroup and we have

$$|f(g)| \ll \Xi^{\mathbb{G}}(g)\sigma(g)^d, \text{ for } g \in \mathbb{G}(F_v);$$

- If v is Archimedean, f is C^∞ and for every $X, Y \in \mathcal{U}(\mathrm{Lie}(\mathbb{G}(F_v)))$,

$$|(L(X)R(Y)f)(g)| \ll_{X,Y} \Xi^{\mathbb{G}}(g)\sigma(g)^d, \text{ for } g \in \mathbb{G}(F_v).$$

When v is Archimedean, $\mathcal{C}^w_d(\mathbb{G}(F_v))$ is naturally a Fréchet space, whereas if v is non-Archimedean, $\mathcal{C}^w_d(\mathbb{G}(F_v))$ is a strict LF space. By definition, the space of tempered functions is $\mathcal{C}^w(\mathbb{G}(F_v)) = \bigcup_{d>0} \mathcal{C}^w_d(\mathbb{G}(F_v))$. It is equipped with the direct limit locally convex topology, and it contains $C^\infty_c(\mathbb{G}(F_v))$ as a dense subspace. (However, note that $C^\infty_c(\mathbb{G}(F_v))$ is not dense in $\mathcal{C}^w_d(\mathbb{G}(F_v))$ for any $d > 0$.)

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8.4.2.
By definition, the local period $\mathcal{P}_{U',v}$ is the linear form

$$\mathcal{P}_{U',v} : \mathcal{C}^w(U(F_v)) \ni f \mapsto \int_{U'(F_v)} f(h)dh,$$

the integral being absolutely convergent by [BP20, Lemma 6.5.1(i)].

Moreover, it is shown in [BP20, Proposition 7.1.1] that the linear form

$$C_c^\infty(\mathcal{G}(F_v)) \ni f_v \mapsto \int_{\mathcal{B}(F_v)} f_v(s)\psi_{\mathcal{B},v}(s)ds$$

extends by continuity to $\mathcal{C}^w(\mathcal{G}(F_v))$. We denote this unique continuous extension by $\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}$ and call it the *local Bessel period*.

A similar argument shows that the linear forms

$$C_c^\infty(G_r(F_v)) \ni f_v \mapsto \int_{N_r(F_v)} f_v(u)\psi_{-r,v}(u)du,$$

$$C_c^\infty(L(F_v)) \ni f_v \mapsto \int_{\mathcal{B}^L(F_v)} f_v(s)\psi^L_{\mathcal{B},v}(s)ds$$

extend by continuity to $\mathcal{C}^w(G_r(F_v))$ and $\mathcal{C}^w(L(F_v))$, respectively. We denote these unique continuous extensions by $\mathcal{P}_{N_r,\psi_{-r},v}$, $\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}$ and call them the local *Whittaker period* and *Whittaker-Bessel period*, respectively.

8.4.3.
Let $(\mathbb{G},\mathbb{H}) \in \{(U,U'),(\mathcal{G},(\mathcal{B},\psi_{\mathcal{B}})),(L,(\mathcal{B}^L,\psi^L_{\mathcal{B}})),(G_r,(N_r,\psi_{-r}))\}$. Then, for any tempered irreducible representation σ_v of $\mathbb{G}(F_v)$ equipped with an invariant inner product $(\cdot,\cdot)_v$ and vectors $\phi_v, \phi'_v \in \sigma_v$, the matrix coefficient

$$f_{\phi_v,\phi'_v} : g \in \mathbb{G}(F_v) \mapsto (\sigma_v(g)\phi_v, \phi'_v)_v$$

belongs to $\mathcal{C}^w(\mathbb{G}(F_v))$. We set

$$\mathcal{P}_{\mathbb{H},v}(\phi_v,\phi'_v) := \mathcal{P}_{\mathbb{H},v}(f_{\phi_v,\phi'_v}).$$

8.5. Relation between global periods

8.5.1.

Proposition 8.5.1.1. *Let $\phi_n \in \mathcal{A}_{Q_n}(U_n)$ and $\phi_{n+1} \in \mathcal{A}_{cusp}(U_{n+1})$. Then, there exists $c > 0$ such that for $s \in \mathcal{H}_{>c}$, we have the identity*

$$\mathcal{P}_{U'}(E^{U_n}_{Q_n}(\phi_n,s)\otimes\phi_{n+1}) = \int_{\mathcal{B}'(\mathbb{A})\backslash U'(\mathbb{A})} \mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}}}\left(R(h)(\phi_{n,s}\otimes\phi_{n+1})\right)dh,$$

where the right expression is absolutely convergent.

Proof. For $\Re(s)$ sufficiently large, we have

$$E^{U_n}_{Q_n}(h,\phi_n,s) = \sum_{\gamma\in Q_n(F)\backslash U_n(F)} \phi_{n,s}(\gamma h), \quad h \in [U_n],$$

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so that (the resulting expression being absolutely convergent by cuspidality of ϕ_{n+1})

$$\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s) \otimes \phi_{n+1}) = \int_{[U_n]} \sum_{\gamma \in Q_n(F)\backslash U_n(F)} \phi_{n,s}(\gamma h)\phi_{n+1}(h)dh \tag{8.5.1.1}$$
$$= \int_{Q_n(F)\backslash U_n(\mathbb{A})} \phi_{n,s}(h)\phi_{n+1}(h)dh$$
$$= \int_{L_n(F)V_n(\mathbb{A})\backslash U_n(\mathbb{A})} \phi_{n,s}(h)\phi_{n+1,V_n}(h)dh,$$

where

$$\phi_{n+1,V_n}(h) := \int_{[V_n]} \phi_{n+1}(vh)dv.$$

We claim that we have a Fourier expansion

$$\phi_{n+1,V_n}(h) = \sum_{\delta \in N_r(F)\backslash G_r(F)} \phi_{n+1,N,\psi_N}(\delta h) \tag{8.5.1.2}$$

for every $h \in U_{n+1}(\mathbb{A})$ where we have set

$$\phi_{n+1,N,\psi_N}(h) := \int_{[N_{r+1}]} \phi_{n+1,V_n}(uh)\psi_{r+1}(u)du = \int_{[N]} \phi_{n+1}(uh)\psi_N(u)du.$$

Indeed, the subgroup G_rN of U_{n+1} contains V_n as a normal subgroup, and the quotient G_rN/V_n can be identified with the mirabolic subgroup P_{r+1} of G_{r+1} via restriction to the subspace $\langle x_1, \ldots, x_r, x_{r+1}\rangle$, where we have set $x_{r+1} = v_0$. It readily follows from the cuspidality of ϕ_{n+1} that for any $h \in U_{n+1}(\mathbb{A})$, the function $p \in [P_{r+1}] \mapsto \phi_{n+1,V_n}(ph)$ is cuspidal in the sense of [Cog08, §1.1] and therefore by *loc. cit.* that we have the Fourier expansion (8.5.1.2).

Replacing ϕ_{n+1,V_n} by its Fourier expansion (8.5.1.2) in (8.5.1.1), we formally obtain (remembering the decomposition (8.1.3.3) for L_n)

$$\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s) \otimes \phi_{n+1}) = \int_{N_r(F)U_m(F)V_n(\mathbb{A})\backslash U_n(\mathbb{A})} \phi_{n,s}(h)\phi_{n+1,N,\psi_N}(h)dh$$
$$= \int_{\mathcal{B}'(\mathbb{A})\backslash U'(\mathbb{A})} \mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L}\left(R(h)(\phi_{n,s} \otimes \phi_{n+1})\right) dh.$$

To justify this formal computation, it remains to check that the integral

$$\int_{N_r(F)U_m(F)V_n(\mathbb{A})\backslash U_n(\mathbb{A})} \left|\phi_{n,s}(h)\phi_{n+1,N,\psi_N}(h)\right| dh$$

converges for $\Re(s)$ sufficiently large. This can be rewritten as

$$\int_{P'(\mathbb{A})\backslash U'(\mathbb{A})} \int_{[U_m]\times T_r(\mathbb{A})\times[N_r]} \left|\phi_{n,s}(uagh)\phi_{n+1,N,\psi_N}(agh)\right| \delta_{Q_n}(a)^{-1}\delta_{B_r}(a)^{-1} dudadgdh.$$

Thus, as $P'(\mathbb{A})\backslash U'(\mathbb{A})$ is compact, it suffices to check the convergence of

$$\int_{[U_m]\times T_r(\mathbb{A})\times[N_r]} \left|\phi_{n,s}(uag)\phi_{n+1,N,\psi_N}(ag)\right| \delta_{Q_n}(a)^{-1}\delta_{B_r}(a)^{-1} dudadg =$$
$$\int_{[U_m]\times T_r(\mathbb{A})\times[N_r]} \left|\phi_{n}(uag)\phi_{n+1,N,\psi_N}(ag)\right| \delta_{Q_n}(a)^{-1}\delta_{B_r}(a)^{-1} |\det a|_{\mathbb{A}_E}^s dudadg.$$

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Let us embed $T_r(\mathbb{A})$ into $\mathbb{A}_E^r$ in the natural way. Then, by [BPCZ22, Lemma 2.6.1.1] and since ϕ_{n+1} is cuspidal, for every $R > 0$, we have

$$\left|\phi_{n+1,N,\psi_N}(ag)\right| \ll_R \|a\|_{\mathbb{A}_E^r}^{-R}\|g\|_{U_m}^{-R}, \quad (a,g) \in T_r(\mathbb{A}) \times [U_m]. \tag{8.5.1.3}$$

Moreover, since ϕ_n is of moderate growth, we can find $D > 0$ such that

$$|\phi_n(uag)| \ll \|a\|^D\|g\|_{U_m}^D, \quad (u,a,g) \in [N_r] \times T_r(\mathbb{A}) \times [U_m]. \tag{8.5.1.4}$$

Combining (8.5.1.3) with (8.5.1.4), we are eventually reduced to the following readily checked property: for every s large enough, there exists $R > 0$ such that the integral

$$\int_{T_r(\mathbb{A})} \|a\|^D\|a\|_{\mathbb{A}_E^r}^{-R}\delta_{Q_n}(a)^{-1}\delta_{B_r}(a)^{-1}|\det a|_{\mathbb{A}_E}^s da$$

converges. □

8.6. *Relations between local periods*

8.6.1.
Let v be a place of F and let $\tau, \sigma_m, \sigma_{n+1}$ be irreducible representations of $G_r(F_v)$, $U_m(F_v)$ and $U_{n+1}(F_v)$, respectively. We set $\sigma = \sigma_m \boxtimes \sigma_{n+1}$ (an irreducible representation of $\mathcal{G}(F_v) = U_m(F_v) \times U_{n+1}(F_v)$). Let

$$\mathcal{L} \in \mathrm{Hom}_{\mathcal{B}^L(F_v)}(\tau \boxtimes \sigma, \psi_{\mathcal{B},v}^L).$$

By multiplicity one results [ARS10], [GGP12, Corollary 15.3], [JSZ10], $\mathcal{L}$ factors as $\mathcal{L} = \mathcal{L}^W \otimes \mathcal{L}^B$, where $\mathcal{L}^W \in \mathrm{Hom}_{N_r(F_v)}(\tau, \psi_{-r,v})$ and $\mathcal{L}^B \in \mathrm{Hom}_{\mathcal{B}(F_v)}(\sigma, \psi_{\mathcal{B},v})$.

For every $s \in \mathbb{C}$, we set $\tau_s = \tau|\det|_{E_v}^s$ and we denote by $I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m)$ the normalized parabolic induction of $\tau_s \boxtimes \sigma_m$ to $U_n(F_v)$.

Proposition 8.6.1.1. *There exists $c \in \mathbb{R}$ such that the functional*

$$\mathcal{L}_s^{U'} : \phi_{n,s} \otimes \phi_{n+1} \in I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m) \otimes \sigma_{n+1} \mapsto \int_{\mathcal{B}'(F_v)\backslash U'(F_v)} \mathcal{L}(\phi_{n,s}(h) \otimes \sigma_{n+1}(h)\phi_{n+1})dh$$

converges absolutely for $s \in \mathcal{H}_{>c}$. If, moreover, both σ and τ are tempered, we may take $c = -1/2$. Furthermore, for s with sufficiently large real part, the following assertions are equivalent:

1. *There exist $\phi_\tau \in \tau$, $\phi_m \in \sigma_m$ and $\phi_{n+1} \in \sigma_{n+1}$ such that $\mathcal{L}(\phi_\tau \otimes \phi_m \otimes \phi_{n+1}) \neq 0$;*
2. *There exist $\phi_{n,s} \in I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m)$ and $\phi_{n+1} \in \sigma_{n+1}$ such that $\mathcal{L}_s^{U'}(\phi_{n,s} \otimes \phi_{n+1}) \neq 0$.*

Proof. Since $P' = T_r\mathcal{B}'$ is a parabolic subgroup of U', it suffices to check the convergence of the integral

$$\int_{T_r(F_v)} \mathcal{L}(\phi_{n,s}(a) \otimes \sigma_{n+1,v}(a)\phi_{n+1})\delta_{P'}(a)^{-1}da =$$
$$\int_{T_r(F_v)} \mathcal{L}((\tau(a) \otimes \sigma(a))\phi_{n,s}(1) \otimes \phi_{n+1})|\det a|_{E_v}^s\delta_{Q_n}(a)^{1/2}\delta_{P'}(a)^{-1}da$$

for $\Re(s) \gg 0$.

Lemma 8.6.1.2. *There exists $D > 0$ such that for every $\phi \in \tau \boxtimes \sigma$ and $R > 0$, we have*

$$|\mathcal{L}((\tau(a) \otimes \sigma(a))\phi)| \ll_R \|a\|_{E_v^r}^{-R}\|a\|^D, \quad \textit{for } a \in T_r(F_v). \tag{8.6.1.1}$$

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If, moreover, σ and τ are tempered, for every $\phi \in \tau \boxtimes \sigma$ and $R > 0$, we have

$$|\mathcal{L}((\tau(a) \otimes \sigma(a))\phi)| \ll_R \|a\|_{E_v^r}^{-R} \Xi^{G_{r,v}}(a)\Xi^{U_{n+1,v}}(a), \quad \text{for } a \in T_r(F_v). \tag{8.6.1.2}$$

Proof. It suffices to prove the lemma when ϕ is a pure tensor (i.e., it is of the form $\phi = \phi_\tau \otimes \phi_m \otimes \phi_{n+1}$, where $\phi_\tau \in \tau$, $\phi_m \in \sigma_m$ and $\phi_{n+1} \in \sigma_{n+1}$). Indeed, in the case where v is non-Archimedean, every vector in $\tau \boxtimes \sigma$ is a sum of pure tensors, whereas if v is Archimedean, the claimed inequalities would automatically extend from the algebraic tensor product to the completed tensor product by the Banach-Steinhaus theorem (see, for example, [Trè67, Theorem 34.1]).

Using the equality $\mathcal{L} = \mathcal{L}^W \otimes \mathcal{L}^B$, we are therefore reduced to show the existence of $D > 0$ such that for every $R > 0$ (resp. that when τ, σ are tempered for every $R > 0$), we have

$$\left|\mathcal{L}^W(\tau(a)\phi_\tau)\right| \ll \|a\|^D \quad (\text{resp. } \left|\mathcal{L}^W(\tau(a)\phi_\tau)\right| \ll \Xi^{G_{r,v}}(a)\ ) \tag{8.6.1.3}$$

and

$$\left|\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1})\right| \ll_R \|a\|_{E_v^r}^{-R}\|a\|^D \tag{8.6.1.4}$$

$$(\text{resp. } \left|\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1})\right| \ll_R \|a\|_{E_v^r}^{-R}\Xi^{U_{n+1,v}}(a)\ )$$

for $a \in T_r(F_v)$.

The estimates (8.6.1.3) and (8.6.1.4) can be established along the same lines as Lemma B.2.1 and Lemma 7.3.1 (i) of [BP20], respectively. More precisely, in the tempered case, (8.6.1.3) is a direct application of Lemma B.2.1 of loc.cit. The same inequality for general representations (and for a suitable D) is a consequence of the continuity of $\mathcal{L}^W$ when v is Archimedean, whereas, for v non-Archimedean, by the same argument as in [BP20, Lemma B.2.1], we can bound the function $a \in T_r(F_v) \mapsto \left|\mathcal{L}^W(\tau(a)\phi_\tau)\right|$ by a matrix coefficient of τ_v which is in turn essentially bounded by $\|a\|^D$ for some $D > 0$.

As for (8.6.1.4), we first note that

$$\psi_{\mathcal{B},v}(e^{\mathrm{Ad}(a)X})\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1}) = \mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(ae^X)\phi_{n+1}) \tag{8.6.1.5}$$

for every $(a, X) \in T_r(F_v) \times \mathrm{Lie}(\mathcal{B}(F_v))$. Furthermore, when v is Archimedean, differentiating the above identities yields

$$d\psi_{\mathcal{B},v}(\mathrm{Ad}(a)X)\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1}) = \mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\sigma_{n+1}(X)\phi_{n+1}), \tag{8.6.1.6}$$

where $d\psi_{\mathcal{B},v} : Lie(\mathcal{B}(F_v)) \to \mathbb{C}$ denotes the differential of $\psi_{\mathcal{B},v}$ at 1. From (8.6.1.6), we deduce that for every linear form $\lambda : E_v^r \to F_v$, there exists $X_\lambda \in Lie(\mathcal{B}(F_v))$ such that

$$\lambda(a)\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1}) = \mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\sigma_{n+1}(X_\lambda)\phi_{n+1})$$

for every $a \in T_r(F_v)$. Similarly, when v is non-Archimedean, we deduce from (8.6.1.5) and the smoothness of ϕ_{n+1} the existence of $C > 0$ such that $\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1}) = 0$ unless $\|a\|_{E_v^r} \leqslant C$. Combining these two facts, we are now reduced to show the existence of $D > 0$ such that (resp. that for σ tempered we have)

$$\left|\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1})\right| \ll \|a\|^D \quad (\text{resp. } \left|\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1})\right| \ll \Xi^{U_{n+1,v}}(a))$$

for $a \in T_r(F_v)$. The case where σ is tempered is a direct application of [BP20, Lemma 7.3.1(i)], whereas the case of a general representation follows from continuity of $\mathcal{L}^B$ in the Archimedean case or an argument similar to that of loc. cit. to show that $a \mapsto \mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(a)\phi_{n+1})$ is bounded by a matrix coefficient of σ in the non-Archimedean case. □

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By the lemma, and since there exists $d > 0$ such that (see [Wal03, Lemme II.1.1])

$$\Xi^{G_{r,v}}(a)\Xi^{U_{n+1,v}}(a) \ll \delta_{B_r}(a)^{1/2}\delta_P(a)^{1/2}\sigma(a)^d, \quad \text{for } a \in T_r(F_v),$$

the convergence part of the proposition reduces to the two following readily checked facts:

- Let $D > 0$. Then, we can find $c > 0$ such that for every $s > c$ and for suitable $R > 0$, the integral

$$\int_{T_r(F_v)} \|a\|^D \|a\|_{E_v^r}^{-R} |\det a|_{E_v}^s \delta_{Q_n}(a)^{1/2}\delta_{P'}(a)^{-1} da$$

  converges;
- For every $s > -1/2$, we can find $R > 0$ such that the integral

$$\int_{T_r(F_v)} \|a\|_{E_v^r}^{-R} |\det a|_{E_v}^s \delta_{B_r}(a)^{1/2}\delta_P(a)^{1/2}\delta_{Q_n}(a)^{1/2}\delta_{P'}(a)^{-1}\sigma(a)^d da =$$
$$\int_{T_r(F_v)} \|a\|_{E_v^r}^{-R} |\det a|_{E_v}^{s+1/2}\sigma(a)^d da$$

  converges.

The implication 2. $\Rightarrow$ 1. is clear from the definition of $\mathcal{L}_s^{U'}$. Let us show the converse. Thus, we assume that $\mathcal{L}$ is not identically zero, and we aim to prove that the same holds for $\mathcal{L}_s^{U'}$ for $\Re(s) \gg 0$. First, by the equality

$$\mathcal{L}_s^{U'}(\phi_{n,s} \otimes \phi_{n+1})$$
$$= \int_{Q_n(F_v)\backslash U_n(F_v)} \int_{N_r(F_v)\backslash G_r(F_v)} \mathcal{L}((\tau(g) \otimes \sigma(gh))\phi_{n,s}(h) \otimes \phi_{n+1})|\det(g)|_{E_v}^s \delta_{Q_n}(g)^{-1/2} dg dh$$

and since the space of $I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m)$ is stable by multiplication by functions in $C^\infty(Q_n(F_v)\backslash U_n(F_v))$, it suffices to show the existence $\phi_\tau \in \tau$, $\phi_m \in \sigma_m$ and $\phi_{n+1} \in \sigma_{n+1}$ such that

$$\int_{N_r(F_v)\backslash G_r(F_v)} \mathcal{L}(\tau(g)\phi_\tau \otimes \phi_m \otimes \sigma_{n+1}(g)\phi_{n+1})|\det(g)|_{E_v}^s \delta_{Q_n}(g)^{-1/2} dg \neq 0.$$

Let $f \in \mathcal{S}(V_{n+1}(F_v))$, $g \in G_r(F_v)$, $\phi_m \in \sigma_m$ and $\phi_{n+1} \in \sigma_{n+1}$. From the equivariance property of $\mathcal{L}^B$, we deduce

$$\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(g)\sigma_{n+1}(f)\phi_{n+1}) = \hat{f}(g^* y_r)\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(g)\phi_{n+1}),$$

where for $y \in Y_r(F_v)$, we have set

$$\hat{f}(y) = \int_{V_{n+1}(F_v)} f(u)\psi_v(h_{n+1}(uv_0, y))du.$$

By the theory of Fourier transform, $f \mapsto \hat{f}$ induces a surjective map $\mathcal{S}(V_{n+1}(F_v)) \to \mathcal{S}(Y_r(F_v))$, whereas the map $g \mapsto g^* y_r$ induces an embedding $\mathcal{S}(P_r(F_v)\backslash G_r(F_v)) \hookrightarrow \mathcal{S}(Y_r(F_v))$ (where we recall that P_r denotes the mirabolic subgroup of G_r). Consequently, it suffices to prove the existence of $\phi_\tau \in \tau$, $\phi_m \in \sigma_m$, $\phi_{n+1} \in \sigma_{n+1}$ and $f \in \mathcal{S}(P_r(F_v)\backslash G_r(F_v))$ such that

$$\int_{P_r(F_v)\backslash G_r(F_v)} f(g) \int_{N_r(F_v)\backslash P_r(F_v)} \mathcal{L}^W(\tau(pg)\phi_\tau)$$
$$\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(pg)\phi_{n+1})|\det(pg)|_{E_v}^s \delta_{Q_n}(pg)^{-1/2}|\det(p)|_{E_v}^{-1} dp dg \neq 0$$

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or, equivalently, the existence of $\phi_\tau \in \tau$, $\phi_m \in \sigma_m$, $\phi_{n+1} \in \sigma_{n+1}$ such that

$$\int_{N_{r-1}(F_v)\backslash G_{r-1}(F_v)} \mathcal{L}^W(\tau(h)\phi_\tau)\mathcal{L}^B(\phi_m \otimes \sigma_{n+1}(h)\phi_{n+1})|\det(h)|_{E_v}^{s-1}\delta_{Q_n}(h)^{-1/2}dh \neq 0.$$

By [GK75, Theorem 6] and [Kem15, Theorem 1], for every $f \in C_c^\infty(N_{r-1}(F_v)\backslash G_{r-1}(F_v))$, we can find $\phi_\tau \in \tau$ such that $\mathcal{L}^W(\tau(h)\phi_\tau) = f(h)$ for $h \in G_{r-1}(F_v)$, and from this, the claim follows readily from the nonvanishing of $\mathcal{L}^B$. □

8.6.2.

Proposition 8.6.2.1. *Assume that σ and τ are tempered. Then, for every $\phi_n, \phi'_n \in I_{Q_n(F_v)}^{U_n(F_v)}(\tau \boxtimes \sigma_m)$ and $\phi_{n+1}, \phi'_{n+1} \in \sigma_{n+1}$, we have*

$$\begin{aligned}&\mathcal{P}_{U',v}(\phi_n \otimes \phi_{n+1}, \phi'_n \otimes \phi'_{n+1}) \qquad (8.6.2.1)\\ &\quad = \nu(G_{r,v})^{-1}\int_{(\mathcal{B}'(F_v)\backslash U'(F_v))^2} \mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L,v}(\phi_n(h_1)\otimes\sigma_{n+1}(h_1)\phi_{n+1}, \phi'_n(h_2)\otimes\sigma_{n+1}(h_2)\phi'_{n+1})dh_1dh_2.\end{aligned}$$

Proof. By definition of the local period $\mathcal{P}_{U',v}$ and of the invariant inner product on $I_{Q_n(F_v)}^{U_n(F_v)}(\tau \boxtimes \sigma_m)$, we have

$$\begin{aligned}&\mathcal{P}_{U'_v}(\phi_n \otimes \phi_{n+1}, \phi'_n \otimes \phi'_{n+1}) \qquad (8.6.2.2)\\ &\quad = \int_{U'(F_v)}\int_{Q_n(F_v)\backslash U_n(F_v)} (\phi_n(h_2h_1), \phi'_n(h_2))dh_2(\sigma_{n+1}(h_1)\phi_{n+1}, \phi'_{n+1})dh_1.\end{aligned}$$

The above double integral is absolutely convergent. Indeed, from [CHH88, Theorem 2] and [Wal03, Lemme II.1.6], we have

$$\int_{Q_n(F_v)\backslash U_n(F_v)} \left|(\phi_n(h_2h_1), \phi'_n(h_2))\right| dh_2 \ll \Xi^{U_{n,v}}(h_1), \quad h_1 \in U_n(F_v),$$

and $(\sigma_{n+1}(h_1)\phi_{n+1}, \phi'_{n+1}) \ll \Xi^{U_{n+1,v}}(h_1)$, whereas $\Xi^{U_v} = \Xi^{U_{n,v}}\Xi^{U_{n+1,v}}$ is integrable on $U'(F_v)$ ([BP20, Lemme 6.5.1(i)]).

For the next lemma, we set $Q' := Q_n$ that we consider as a subgroup of $G_r \times \mathcal{G} = L_n \times U_{n+1}$ via the product of the natural surjection $Q_n \twoheadrightarrow L_n$ and inclusion $Q_n \subset U_{n+1}$.

Lemma 8.6.2.2. *For every $f \in \mathcal{C}^w(G_r(F_v) \times \mathcal{G}(F_v))$, we have the identity*

$$\int_{Q'(F_v)} f(q)\delta_{Q'}(q)^{1/2}d_Lq = \nu(G_{r,v})^{-1}\int_{(N_r(F_v)\backslash G_r(F_v))^2} \mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L,v}(L(g_1)R(g_2)f)\delta_{Q_n}(g_1g_2)^{-1/2}dg_1dg_2, \qquad (8.6.2.3)$$

where both sides are absolutely convergent.

Indeed, assuming the lemma for the moment, from (8.6.2.2), we obtain

$$\begin{aligned}
&\nu(G_{r,v})\mathcal{P}_{U'_v}(\phi_n\otimes\phi_{n+1},\phi'_n\otimes\phi'_{n+1})\\
&=\nu(G_{r,v})\int_{(Q_n(F_v)\backslash U_n(F_v))^2}\int_{Q_n(F_v)}(\phi_n(qh_1),\phi'_n(h_2))(\sigma_{n+1}(qh_1)\phi_{n+1},\sigma_{n+1}(h_2)\phi'_{n+1})d_Lq\,dh_1dh_2\\
&=\int_{(Q_n(F_v)\backslash U_n(F_v))^2}\int_{(N_r(F_v)\backslash G_r(F_v))^2}\\
&\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}(\phi_n(g_1h_1)\otimes\sigma_{n+1}(g_1h_1)\phi_{n+1},\phi'_n(g_2h_2)\otimes\sigma_{n+1}(g_2h_2)\phi'_{n+1})\delta_{Q_n}(g_1g_2)^{-1}dg_1dg_2dh_1dh_2\\
&=\int_{(\mathcal{B}'(F_v)\backslash U'(F_v))^2}\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}(\phi_n(h_1)\otimes\sigma_{n+1}(h_1)\phi_{n+1},\phi'_n(h_2)\otimes\sigma_{n+1}(h_2)\phi'_{n+1})dh_1dh_2,
\end{aligned}$$

where in the second equality, we have applied the lemma thanks to the fact that the function

$$(g_r,g_m,g_{n+1})\mapsto\big((\tau(g_r)\otimes\sigma_m(g_m))\phi_n(h_1),\phi'_n(h_2)\big)\big(\sigma_{n+1}(g_{n+1})\sigma_{n+1}(h_1)\phi_{n+1},\sigma_{n+1}(h_2)\phi'_{n+1}\big)$$

belongs to $\mathcal{C}^w(G_r(F_v)\times\mathcal{G}(F_v))$ and the proposition is therefore established.

Proof. (of Lemma 8.6.2.2). First, we check that both sides are convergent and define continuous functionals on $\mathcal{C}^w(G_r(F_v)\times\mathcal{G}(F_v))$.

For the left-hand side, we can use the identity ([Wal03, Lemme II.1.6])

$$\Xi^{U_v}(g)=\int_{K_v^{U'}}\Xi^{G_{r,v}\times\mathcal{G}_v}(l(kg))\delta_Q(l(kg))^{1/2}dk,\quad\text{for } g\in U(F_v),$$

where, for $g\in U(F_v)$, $l(g)$ denotes any element in $L(F_v)=G_r(F_v)\times\mathcal{G}(F_v)$ such that $l(g)^{-1}g\in(V_n(F_v)\times 1)K_v^U$, which in turn implies

$$\begin{aligned}
\int_{Q'(F_v)}\Xi^{G_{r,v}\times\mathcal{G}_v}(q)\sigma(q)^d\delta_{Q'}(q)^{1/2}d_Lq&=\int_{U'(F_v)}\Xi^{G_{r,v}\times\mathcal{G}_v}(l(h))\sigma(l(h))^d\delta_Q(l(h))^{1/2}dh\\
&\ll\int_{U'(F_v)}\Xi^{U_v}(h)\sigma(h)^d dh<\infty
\end{aligned}$$

for every $d>0$.

For the right-hand side, as in the proof of Proposition 8.6.1.1, it suffices to show for every $d>0$ the existence of $d'>0$ as well as a continuous semi-norm ν_d on $\mathcal{C}^w_d(G_r(F_v)\times\mathcal{G}(F_v))$ such that

$$\left|\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}(L(a_1)R(a_2)f)\right|\ll\Xi^{G_{r,v}\times U_{n+1,v}}(a_1)\Xi^{G_{r,v}\times U_{n+1,v}}(a_2)\sigma(a_1)^{d'}\sigma(a_2)^{d'}\nu_d(f)\qquad(8.6.2.4)$$

for $f\in\mathcal{C}^w_d(G_r(F_v)\times\mathcal{G}(F_v))$ and $a_1,a_2\in T_r(F_v)$. Such an inequality can be proved along the same lines as [BP20, Lemma 7.3.1(ii)].

Now that we know that both sides of (8.6.2.3) define continuous functionals on $\mathcal{C}^w(G_r(F_v)\times\mathcal{G}(F_v))$, by density it suffices to check the claimed identity for $f=f_1\otimes f_2\in C_c^\infty(G_r(F_v))\otimes C_c^\infty(\mathcal{G}(F_v))$. The subgroup $G_r\mathcal{B}$ of U_{n+1} contains $U_m\ltimes V_n$ as a normal subgroup, and we may define a function $f_3\in C_c^\infty(G_r(F_v)\mathcal{B}(F_v)/U_m(F_v)V_n(F_v))$ by

$$f_3(p)=\delta_{Q_n}(p)^{1/2}\int_{U_m(F_v)\times V_n(F_v)}f_2(phu)du\,dh,\quad p\in G_r(F_v)\mathcal{B}(F_v).$$

Moreover, restriction to the susbpace $\langle x_1,\dots,x_{r+1}\rangle$, where we have again set $x_{r+1}=v_0$, induces an identification $G_r\mathcal{B}/U_mV_n\simeq P_{r+1}$. Seeing f_3 as a test function on $P_{r+1}(F_v)$ in this way, the identity of

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the lemma becomes

$$\int_{G_r(F_v)} f_1(g)f_3(g)dg = \nu(G_{k,v})^{-1}\int_{(N_r(F_v)\backslash G_r(F_v))^2} W_{f_1}(g_1,g_2)W_{f_3}(g_1,g_2)dg_2dg_1, \qquad (8.6.2.5)$$

where we have set

$$W_{f_1}(g_1,g_2) = \int_{N_r(F_v)} f_1(g_1^{-1}ug_2)\psi_r(u)^{-1}du, \quad W_{f_3}(g_1,g_2) = \int_{N_{r+1}(F_v)} f_3(g_1^{-1}ug_2)\psi_{r+1}(u)du.$$

But (8.6.2.5) is now a direct consequence of the Fourier inversion formula (8.2.2.1). □

□

8.7. Unramified computation

8.7.1.

We continue with the setting of the previous subsection assuming, moreover, that $v \notin S$ and the representations $\tau, \sigma_m, \sigma_{n+1}$ are unramified in the sense that $\tau^{K_{r,v}} \neq 0$, $\sigma_m^{K^U_{m,v}} \neq 0$ and $\sigma_{n+1}^{K^U_{n+1,v}} \neq 0$. Note that this implies $I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m)^{K^U_{n,v}} \neq 0$ (i.e., $I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m)$ is also unramified).

8.7.2.

Proposition 8.7.2.1. *For $\Re(s)$ sufficiently large, $\phi^\circ_{n,s} \in I_{Q_n(F_v)}^{U_n(F_v)}(\tau_s \boxtimes \sigma_m)^{K^U_{n,v}}$ and $\phi^\circ_{n+1} \in \sigma_{n+1}^{K^U_{n+1,v}}$, we have*

$$\mathcal{L}_s^{U'}(\phi^\circ_{n,s}\otimes\phi^\circ_{n+1}) = \frac{\mathrm{vol}(K^U_{n,v})}{\mathrm{vol}(K^U_{n,v}\cap\mathcal{B}'(F_v))}\frac{L\left(\frac{1}{2}+s,\tau\times\sigma_{n+1}\right)}{L(1+s,\tau^c\times\sigma_m)L(1+2s,\tau,\mathrm{As}^{(-1)^m})}\mathcal{L}(\phi^\circ_{n,s}(1)\otimes\phi^\circ_{n+1}). \qquad (8.7.2.1)$$

Proof. Let $\phi^\circ_\tau \in \tau^{K_{r,v}}$ and $\phi^\circ_m \in \sigma_m^{K^U_{m,v}}$ be such that $\phi^\circ_n(1) = \phi^\circ_\tau\otimes\phi^\circ_m$. Recall the factorization $\mathcal{L} = \mathcal{L}^W \otimes \mathcal{L}^B$. Up to scaling, we may assume, without loss of generality, that

$$\mathcal{L}^W(\phi^\circ_\tau) = 1 \text{ and } \mathcal{L}^B(\phi^\circ_m\otimes\phi^\circ_{n+1}) = 1.$$

Let $P' = P\cap U' \subset U'$ be the parabolic subgroup stabilizing the flag

$$0 = X_0 \subset X_1 \subset \ldots \subset X_r.$$

Then, we have $P' = \mathcal{B}' \rtimes T_r$ and from the Iwasawa decomposition $U'(F_v) = P'(F_v)K^U_{n,v}$, we obtain

$$\mathcal{L}_s^{U'}(\phi^\circ_{n,s}\otimes\phi^\circ_{n+1}) = \frac{vol(K^U_{n,v})}{vol(K^U_{n,v}\cap\mathcal{B}'(F_v))}\sum_{t\in\Lambda_r}\mathcal{L}(\phi^\circ_{n,s}(t)\otimes\sigma_{n+1}(t)\phi^\circ_{n+1})\delta_{P'}(t)^{-1} \qquad (8.7.2.2)$$

$$= \frac{vol(K^U_{n,v})}{vol(K^U_{n,v}\cap\mathcal{B}'(F_v))}\sum_{t\in\Lambda_r}\mathcal{L}^W(\tau(t)\phi^\circ_\tau)\mathcal{L}^B(\phi^\circ_m\otimes\sigma_{n+1}(t)\phi^\circ_{n+1})|\det t|^s_{E_v}\delta_{Q_n}(t)^{1/2}\delta_{P'}(t)^{-1},$$

where we have set $\Lambda_r = T_r(F_v)/T_r(\mathcal{O}_v)$, which we will identify with the cocharacter lattice $X_*(T_{r,v})$ via the map $\lambda \mapsto \lambda(\varpi_F)$. Let $\Lambda_r^+ \subset \Lambda_r$ be the cone of dominant cocharacters with respect to B_r and $\Lambda_r^{++} \subset \Lambda_r^+$ the subcone of cocharacters that are moreover dominant with respect to B_{r+1} through the embedding $g \in G_r \mapsto \begin{pmatrix} g & \\ & 1\end{pmatrix} \in G_{r+1}$. To lighten the computations a bit, we will assume from now on that the local measures at v have been chosen such that $vol(K^U_{n,v}) = vol(K^U_{n,v}\cap\mathcal{B}'(F_v)) = 1$.

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Before proceeding, it will be convenient to introduce some notation pertaining to complex dual groups:

- For $\ell \in \{m, m+1, n+1\}$, we identify the dual group $\widehat{U}_\ell$ of U_ℓ (resp. $\widehat{G}_r$ of G_r) with $\mathrm{GL}_\ell(\mathbb{C})$ (resp. $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$) equipped with its standard pinning. Its L-group can be written

$$^L U_\ell = \mathrm{GL}_\ell(\mathbb{C}) \rtimes \mathrm{Gal}(E/F) \quad (\text{resp. } ^L G_r = (\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})) \rtimes \mathrm{Gal}(E/F)),$$

where the Galois action is given by $c(g) = g^\star$ (resp. $c(g_1, g_2) = (g_2, g_1)$), where the involution $g \in \mathrm{GL}_\ell(\mathbb{C}) \mapsto g^\star$ is

$$g^\star = J_\ell {}^t g^{-1} J_\ell^{-1}, \quad J_\ell = \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{\ell-1} & & \end{pmatrix}.$$

We will also denote by $S \mapsto S^\star$ the automorphism of $^L G_r$ which is the identity on $\mathrm{Gal}(E/F)$ and given by $(g_1, g_2) \mapsto (g_2^\star, g_1^\star)$ on $\widehat{G}_r$.

- We will write $(\widehat{T}_r, \widehat{B}_r)$ for the standard Borel pair in $\widehat{G}_r$ and $(\widehat{T}_\ell^U, \widehat{B}_\ell^U)$ for that in $\widehat{U}_\ell$. The corresponding semi-direct products with $\mathrm{Gal}(E/F)$ will be denoted

$$^L T_r = \widehat{T}_r \rtimes \mathrm{Gal}(E/F),\ ^L B_r = \widehat{B}_r \rtimes \mathrm{Gal}(E/F),\ ^L T_\ell^U = \widehat{T}_\ell^U \rtimes \mathrm{Gal}(E/F),\ ^L B_\ell^U = \widehat{B}_\ell^U \rtimes \mathrm{Gal}(E/F).$$

- The dual groups of $\mathcal{G}$, $\widetilde{U}$ are

$$\widehat{\mathcal{G}} = \widehat{U}_m \times \widehat{U}_{n+1},\ \widehat{\widetilde{U}} = \widehat{U}_m \times \widehat{U}_{m+1},$$

and writing $\times_\Gamma$ for the fibered products over $\mathrm{Gal}(E/F)$, their L-groups are

$$^L \mathcal{G} = {}^L U_m \times_\Gamma {}^L U_{n+1},\ ^L \widetilde{U} = {}^L U_m \times_\Gamma {}^L U_{m+1},$$

respectively. We let $\widehat{B} = \widehat{B}_m^U \times \widehat{B}_{n+1}^U$, $\widehat{T} = \widehat{T}_m^U \times \widehat{T}_{n+1}^U$ (resp. $\widehat{\widetilde{B}} = \widehat{B}_m^U \times \widehat{B}_{m+1}^U$, $\widehat{\widetilde{T}} = \widehat{T}_m^U \times \widehat{T}_{m+1}^U$) be the standard Borel and maximal torus in $\widehat{\mathcal{G}}$ (resp. in $\widehat{\widetilde{U}}$). We also write

$$^L T = \widehat{T} \rtimes \mathrm{Gal}(E/F),\ ^L \widetilde{T} = \widehat{\widetilde{T}} \rtimes \mathrm{Gal}(E/F).$$

- The parabolic subgroup $Q_{n+1} \subset U_{n+1}$ stabilizing the isotropic subspace X_r corresponds to a standard parabolic subgroup $\widehat{Q}_{n+1}$ of $\widehat{U}_{n+1}$ with standard Levi

$$\widehat{L}_{n+1} = \begin{pmatrix} \mathrm{GL}_r(\mathbb{C}) & & \\ & \mathrm{GL}_{m+1}(\mathbb{C}) & \\ & & \mathrm{GL}_r(\mathbb{C}) \end{pmatrix}.$$

The corresponding L-group $^L L_{n+1} = \widehat{L}_{n+1} \rtimes \mathrm{Gal}(E/F)$ is isomorphic to $^L G_r \times_\Gamma {}^L U_{m+1}$ via the map which is the identity on $\mathrm{Gal}(E/F)$ and

$$\begin{pmatrix} g_r^{(1)} & & \\ & g_{m+1} & \\ & & g_r^{(2)} \end{pmatrix} \mapsto ((g_r^{(1)}, g_r^{(2)\star}), g_{m+1})$$

on $\widehat{L}_{n+1}$. For $S \in {}^L L_{n+1}$, we denote by $(S^{(r)}, S^{(m+1)}) \in {}^L G_r \times_\Gamma {}^L U_{m+1}$ its image by this isomorphism.

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- It will also be convenient to use the parabolic subgroup $\mathcal{Q} = U_m \times Q_{n+1}$ of $\mathcal{G}$. We set $\widehat{\mathcal{Q}} = \widehat{U}_m \times \widehat{Q}_{n+1}$, $\widehat{\mathcal{L}} = \widehat{U}_m \times \widehat{L}_{n+1}$ and ${}^L\mathcal{L} = \widehat{\mathcal{L}} \rtimes \mathrm{Gal}(E/F)$. There are two natural decompositions

$$ {}^L\mathcal{L} = {}^LU_m \times_\Gamma {}^LL_{n+1} \text{ and } {}^L\mathcal{L} = {}^L\widetilde{U} \times_\Gamma {}^LG_r, $$

  and for $S \in {}^L\mathcal{L}$, we will denote by (S_m, S_{n+1}) and $(\widetilde{S}, S_{n+1}^{(r)})$ the corresponding respective decompositions of S.
- For every complex Lie group ${}^L\mathbb{G}$ with a subgroup ${}^L\mathbb{Q}$ and respective identity components $\widehat{\mathbb{G}}, \widehat{\mathbb{Q}}$, we set

$$ D_{\widehat{\mathbb{G}}/\widehat{\mathbb{Q}}}(S) = \det(1 - Ad(S) \mid Lie(\widehat{\mathbb{G}})/Lie(\widehat{\mathbb{Q}})), \text{ for } S \in {}^L\mathbb{Q}. $$

- For $\mathbb{G} \in \{G_r, U_\ell, \mathcal{G}, \mathcal{L}, L_{n+1}, T_r, T_\ell^U, T\}$, we denote by ${}^L\mathbb{G}_v$ the L-group of $\mathbb{G}_v = \mathbb{G} \times_F F_v$ that is

$$ {}^L\mathbb{G}_v = \begin{cases} {}^L\mathbb{G} & \text{if } v \text{ is inert in } E; \\ \widehat{\mathbb{G}} & \text{if } v \text{ splits in } E. \end{cases} $$

  Also, for $\mathbb{G} \in \{G_r, U_\ell, \mathcal{G}, \mathcal{L}, L_{n+1}\}$, we write $W(\mathbb{G}_v)$ for the Weyl group $Norm_{\widehat{\mathbb{G}}}({}^L\mathbb{T}_v)/\widehat{\mathbb{T}}$, where $\widehat{\mathbb{T}} \subset \widehat{\mathbb{G}}$ is the standard maximal torus.
- The choice of the Borel pair (T_r, B_r) allows to identify Λ_r with the group of characters of ${}^LT_{r,v}$ that are trivial on $\mathrm{Gal}(E/F)$, and we will denote by $\Lambda_r \ni t \mapsto \chi_t$ this identification. For $t \in \Lambda_r^+$, we write ch_t for the character of the irreducible representation of ${}^LG_r$ with highest weight χ_t (see Appendix A).
- For $k, \ell \in \mathbb{N}$, we define the representation

$$ {}^L(G_k \times G_\ell) = {}^LG_k \times_\Gamma {}^LG_\ell \to \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^\ell \oplus \mathbb{C}^k \otimes \mathbb{C}^\ell) $$

$$ (S_k, S_\ell) \mapsto S_k \overset{\mathrm{I}}{\otimes} S_\ell, $$

  which sends $((g_k^{(1)}, g_k^{(2)}), (g_\ell^{(1)}, g_\ell^{(2)})) \in \widehat{G_k} \times \widehat{G_\ell}$ to $g_k^{(1)} \otimes g_\ell^{(1)} \oplus g_k^{(2)} \otimes g_\ell^{(2)}$ and $c \in \mathrm{Gal}(E/F)$ to the operator ι that swaps the two copies of $\mathbb{C}^k \otimes \mathbb{C}^\ell$.
- Composing this representation with the embeddings ${}^LU_i \to {}^LG_i$, $g \in \widehat{U}_i \mapsto (g, g^\star)$, $(i = k, \ell)$, we obtain representations

$$ {}^L(U_k \times G_\ell) \to \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^\ell \oplus \mathbb{C}^k \otimes \mathbb{C}^\ell) $$

  and

$$ {}^L(U_k \times U_\ell) \to \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^\ell \oplus \mathbb{C}^k \otimes \mathbb{C}^\ell) $$

  that for simplicity we will also denote by the symbol $\overset{\mathrm{I}}{\otimes}$.
- In particular, we have two representations

$$ \mathcal{R} : {}^L\mathcal{G} = {}^L(U_m \times U_{n+1}) \to \mathrm{GL}(\mathbb{C}^m \otimes \mathbb{C}^{n+1} \oplus \mathbb{C}^m \otimes \mathbb{C}^{n+1}),\ (S_m, S_{n+1}) \mapsto S_m \overset{\mathrm{I}}{\otimes} S_{n+1} $$

$$ \widetilde{\mathcal{R}} : {}^L\widetilde{U} = {}^L(U_m \times U_{m+1}) \to \mathrm{GL}(\mathbb{C}^m \otimes \mathbb{C}^{m+1} \oplus \mathbb{C}^m \otimes \mathbb{C}^{m+1}),\ (S_m, S_{m+1}) \mapsto S_m \overset{\mathrm{I}}{\otimes} S_{m+1} $$

  that we will denote by $\mathcal{R}$ and $\widetilde{\mathcal{R}}$, respectively. The subspace

$$ V_- := \left\langle (e_i \otimes e_j, 0) \mid \substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n+1 \\ i+j > m+r+1} \right\rangle \oplus \left\langle (0, e_i \otimes e_j) \mid \substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n+1 \\ i+j > m+r+1} \right\rangle $$

$$ \left(\text{resp. } \widetilde{V}_- := \left\langle (e_i \otimes e_j, 0) \mid \substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant m+1 \\ i+j > m+1} \right\rangle \oplus \left\langle (0, e_i \otimes e_j) \mid \substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant m+1 \\ i+j > m+1} \right\rangle \right), $$

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where we denote by $(e_i)_{i=1}^k$ the standard basis of $\mathbb{C}^k$ for any k, is stable by ${}^LT$ (resp. by ${}^L\widetilde{T}$), and we set $\mathcal{R}_-(S) := \mathcal{R}(S)\,|_{V_-}$ (resp. $\widetilde{\mathcal{R}}_-(\widetilde{S}) := \widetilde{\mathcal{R}}(\widetilde{S})_{|\widetilde{V}_-}$) for $S \in {}^LT$ (resp. $\widetilde{S} \in {}^L\widetilde{T}$).

- We will also denote by As_m the representation

$$\mathrm{As}_m : {}^LG_r \to \mathrm{GL}(\mathbb{C}^r \otimes \mathbb{C}^r)$$

given by $\mathrm{As}_m(g^{(1)}, g^{(2)}) = g^{(1)} \otimes g^{(2)}$ for $(g^{(1)}, g^{(2)}) \in \widehat{G}_r$ and $\mathrm{As}_m(c) = (-1)^m s$, where $s : \mathbb{C}^r \otimes \mathbb{C}^r \to \mathbb{C}^r \otimes \mathbb{C}^r$ is defined by $s(u \otimes v) = v \otimes u$ for $u, v \in \mathbb{C}^r$.

- The Satake parameters of the unramified representations τ, σ_m, σ_{n+1} will be denoted by S_τ, S_m and S_{n+1}, respectively. These are semisimple conjugacy classes in ${}^LG_r$, ${}^LU_m$ and ${}^LU_{n+1}$, and to simplify some arguments, we will choose representatives of them in ${}^LT_r$, ${}^LT_m^U$ and ${}^LT_{n+1}^U$, respectively. Thus, denoting by $Frob_v \in \mathrm{Gal}(E/F)$ the Frobenius at v, we have

$$S_\tau \in \widehat{T}_r Frob_v, \quad S_m \in \widehat{T}_m^U Frob_v \quad \text{and} \quad S_{n+1} \in \widehat{T}_{n+1}^U Frob_v.$$

We will also write $S = (S_m, S_{n+1}) \in {}^LT$ for the Satake parameter of $\sigma = \sigma_m \boxtimes \sigma_{n+1}$.

By Shintani and Casselman-Shalika's formula [Shi76] [CS80], we have

$$\mathcal{L}^W(\tau(t)\phi_\tau^\circ) = \begin{cases} \delta_{B_r}(t)^{1/2} ch_t(S_\tau) & \text{if } t \in \Lambda_r^+, \\ 0 & \text{otherwise.} \end{cases} \tag{8.7.2.3}$$

Moreover, according to the formulas given in [Kho08, Theorem 11.4], [Liu16, Proposition 6.4] and [Zha18], when $S \in {}^LT$ is regular, we have

$$\begin{aligned} &\mathcal{L}^B(\phi_m^\circ \otimes \sigma_{n+1}(t)\phi_{n+1}^\circ) \\ &= \begin{cases} (\Delta_{m,v}^U)^{-1} \sum_{w \in W(\mathcal{G}_v)} \frac{\det(1-q^{-1/2}\mathcal{R}_-(wS))}{D_{\widehat{\mathcal{G}}/\widehat{\mathcal{B}}}(wS)} \chi_t((wS_{n+1})^{(r)})\delta_P(t)^{1/2} & \text{if } t \in \Lambda_r^{++} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{8.7.2.4}$$

Moreover, the above sum over $W(\mathcal{G}_v)$ extends to a regular function on ${}^LT$, and the formula is still valid when we interpret the right-hand side in terms of this extension. We will now prove the formula of the proposition assuming that S is regular but, as it is an identity between rational functions, the extension to the non-regular case will follow.

Lemma 8.7.2.2. *For $t \in \Lambda_r^{++}$, we have*

$$\mathcal{L}^B(\phi_m^\circ \otimes \sigma_{n+1}(t)\phi_{n+1}^\circ) = \sum_{w \in W(U_{n+1,v})} \frac{\det(1 - q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{B}_{n+1}^U}(wS_{n+1})} ch_t((wS_{n+1})^{(r)})\delta_P(t)^{1/2}. \tag{8.7.2.5}$$

Proof. First, we note that the function

$$S_{n+1} \in \widehat{T}_{n+1}^U Frob_v \mapsto \det(1 - q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) ch_t(S_{n+1}^{(r)})$$

is invariant under the action of $W(L_{n+1,v})$. Combining this with the identity (see Corollary A.0.2.2)

$$\sum_{w \in W(L_{n+1,v})} D_{\widehat{U}_{n+1}/\widehat{B}_{n+1}^U}(wS_{n+1})^{-1} = D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(S_{n+1})^{-1},$$

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we see that the right-hand side of (8.7.2.5) can be rewritten as

$$\sum_{w\in W(L_{n+1,v})\backslash W(U_{n+1,v})} \frac{\det(1-q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(wS_{n+1})} ch_t((wS_{n+1})^{(r)})\delta_P(t)^{1/2}.$$

The natural projection $W(\mathcal{G}_v) \to W(U_{n+1,v})$ induces a bijection

$$W(\mathcal{L}_v)\backslash W(\mathcal{G}_v) \simeq W(L_{n+1,v})\backslash W(U_{n+1,v}),$$

and thus by (8.7.2.4), we just need to establish, for every regular $S = (S_m, S_{n+1}) \in {}^LT$ and $t \in \Lambda_r$, the identity

$$(\Delta^U_{m,v})^{-1} \sum_{w\in W(\mathcal{L}_v)} \frac{\det(1-q^{-1/2}\mathcal{R}_-(wS))}{D_{\widehat{\mathcal{G}}/\widehat{\mathcal{B}}}(wS)} \chi_t((wS_{n+1})^{(r)}) = \frac{\det(1-q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(S_{n+1})} ch_t(S_{n+1}^{(r)}). \tag{8.7.2.6}$$

We have decompositions

$$D_{\widehat{\mathcal{G}}/\widehat{\mathcal{B}}}(S) = D_{\widehat{\mathcal{G}}/\widehat{\mathcal{Q}}}(S) D_{\widehat{\mathcal{L}}/\widehat{B}_{\mathcal{L}}}(S) = D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(S_{n+1}) D_{\widehat{\widetilde{U}}/\widehat{\widetilde{B}}}(\widetilde{S}) D_{\widehat{G}_r/\widehat{B}_r}(S_{n+1}^{(r)}),$$

$$\mathcal{R}_-(S) = S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star} \oplus \widetilde{\mathcal{R}}_-(\widetilde{S}).$$

and

$$W(\mathcal{L}_v) = W(\widetilde{U}_v) \times W(G_{r,v}).$$

This leads to the following expression for the left-hand side of (8.7.2.6):

$$\frac{\det(1-q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(S_{n+1})} \times (\Delta^U_{m,v})^{-1} \sum_{\widetilde{w}\in W(\widetilde{U}_v)} \frac{\det(1-q^{-1/2}\widetilde{\mathcal{R}}_-(\widetilde{w}\widetilde{S}))}{D_{\widehat{\widetilde{U}}/\widehat{\widetilde{B}}}(\widetilde{w}\widetilde{S})} \sum_{w_r\in W(G_{r,v})} \frac{\chi_t(w_r S_{n+1}^{(r)})}{D_{\widehat{G_r}/\widehat{B_r}}(w_r S_{n+1}^{(r)})}.$$

Furthermore, Weyl's character formula implies (see Proposition A.0.2.1)

$$\sum_{w_r\in W(G_{r,v})} \frac{\chi_t(w_r S_{n+1}^{(r)})}{D_{\widehat{G_r}/\widehat{B_r}}(w_r S_{n+1}^{(r)})} = ch_t(S_{n+1}^{(r)}),$$

while, according to [Liu16, Proposition 6.4], we have

$$(\Delta^U_{m,v})^{-1} \sum_{\widetilde{w}\in W(\widetilde{U}_v)} \frac{\det(1-q^{-1/2}\widetilde{\mathcal{R}}_-(\widetilde{w}\widetilde{S}))}{D_{\widehat{\widetilde{U}}/\widehat{\widetilde{B}}}(\widetilde{w}\widetilde{S})} = 1.$$

This shows the formula (8.7.2.6) and ends the proof of the lemma. □

The next lemma is a consequence of the Cauchy identity [Bum04, Theorem 43.3].

Lemma 8.7.2.3. *Let $S_1, S_2 \in \widehat{T_r} Frob_v$. Then, for $\Re(s)$ sufficiently large, we have*

$$\sum_{t\in\Lambda_r^{++}} |\det t|_{E_v}^{1/2+s} ch_t(S_1) ch_t(S_2) = \det(1-q^{-1/2}S_{1,s} \overset{\mathrm{I}}{\otimes} S_2)^{-1},$$

where we have set $S_{1,s} := q^{-s}S_1$

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Combining the two above lemmas with (8.7.2.2), (8.7.2.3) as well as the identity

$$\delta_P(t)^{1/2}\delta_{B_r}(t)^{1/2}\delta_{Q_n}(t)^{1/2}\delta_{P'}(t)^{-1} = |\det t|_{E_v}^{1/2}, \quad t \in \Lambda_r,$$

we obtain

$$\mathcal{L}_s^{U'}(\phi_{n,s}^{\circ} \otimes \phi_{n+1}^{\circ}) = \sum_{w \in W(U_{n+1,v})} \frac{\det(1 - q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{B}_{n+1}^U}(wS_{n+1})} \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)})^{-1}. \tag{8.7.2.7}$$

For every $w \in W(U_{n+1,v})$, we have the identity

$$S_{\tau,s} \overset{\mathrm{I}}{\otimes} wS_{n+1} = S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)} \oplus S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(m+1)} \otimes S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}.$$

It follows that

$$L\left(\frac{1}{2}, \tau_s \times \sigma_{n+1}\right)^{-1} = \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1})$$
$$= \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)}) \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(m+1)}) \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}).$$

Thus, (8.7.2.7) can be rewritten as

$$\mathcal{L}_s^{U'}(\phi_{n,s}^{\circ} \otimes \phi_{n+1}^{\circ}) = L\left(\frac{1}{2}, \tau_s \times \sigma_{n+1}\right) \sum_{w \in W(U_{n+1,v})}$$
$$\frac{\det(1 - q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{B}_{n+1}^U}(wS_{n+1})} \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}) \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(m+1)}).$$

Since we have

$$L(1, \tau_s^c \times \sigma_m) = \det(1 - q^{-1}S_{\tau,s}^c \overset{\mathrm{I}}{\otimes} S_m)^{-1}, \quad L(1, \tau_s, \mathrm{As}^{(-1)^m}) = \det(1 - q^{-1}\mathrm{As}_m(S_{\tau,s}))^{-1},$$

we see that the proposition is now reduced to the equality

$$\det(1 - q^{-1}S_{\tau,s}^c \overset{\mathrm{I}}{\otimes} S_m) \det(1 - q^{-1}\mathrm{As}_m(S_{\tau,s})) = \sum_{w \in W(U_{n+1,v})} \tag{8.7.2.8}$$
$$\frac{\det(1 - q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{B}_{n+1}^U}(wS_{n+1})} \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}) \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(m+1)}).$$

To prove the above identity, we first show that the right-hand side of (8.7.2.8) considered as a function of the regular element $S_{n+1} \in \widehat{T}_{n+1}^U Frob_v$ is constant. For this, we first note that the function

$$S_{n+1} \in \widehat{T}_{n+1}^U Frob_v \mapsto$$
$$\det(1 - q^{-1/2}S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}) \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}) \det(1 - q^{-1/2}S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(m+1)})$$

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is (the restriction of) a linear combination of characters of ${}^L T^U_{n+1}$. Thus, by Proposition A.0.2.1, it suffices to check that for every character χ appearing in this linear combination the sum $\rho + \chi\,|_{\widehat{T}^U_{n+1}}$, where $\rho \in X^*(\widehat{T}^U_{n+1})$ stands for the half-sum of positive roots (with respect to $\widehat{B}^U_{n+1}$), is either singular or conjugate, under the full Weyl group of $(\widehat{U}_{n+1}, \widehat{T}^U_{n+1})$, to ρ. Let χ be such a character. Using the natural isomorphism $X^*(\widehat{T}^U_{n+1}) \simeq \mathbb{Z}^{n+1}$, we have $\chi\,|_{\widehat{T}^U_{n+1}} = (\lambda_1, \dots, \lambda_{n+1})$, where the λ_i's are integers satisfying

$$-m-r \leqslant \lambda_i \leqslant 0 \text{ for } 0 \leqslant i \leqslant r,$$
$$-r \leqslant \lambda_i \leqslant r \text{ for } r+1 \leqslant i \leqslant r+m+1,$$
$$0 \leqslant \lambda_i \leqslant m+r \text{ for } r+m+2 \leqslant i \leqslant n+1.$$

Furthermore, we have $\rho = (\frac{n}{2}, \frac{n-2}{2}, \dots, -\frac{n}{2})$, and from the above inequality, we see that the coordinates of $\rho + \chi\,|_{\widehat{T}^U_{n+1}}$ are all integers or half-integers between $-\frac{n}{2}$ and $\frac{n}{2}$. The claim about $\chi\,|_{\widehat{T}^U_{n+1}} + \rho$ readily follows, and we therefore deduce that the right-hand side of (8.7.2.8) is indeed independent of S_{n+1}.

Let $A, B \in \mathrm{GL}_r(\mathbb{C})$ be such that $S_{\tau,s} = (A, B)Frob_v$. Then, plugging

$$S_{n+1} = \begin{pmatrix} q^{1/2}A^\star & & \\ & A_{m+1} & \\ & & q^{-1/2}B \end{pmatrix} Frob_v \in \widehat{T}^U_{n+1} Frob_v$$

in the right-hand side of (8.7.2.8), where the matrix $A_{m+1} \in \mathrm{GL}_{m+1}(\mathbb{C})$ is chosen such that S_{n+1} is regular, the term indexed by $w \in W(U_{n+1,v})$ is nonzero only when $w \in W(L_{n+1,v})$. It follows that this sum equals

$$\sum_{w \in W(L_{n+1,v})} \frac{\det(1 - q^{-1/2} S_m \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star})}{D_{\widehat{U}_{n+1}/\widehat{B}^U_{n+1}}(wS_{n+1})} \det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(r)\star}) \times$$
$$\det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} (wS_{n+1})^{(m+1)})$$
$$= \det(1 - q^{-1/2} S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) \det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) \det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1}^{(m+1)}) \times$$
$$\sum_{w \in W(L_{n+1,v})} D_{\widehat{U}_{n+1}/\widehat{B}^U_{n+1}}(wS_{n+1})^{-1}$$
$$= \det(1 - q^{-1/2} S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) \det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) \times$$
$$\det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1}^{(m+1)}) D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(S_{n+1})^{-1},$$

where the last equality follows from Corollary A.0.2.2. By direct computation, we have

$$\det(1 - q^{-1/2} S_m \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) = \det(1 - q^{-1} S_m \overset{\mathrm{I}}{\otimes} S^c_{\tau,s}),$$
$$\det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1}^{(r)\star}) = \det(1 - q^{-1} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S^c_{\tau,s}),$$
$$\det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{n+1}^{(m+1)}) = \det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{m+1}),$$
$$D_{\widehat{U}_{n+1}/\widehat{Q}_{n+1}}(S_{n+1}) = \det(1 - q^{-1} As_{m+1}(S_{\tau,s})) \det(1 - q^{-1/2} S_{\tau,s} \overset{\mathrm{I}}{\otimes} S_{m+1}),$$

where we have set $S_{m+1} = A_{m+1} Frob_v \in \widehat{T}^U_{m+1} Frob_v$. Since $S_{\tau,s} \overset{\mathrm{I}}{\otimes} S^c_{\tau,s} = \mathrm{As}_m(S_{\tau,s}) \oplus \mathrm{As}_{m+1}(S_{\tau,s})$, this concludes the proof of (8.7.2.8) and therefore of the proposition. □

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8.8. *Reduction to the corank one case*

8.8.1.
In this subsection, we let

- σ_m and σ_{n+1} be cuspidal automorphic representations of $U_m(\mathbb{A})$ and $U_{n+1}(\mathbb{A})$, respectively;
- τ be an irreducible automorphic representation of $G_r(\mathbb{A})$ that is induced from a unitary cuspidal representation, meaning that there exist a parabolic subgroup $R = MN_R \subset G_r$ as well as a unitary cuspidal automorphic representation κ of $M(\mathbb{A})$ such that

$$\tau = \{E_R^{G_r}(\phi, 0) \mid \phi \in I_{R(\mathbb{A})}^{G_r(\mathbb{A})}(\kappa)\}.$$

Moreover, we henceforth identify $\mathcal{A}_{Q_n,\tau\boxtimes\sigma_m}(U_n)$ with the parabolic induction $\sigma_n := I_{Q_n(\mathbb{A})}^{U_n(\mathbb{A})}(\tau\boxtimes\sigma_m)$ so that for $\phi_n \in \sigma_n$ and any $s \in \mathbb{C}$, we have $\phi_{n,s} \in \sigma_{n,s} := I_{Q_n(\mathbb{A})}^{U_n(\mathbb{A})}(\tau_s \boxtimes \sigma_m)$. Then, by Proposition 8.5.1.1 and Proposition 8.7.2.1, for $\phi_n \in \sigma_n$ and $\phi_{n+1} \in \sigma_{n+1}$, we have

$$\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s)\otimes\phi_{n+1}) = \frac{\mathrm{vol}(K_n^{U,T})}{\mathrm{vol}(K_n^{U,T}\cap\mathcal{B}'(\mathbb{A}^T))}\frac{L^T\left(\frac{1}{2}+s, \tau\times\sigma_{n+1}\right)}{L^T(1+s,\tau^c\times\sigma_m)L^T(1+2s,\tau,\mathrm{As}^{(-1)^m})} \tag{8.8.1.1}$$
$$\times\int_{\mathcal{B}'(F_T)\backslash U'(F_T)}\mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L}(\phi_{n,s}(h)\otimes\sigma_{n+1}(h)\phi_{n+1})dh$$

for $\Re(s) \gg 0$ and where $T \supset S$ is any sufficiently large finite set of places such that ϕ_n is $K_n^{U,T}$-invariant and ϕ_{n+1} is $K_{n+1}^{U,T}$-invariant.

8.8.2.

Proposition 8.8.2.1. *The following are equivalent:*

1. *There exist* $\phi_m \in \sigma_m$ *and* $\phi_{n+1} \in \sigma_{n+1}$ *such that* $\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}}}(\phi_m\otimes\phi_{n+1}) \neq 0$;
2. *There exist* $\phi_n \in \sigma_n$, $\phi_{n+1} \in \sigma_{n+1}$ *and* $s \in \mathbb{C}$ *such that* $E_{Q_n}^{U_n}(\phi_n, .)$ *has no pole at* s *and* $\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s)\otimes\phi_{n+1}) \neq 0$.

Proof. First, we remark that since there exists $\phi_\tau \in \tau$ whose Whittaker period $\int_{[N_r]}\phi_\tau(u)\psi_{-r}(u)du$ is nonzero, assertion 1 is equivalent to the nonvanishing of $\mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L}$ on $\tau\boxtimes\sigma_m\boxtimes\sigma_{n+1}$. Then, the equivalence 2. ⇔ 1. follows from (8.8.1.1) and the last part of Proposition 8.6.1.1. □

8.8.3.
Choose isomorphisms $\tau \simeq \bigotimes_v' \tau_v$, $\sigma_m \simeq \bigotimes_v' \sigma_{m,v}$ and $\sigma_{n+1} \simeq \bigotimes_v' \sigma_{n+1,v}$. This induces an isomorphism $\sigma_n \simeq \bigotimes_v' \sigma_{n,v}$, where $\sigma_{n,v} := I_{Q_n(F_v)}^{U_n(F_v)}(\tau_v\boxtimes\sigma_{m,v})$. We assume henceforth that for every place v, the representations τ_v, $\sigma_{m,v}$ and $\sigma_{n+1,v}$ are all tempered. Let $\phi_\tau \in \tau$, $\phi_m \in \sigma_m$, $\phi_n \in \sigma_n$ and $\phi_{n+1} \in \sigma_{n+1}$ be factorizable vectors – that is,

$$\phi_\tau = \bigotimes_v{}' \phi_{\tau,v},\ \phi_m = \bigotimes_v{}' \phi_{m,v},\ \phi_n = \bigotimes_v{}' \phi_{n,v},\ \phi_{n+1} = \bigotimes_v{}' \phi_{n+1,v},$$

where $\phi_{\tau,v} \in \tau_v$, $\phi_{m,v} \in \sigma_{m,v}$, $\phi_{n,v} \in \sigma_{n,v}$ and $\phi_{n+1,v} \in \sigma_{n+1,v}$.

We equip σ_m, σ_{n+1}, τ and σ_n with invariant inner products as follows:

- We endow σ_m, σ_{n+1} with the Petersson inner products $\langle .,.\rangle_{Pet}$ (i.e., the L^2 inner products with respect to the Tamagawa measures on $[U_m]$ and $[U_{n+1}]$, respectively).

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- On τ, we put the inner product defined by

$$\langle E_R^{G_r}(\phi,0), E_R^{G_r}(\phi',0)\rangle_\tau = \int_{R(\mathbb{A})\backslash G_r(\mathbb{A})}\int_{[M]^1}\phi(mg)\overline{\phi'(mg)}dmdg$$

for $\phi,\phi'\in I_{R(\mathbb{A})}^{G_r(\mathbb{A})}(\kappa)$.
- σ_n is equipped with the inner product induced from that on $\tau\boxtimes\sigma_m$, here denoted $\langle .,.\rangle_{\tau\boxtimes\sigma_m}$; that is,

$$\langle\phi,\phi'\rangle_{\sigma_n} = \int_{Q_n(\mathbb{A})\backslash U_n(\mathbb{A})}\langle\phi(g),\phi'(g)\rangle_{\tau\boxtimes\sigma_m}dg, \text{ for } \phi,\phi'\in\sigma_n.$$

We also fix factorizations of these inner products on σ_m, σ_{n+1}, τ and σ_n into local invariant inner products. Following Section 8.4, this allows to define, for every place v of F, local periods $\mathcal{P}_{U',v}$, $\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}$, $\mathcal{P}_{N_r,\psi_{-r},v}$ and $\mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L,v}$ on $\sigma_{n,v}\boxtimes\sigma_{n+1,v}$, $\sigma_{m,v}\boxtimes\sigma_{n,v}$, τ_v and $\tau_v\boxtimes\sigma_{m,v}\boxtimes\sigma_{n+1,v}$, respectively. Furthermore, for almost all v, we have

$$\mathcal{P}_{U',v}(\phi_{n,v}\otimes\phi_{n+1,v},\phi_{n,v}\otimes\phi_{n+1,v}) = \Delta_{U_{n+1},v}\frac{L\left(\frac{1}{2},\sigma_{n,v}\times\sigma_{n+1,v}\right)}{L(1,\sigma_{n,v},\mathrm{Ad})L(1,\sigma_{n+1,v},\mathrm{Ad})}$$

$$\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}(\phi_{m,v}\otimes\phi_{n+1,v},\phi_{m,v}\otimes\phi_{n+1,v}) = \Delta_{U_{n+1},v}\frac{L\left(\frac{1}{2},\sigma_{m,v}\times\sigma_{n+1,v}\right)}{L(1,\sigma_{m,v},\mathrm{Ad})L(1,\sigma_{n+1,v},\mathrm{Ad})}$$

$$\mathcal{P}_{N_r,\psi_{-r},v}(\phi_{\tau,v},\phi_{\tau,v}) = \frac{\Delta_{G_r,v}}{L(1,\tau_v,\mathrm{Ad})},$$

where we have set $\Delta_{U_{n+1},v} = \prod_{i=1}^{n+1}L(i,\eta_{E/F,v}^i)$ and $\Delta_{G_r,v} = \prod_{i=1}^r\zeta_{E_v}(i)$. Note that the last two equalities above imply that

$$\mathcal{P}_{\mathcal{B}^L,\psi_{\mathcal{B}}^L,v}(\phi_{\tau,v}\otimes\phi_{m,v}\otimes\phi_{n+1,v},\phi_{\tau,v}\otimes\phi_{m,v}\otimes\phi_{n+1,v}) =$$
$$\Delta_{U_{n+1},v}\Delta_{G_r,v}\frac{L\left(\frac{1}{2},\sigma_{m,v}\times\sigma_{n+1,v}\right)}{L(1,\sigma_{m,v},\mathrm{Ad})L(1,\sigma_{n+1,v},\mathrm{Ad})L(1,\tau_v,\mathrm{Ad})}$$

for almost all v. Given all these identities, it makes sense to define

$$\prod_v{}'\mathcal{P}_{U',v}(\phi_{n,v}\otimes\phi_{n+1,v},\phi_{n,v}\otimes\phi_{n+1,v}) :=$$
$$\Delta_{U_{n+1}}^T\frac{L^T\left(\frac{1}{2},\sigma_n\times\sigma_{n+1}\right)}{L^{T,*}(1,\sigma_n,\mathrm{Ad})L^T(1,\sigma_{n+1},\mathrm{Ad})}\prod_{v\in T}\mathcal{P}_{U',v}(\phi_{n,v}\otimes\phi_{n+1,v},\phi_{n,v}\otimes\phi_{n+1,v}),$$

$$\prod_v{}'\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}(\phi_{m,v}\otimes\phi_{n+1,v},\phi_{m,v}\otimes\phi_{n+1,v}) :=$$
$$\Delta_{U_{n+1}}^T\frac{L^T\left(\frac{1}{2},\sigma_m\times\sigma_{n+1}\right)}{L^T(1,\sigma_m,\mathrm{Ad})L^T(1,\sigma_{n+1},\mathrm{Ad})}\prod_{v\in T}\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}(\phi_{m,v}\otimes\phi_{n+1,v},\phi_{m,v}\otimes\phi_{n+1,v}),$$

$$\prod_v{}'\mathcal{P}_{N_r,\psi_{-r},v}(\phi_{\tau,v},\phi_{\tau,v}) := \frac{\Delta_{G_r}^{T,*}}{L^{T,*}(1,\tau,\mathrm{Ad})}\prod_{v\in T}\mathcal{P}_{N_r,\psi_{-r},v}(\phi_{\tau,v},\phi_{\tau,v})$$

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for any sufficiently large finite set T of places where we have set

$$\Delta_{U_{n+1}}^T = \prod_{i=1}^{n+1} L^T(i, \eta_{E/F}^i), \quad \Delta_{G_r}^{T,*} = \zeta_E^{T,*}(1) \prod_{i=2}^{r} \zeta_E^T(i)$$

and $L^{T,*}(1, \sigma_n, \mathrm{Ad})$, $L^{T,*}(1, \tau, \mathrm{Ad})$ stand for the regularized values

$$L^{T,*}(1, \sigma_n, \mathrm{Ad}) := \left((s-1)^a L^T(s, \sigma_n, \mathrm{Ad})\right)_{s=1}, \quad L^{T,*}(1, \tau, \mathrm{Ad}) := \left((s-1)^a L^T(s, \tau, \mathrm{Ad})\right)_{s=1}$$

with $a = \dim(A_M)$, respectively. We define similarly

$$\prod_v{}' \mathcal{P}_{\mathcal{B}^L, \psi_{\mathcal{B}}^L, v}(\phi_{\tau,v} \otimes \phi_{m,v} \otimes \phi_{n+1,v}, \phi_{\tau,v} \otimes \phi_{m,v} \otimes \phi_{n+1,v}).$$

Of course, the previous discussion applies verbatim when we replace τ and σ_n by τ_s and $\sigma_{n,s}$ for every $s \in i\mathbb{R}$.

Proposition 8.8.3.1. *Let $c \in \mathbb{C}$ and assume that for every factorizable vectors $\phi_n \in \sigma_n$, $\phi_{n+1} \in \sigma_{n+1}$ and every $s \in i\mathbb{R}$, we have*

$$\left|\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s) \otimes \phi_{n+1})\right|^2 = c \prod_v{}' \mathcal{P}_{U',v}(\phi_{n,s,v} \otimes \phi_{n+1,v}, \phi_{n,s,v} \otimes \phi_{n+1,v}).$$

Then, for every factorizable vectors $\phi_m \in \sigma_m$ and $\phi_{n+1} \in \sigma_{n+1}$, we have

$$\left|\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}}}(\phi_m \otimes \phi_{n+1})\right|^2 = c \prod_v{}' \mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}(\phi_{m,v} \otimes \phi_{n+1,v}, \phi_{m,v} \otimes \phi_{n+1,v}).$$

Proof. Let $\phi_n \in \sigma_n$ and $\phi_{n+1} \in \sigma_{n+1}$ be factorizable vectors and let T be a finite set of places of F that we will assume throughout to be sufficiently large. By (8.8.1.1), we have

$$\left|\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s) \otimes \phi_{n+1})\right|^2 = \left|\frac{L^T\left(\frac{1}{2}+s, \tau \times \sigma_{n+1}\right)}{L^T(1+s, \tau^c \times \sigma_m) L^T(1+2s, \tau, \mathrm{As}^{(-1)^m})}\right|^2 \tag{8.8.3.1}$$

$$\int_{(\mathcal{B}'(F_T)\backslash U'(F_T))^2} \mathcal{P}_{\mathcal{B}^L, \psi_{\mathcal{B}}^L}(\phi_{n,s}(h_1) \otimes \sigma_{n+1}(h_1)\phi_{n+1}) \overline{\mathcal{P}_{\mathcal{B}^L, \psi_{\mathcal{B}}^L}(\phi_{n,s}(h_2) \otimes \sigma_{n+1}(h_2)\phi_{n+1})} dh_2 dh_1$$

for $s \in i\mathbb{R}$. However, from the hypothesis, Proposition 8.6.2.1 and (8.2.1.1), we obtain

$$\left|\mathcal{P}_{U'}(E_{Q_n}^{U_n}(\phi_n, s) \otimes \phi_{n+1})\right|^2 = c \Delta_{G_r}^{T,*} \Delta_{U_{n+1}}^T \frac{L^T\left(\frac{1}{2}, \sigma_{n,s} \times \sigma_{n+1}\right)}{L^{T,*}(1, \sigma_{n,s}, \mathrm{Ad}) L^T(1, \sigma_{n+1}, \mathrm{Ad})} \tag{8.8.3.2}$$

$$\prod_{v \in T} \int_{(\mathcal{B}'(F_v)\backslash U'(F_v))^2} \mathcal{P}_{\mathcal{B}^L, \psi_{\mathcal{B}}^L, v}(\phi_{n,s,v}(h_1) \otimes \sigma_{n+1,v}(h_2)\phi_{n+1,v}, \phi_{n,s,v}(h_1) \otimes \sigma_{n+1,v}(h_2)\phi_{n+1,v}) dh_2 dh_1$$

for $s \in i\mathbb{R}$. From (8.8.3.1), (8.8.3.2), the last part of Proposition 8.6.1.1 as well as the identity of partial L-functions for $s \in i\mathbb{R}$

$$\frac{L^T\left(\frac{1}{2}, \sigma_{n,s} \times \sigma_{n+1}\right)}{L^{T,*}(1, \sigma_{n,s}, \mathrm{Ad})} = \frac{L^T\left(\frac{1}{2}, \sigma_m \times \sigma_{n+1}\right)}{L^{T,*}(1, \tau, \mathrm{Ad}) L^T(1, \sigma_m, \mathrm{Ad})} \left|\frac{L^T\left(\frac{1}{2}+s, \tau \times \sigma_{n+1}\right)}{L^T(1+s, \tau^c \times \sigma_m) L^T(1+2s, \tau, \mathrm{As}^{(-1)^m})}\right|^2,$$

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we deduce the equality

$$\left|\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}}}(\phi\otimes\phi_{n+1})\right|^2 = c\Delta^{T,*}_{G_r}\Delta^T_{U_{n+1}}\frac{L^T\left(\frac{1}{2},\sigma_m\times\sigma_{n+1}\right)}{L^{T,*}(1,\tau,\mathrm{Ad})L^T(1,\sigma_m,\mathrm{Ad})L^T(1,\sigma_{n+1},\mathrm{Ad})} \times\prod_{v\in T}\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}(\phi_v\otimes\phi_{n+1,v},\phi_v\otimes\phi_{n+1,v}) = c\prod_v{}' \mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}(\phi_v\otimes\phi_{n+1,v},\phi_v\otimes\phi_{n+1,v}) \tag{8.8.3.3}$$

for every factorizable vector $\phi=\bigotimes'_v\phi_v\in\tau\boxtimes\sigma_m$. Furthermore, for $\phi_\tau\in\tau$ and $\phi_m\in\sigma_m$ two factorizable vectors, we have

$$\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}}}(\phi_\tau\otimes\phi_m\otimes\phi_{n+1}) = \mathcal{P}_{N_r,\psi_{-r}}(\phi_\tau)\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}}}(\phi_m\otimes\phi_{n+1})$$

and

$$\mathcal{P}_{\mathcal{B}^L,\psi^L_{\mathcal{B}},v}(\phi_{\tau,v}\otimes\phi_{m,v}\otimes\phi_{n+1,v},\phi_{\tau,v}\otimes\phi_{m,v}\otimes\phi_{n+1,v}) = \mathcal{P}_{N_r,\psi_{-r},v}(\phi_{\tau,v},\phi_{\tau,v})\mathcal{P}_{\mathcal{B},\psi_{\mathcal{B}},v}(\phi_{m,v}\otimes\phi_{n+1,v},\phi_{m,v}\otimes\phi_{n+1,v})$$

for every place v as well as the identity (cf. [FLO12, Eq. (11.3)])

$$\left|\mathcal{P}_{N_r,\psi_{-r}}(\phi_\tau)\right|^2 = \prod_v{}'\mathcal{P}_{N_r,\psi_{-r},v}(\phi_{\tau,v},\phi_{\tau,v}). \tag{8.8.3.4}$$

The proposition can now be deduced by dividing the identity (8.8.3.4) by (8.8.3.3). Note that such a deduction is valid as the Whittaker period $\mathcal{P}_{N_r,\psi_{-r}}$ is known to be nonzero on τ. □

A. Weyl character formula for non-connected groups

A.0.1.
Let $\widehat{G}$ be a connected complex reductive group, $\widehat{B}\subset\widehat{G}$ be a Borel subgroup and $\widehat{T}\subset\widehat{B}$ be a maximal torus. Let Γ be a *cyclic* finite group acting on $\widehat{G}$ by holomorphic automorphisms preserving the Borel pair $(\widehat{B},\widehat{T})$. Set ${}^LG=\widehat{G}\rtimes\Gamma$, ${}^LT=\widehat{T}\rtimes\Gamma$ and ${}^LB=\widehat{B}\rtimes\Gamma$. We say that an element $S\in{}^LT$ is *regular* if the neutral component of its centralizer is contained in $\widehat{T}$.

Let $X^*(\widehat{T})$ be the group of algebraic characters of $\widehat{T}$ and $X^*(\widehat{T})^+\subset X^*(\widehat{T})$ be the subset of dominant elements (with respect to the chosen Borel $\widehat{B}$). For every $\chi\in X^*(\widehat{T})^+$, we denote by (π_χ,V_χ) an algebraic irreducible representation of $\widehat{G}$ with highest weight χ (it is unique up to isomorphism). Note that the subgroup of Γ-invariant characters $X^*(\widehat{T})^\Gamma$ can be identified with the set of algebraic characters ${}^LT\to\mathbb{C}^\times$ that are trivial on Γ. Moreover, for every $\chi\in X^*(\widehat{T})^+\cap X^*(\widehat{T})^\Gamma$, the representation (π_χ,V_χ) can be extended in a unique way to a representation of ${}^LG$ such that Γ acts trivially on the line of highest weight vectors. We shall denote by ch_χ the character of that representation of ${}^LG$ (i.e., $ch_\chi(S)=\mathrm{Tr}\pi_\chi(S)$ for $S\in{}^LG$).

A.0.2.
Let $\widehat{W}=\mathrm{Norm}_{\widehat{G}}(\widehat{T})/\widehat{T}$ be the Weyl group of $\widehat{T}$ in $\widehat{G}$. Then, Γ acts in a natural way on $\widehat{W}$ and the subgroup of fixed points $\widehat{W}^\Gamma$ can be identified with $W=\mathrm{Norm}_{\widehat{G}}({}^LT)/\widehat{T}$. Let $\rho\in\frac{1}{2}X^*(\widehat{T})$ be the half-sum of the positive roots with respect to $\widehat{B}$. The dot action of $\widehat{W}$ on $X^*(\widehat{T})$ is defined by

$$w\cdot\chi=w(\chi+\rho)-\rho,\quad (w,\chi)\in\widehat{W}\times X^*(\widehat{T}).$$

For $\chi \in X^*(\widehat{T})$, we recall the following alternative. Either

- $\chi + \rho$ is singular (i.e., there exists a coroot $\alpha^\vee$ such that $\langle \alpha^\vee, \chi + \rho \rangle = 0$);
- or there exists a unique $w_\chi \in \widehat{W}$ such that $w_\chi \cdot \chi \in X^*(\widehat{T})^+$.

Note that, by unicity, for $\chi \in X^*(\widehat{T})^\Gamma$ such that $\chi + \rho$ is nonsingular, we have $w_\chi \in W$.

For every standard parabolic subgroup $\widehat{B} \subset \widehat{Q} \subset \widehat{G}$ that is Γ-stable, we set

$$D_{\widehat{G}/\widehat{Q}}(S) = \det(1 - Ad(S)) \mid Lie(\widehat{G})/Lie(\widehat{Q})$$

for $S \in {}^L T$.

Proposition A.0.2.1. *Let $F \in \Gamma$ be a generator and $\chi \in X^*(\widehat{T})^\Gamma$. Then*

1. *If $\chi + \rho$ is singular, we have*

$$\sum_{w \in W} \frac{\chi(wS)}{D_{\widehat{G}/\widehat{B}}(wS)} = 0$$

for every regular $S \in {}^L T$.

2. *If $\chi + \rho$ is nonsingular, setting $\chi^+ = w_\chi \cdot \chi$, there exists a root of unity $\epsilon_\chi \in \mathbb{C}^\times$ such that*

$$\sum_{w \in W} \frac{\chi(wS)}{D_{\widehat{G}/\widehat{B}}(wS)} = \epsilon_\chi ch_{\chi^+}(S)$$

for every regular $S \in \widehat{T}F$. Moreover, if χ is dominant, we have $\epsilon_\chi = 1$.

Proof. Let $X = \widehat{G}/\widehat{B}$ be the flag variety of $\widehat{G}$. The action of $\widehat{G}$ on X naturally extends to ${}^L G$ (e.g., because we can also write $X = {}^L G/{}^L B$). Let $\mathcal{L}_\chi$ be the ${}^L G$-equivariant line bundle on X such that the action of ${}^L B$ on the fiber above $1 \in X$ is given by χ. Then, by the Borel-Weil-Bott theorem, we have

- If $\chi + \rho$ is singular, $H^i(X, \mathcal{L}_\chi) = 0$ for $i \geqslant 0$,
- otherwise (i.e., if $\chi + \rho$ is nonsingular), there exists a unique $i \geqslant 0$ such that $H^i(X, \mathcal{L}_\chi) \neq 0$ and, moreover, $H^i(X, \mathcal{L}_\chi) \simeq V_{\chi^+}$ as $\widehat{G}$-modules.

Let $S \in \widehat{T}F$ be regular. We apply Atiyah-Bott fixed point theorem [AB68, Theorem 4.12] to the action of S on the pair $(X, \mathcal{L}_\chi)$. First, note that the set of fixed points of S in X is precisely the image of the natural embedding $W \subset X$. Moreover, for $w \in W$, the action of S on the fiber $(\mathcal{L}_\chi)_w$ (resp. on the tangent space $T_w X$ identified in a natural way with $Lie(\widehat{G})/Lie(\widehat{B})$) is the multiplication by $\chi(w^{-1}S)$ (resp. the adjoint operator $Ad(w^{-1}S)$). Given this as well as the above description of the cohomology groups of $\mathcal{L}_\chi$, the Atiyah-Bott fixed point theorem implies directly the proposition when $\chi + \rho$ is singular, whereas in the nonsingular, case it gives

$$\sum_{w \in W} \frac{\chi(wS)}{D_{\widehat{G}/\widehat{B}}(wS)} = \mathrm{Tr}(S \mid H^i(X, \mathcal{L}_\chi)).$$

However, since $H^i(X, \mathcal{L}_\chi) \simeq V_{\chi^+}$ as $\widehat{G}$-modules and V_{χ^+} is irreducible, we see that $H^i(X, \mathcal{L}_\chi)$ is isomorphic as a ${}^L G$-representation to a twist of V_{χ^+} by a character of Γ. Denoting by ϵ_χ the value of this character on F, we obtain the second formula of the proposition. □

Corollary A.0.2.2. *Let $\widehat{Q} \subset \widehat{G}$ be a Γ-stable standard parabolic subgroup. Let $\widehat{L} \subset \widehat{Q}$ be the unique Levi component containing $\widehat{T}$ and set $W^L = \mathrm{Norm}_{{}^L L}(\widehat{T})/\widehat{T} \subset W$, where we have set ${}^L L = \widehat{L} \rtimes \Gamma$. Then,*

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we have

$$\sum_{w\in W^L} D_{\widehat{G}/\widehat{B}}(wS)^{-1} = D_{\widehat{G}/\widehat{Q}}(S)^{-1}$$

for every regular $S \in {}^L T$.

Proof. Note that

$$D_{\widehat{G}/\widehat{B}}(S) = D_{\widehat{G}/\widehat{Q}}(S) D_{\widehat{L}/\widehat{B}_L}(S)$$

for $S \in {}^L T$, where we have set $\widehat{B}_L = \widehat{B} \cap \widehat{L}$. The corollary now follows readily from the previous proposition applied to the trivial character and ${}^L L$ instead of ${}^L G$. □

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