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INTERPOLATION IN SEPARABLE FRECHET SPACES WITH APPLICATIONS TO SPACES OF ANALYTIC FUNCTIONS

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Let E be a separable Fréchet space, and let E^* be its topological dual space. We recall that a Fréchet space is, by definition, a complete metrizable locally convex topological vector space. A sequence $\{L_n\}$ of continuous linear functionals is said to be *interpolating* if for every sequence $\{A_n\}$ of complex numbers, there exists an $f \in E$ such that $L_n(f) = A_n$ for $n = 1, 2, 3, \ldots$. In this paper, we give necessary and sufficient conditions that $\{L_n\}$ be an interpolating sequence. They are different from the conditions in [**2**] and don't seem to be easily interderivable with them. Perhaps our main claim to novelty rests on Corollary 3. R. E. Edwards has shown us in a private communication how to prove our main theorem using Theorem 8.6.13 of his book [**1**]. The basic idea of our approach is adapted from [**5**]. We thank B. A. Taylor for valuable assistance. A portion of this work was done at the Summer Institute at Université Laval.

A case of special interest is the space $E = H(G)$, namely the space of all holomorphic functions on an open set G in the complex plane, in the topology of uniform convergence on compact subsets of G. The dual $H(G)^*$ may be identified [**5**] with the space of germs of holomorphic functions on the complement of G that vanish at ∞. The identification is via the Cauchy transform $L^\wedge$ of a functional L in $H(G)^*$. Formally, $L^\wedge(w) = L((z - w)^{-1})$. More correctly,

$$L^\wedge(w) = \left\{ \int \frac{d\mu(z)}{z - w} \right\},$$

where μ is any measure of compact support such that $L(f) = \int f d\mu$ for all $f \in H(G)$. In this special case $E = H(G)$, our necessary and sufficient conditions for interpolating are given a special form that is well suited to computational verification.

Definition. The sequence $\{L_n\}$ is *totally linearly independent* if, first of all, it is linearly independent, and second, on letting V_n be the linear span of

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$L_1, L_2, \ldots, L_n$, then if $T_n \in V_n$, $n = 1, 2, 3, \ldots$, and $T_n \to 0$ weak-* as $n \to \infty$, then there exists an N such that $T_n \in V_N$ for $n = 1, 2, 3, \ldots$.

THEOREM 1. *The sequence $\{L_n\}$ is interpolating if and only if it is totally linearly independent.*

COROLLARY 1. *The sequence $\{L_n\}$ in $H(G)^*$ is interpolating if and only if*

(a) *it is linearly independent, and*

(b) *for every compact set $K \subseteq G$, there exists an integer $N(K)$ such that if $T_n \in V_n$ and $T_n^\wedge$ has an analytic continuation to the complement of K, then $T_n \in V_{N(K)}$.*

This corollary follows from Theorem 1 and its proof on applying the criterion [**5**] for weak-* convergence in terms of the Cauchy transforms. Let $\{z_n\}$ be a sequence of distinct points in G that leaves every compact set. A simple computation re-derives the well-known fact that if $L_n(f) = f(z_n)$ (a finite number of derivations can be introduced at each z_n with no trouble) then $\{L_n\}$ is interpolating. This is because $L_n^\wedge$ has a pole at each z_n. For an apparently new application, let G be the complex plane with a sequence of disjoint discs removed. Then not only can we assign the periods of an analytic function around these holes, but we can at the same time interpolate the values at a sequence of distinct points that marches to the boundary.

COROLLARY 2. *If $\{L_n\}$ is an interpolating sequence and if L is linearly independent of the $\{L_n\}$, then $L, L_1, L_2, \ldots$ is an interpolating sequence also.*

Corollary 2 is proved by virtually retracing the proof of the "only if" part of Theorem 1. We omit the details.

Definition. $\{L_n\}$ is said to *interpolate all rapidly growing sequences* if for every sequence $\{M_n\}$ of positive constants, there exists an $f \in E$ with $L_1(f) \neq 0$ and

$$|L_n(f)| \geqq M_n |L_{n-1}(f)| \text{ for } n = 2, 3, \ldots.$$

COROLLARY 3. *If $\{L_n\}$ interpolates all rapidly growing sequences, then it interpolates all sequences.*

Corollary 3 follows from the proof of Theorem 1.

Proof of Theorem 1. Suppose that $\{L_n\}$ is totally linearly independent. Let S be the subspace of E^* consisting of all finite linear combinations of the $\{L_n\}$, and define $\Lambda : S \to \mathbf{C}$ by

$$\Lambda\left(\sum a_n L_n\right) = \sum a_n A_n,$$

where $\{A_n\}$ is the sequence to be interpolated. By the linear independence of the $\{L_n\}$, Λ is well-defined. We shall show that S is weak-* closed and that Λ is weak-* continuous. This being done, we extend Λ continuously to all of E^* (in the weak-* topology) by the Hahn-Banach Theorem. The extended functional corresponds to an element $f \in E$ since $(E^*, \sigma(E^*, E))^* = E$, [**4**, p. 235] so that $L_n(f) = A_n$ as desired. To prove that S is closed, it is enough, by a corollary of the Banach-Dieudonné Theorem [**4**, p. 273], to prove that it is sequentially closed. Let then $T_n \to T$, where $T_n \in V_n \subseteq S$. By total linear independence, $T_n \in V_N$ for some N and all n, since $n^{-1}T_n \to 0$. But a finite dimensional subspace of a topological vector space is closed [**6**, p. 16], and so $T \in V_N$ and consequently $T \in S$. That Λ is continuous follows in the same way, using the fact that every linear functional on a finite dimensional topological vector space is continuous, and that a linear functional is continuous if and only if its kernel is closed. Thus $\{L_n\}$ is interpolating.

For the converse, we suppose only that $\{L_n\}$ interpolates all rapidly growing sequences. It is clear that the L_n must be linearly independent. We suppose that $\{L_n\}$ is not totally linearly independent and derive a contradiction. For then there would exist

$$T_k = \sum a_n^{(k)} L_n$$

with coefficient of highest index $a_{n_k}^{(k)} \neq 0$, $n_k \to \infty$, such that $T_k(f) \to 0$ for each $f \in E$. In particular,

$$\left|\sum a_n^{(k)} L_n(f)\right| \leqq 1 \quad \text{for } k \geqq k(f).$$

Thus

$$|a_{n_k}^{(k)} L_{n_k}(f) + a_{n_k-1}^{(k)} L_{n_k-1}(f) + \ldots + a_1^{(k)} L_1(f)| \leqq 1,$$

so that

$$|L_{n_k}(f)| \leqq \frac{1}{|a_{n_k}^{(k)}|}[1 + |a_{n_k-1}^{(k)}||L_{n_k-1}(f)| + \ldots + |a_1^{(k)}||L_1(f)|].$$

Choose

$$p_k = \max\left\{\frac{1}{|a_{n_k}^{(k)}|}, \frac{|a_{n_k-1}^{(k)}|}{|a_{n_k}^{(k)}|}, \ldots, \frac{|a_1^{(k)}|}{|a_{n_k}^{(k)}|}\right\}.$$

Then

$$|L_{n_k}(f)| \leqq n_k p_k \max\{1, |L_{n_k-1}(f)|, \ldots, |L_1(f)|\}.$$

Let $P_k = n_k p_k$ and place on f the restriction that $|L_1(f)| > 1$. Then

$$|L_{n_k}(f)| \leqq P_k \max\{|L_{n_k-1}(f)|, \ldots, |L_1(f)|\},$$

for $k \geqq k(f)$. Now choose M_n so that $M_n > n$ and $M_{n_k} > n_k P_k$ for $k = 2, 3, \ldots$ and further choose f so that $|L_1(f)| > 1$ and

$$|L_n(f)| \geqq M_n |L_{n-1}(f)| \quad \text{for } n = 1, 2, 3, \ldots.$$

In particular, $|L_1(f)| \leqq |L_2(f)| \leqq \ldots$, so that

$$|L_{n_k}(f)| \leqq P_k |L_{n_k-1}(f)| \quad \text{for } k \geqq k(f),$$

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$$|L_{n_k}(f)| \geqq n_k P_k |L_{n_{k}-1}(f)| \quad \text{for all } k,$$

which is a contradiction, and the theorem is proved.

We conclude with some remarks. First of all, a *Banach* space admits no interpolating sequences $\{L_n\}$ since $|L_n(f)| \leqq ||L_n||\,||f||$. Second, one could derive from [**7**, Theorem 37.2] the result that $\{L_n\}$ is interpolating if and only if the space of finite linear combinations of the L_n (supposed to be linearly independent) is weak-* closed. For we could take F as the space of all sequences $\{A_n\}$ of complex numbers, in the topology of coordinatewise convergence, and take $u : E \to F$ as given by $u(f) = \{L_n(f)\}$. The dual F^* is the space of all sequences $\{B_n\}$ of complex numbers that terminate after finitely many terms, and then the transpose ${}^t u : F^* \to E^*$ is given by ${}^t u(\{B_n\}) = \sum B_n L_n$, from which the result follows. But this condition that the span be closed does not seem to be as convenient for applications as our condition.

Finally, the condition that $|L_n(f)| \geqq M_n |L_{n-1}(f)|$ in Corollary 3 cannot be replaced by the simpler condition $|L_n(f)| \geqq M_n$, as the following example shows. Take $G = \mathbf{C}$ and $E = H(G)$, the space of all entire functions, and let

$$L_{2n}(f) = f(n+1)$$

$$L_{2n+1}(f) = f(n+1) - f\left(\frac{1}{n}\right).$$

Now by [**3**], there is an entire function f, bounded by 1 on $[0, 1]$ such that $f(n+1)$ is any sequence, for $n = 1, 2, 3, \ldots$. So we can make $|L_n(f)| \geqq M_n$ for all n, but it is clear from Corollary 1 that $\{L_n\}$ is not interpolating.

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