energy, before retracing its path to collide again with $M$. Clearly, the speed of $m$ just before its second collision with $M$ will be the same as that immediately after its first collision with $M$ but, of course, the respective velocities will be in opposite directions. After the second collision between $M$ and $m$, $m$ will again move towards and then rebound from the reflecting surface $Y$ before colliding again with $M$, and this pattern will then occur repeatedly. Now, considering the situation where $m \ll M$, it is clear that the velocity of $M$ will be only slightly affected by each collision with $m$ and the fact that the collision is assumed to be perfectly elastic then implies that the speed of $m$ after each collision will be greater than the speed before the collision by about twice the speed of $M$. In general terms, then, the speed of $m$ will initially continually increase after each collision with $M$, while the speed of $M$ towards the reflecting surface will slowly decrease due to the slowing-down effect of the repeated impacts from $m$. This process whereby $m$ moves continually faster and $M$ moves continually slower will then continue until $M$ is effectively brought to rest and then proceeds to move off in the opposite direction. The effect of further impacts from $m$ will then be to slowly increase the speed of $M$ (now in a direction away from the rigid surface $Y$) while at the same time decreasing the speed of $m$ at each collision. This will then continue until the speed of $M$ exceeds that of $m$, when no further collisions will take place. During this whole sequence of events it is clear that the maximum speed of $m$ will occur when the speed of $M$ is a minimum, and, if this corresponds to $M$ being stationary, then this maximum speed for $m$ will occur when all the original energy of $M$ has been completely transferred to $m$. The above qualitative analysis may be quantified by formulating the difference equations governing the speeds ($u_n$ and $v_n$ respectively) of $m$ and $M$ after their $n$th collision. The solution of these equations then yields $u_n = \sqrt{M/m} V \sin 2n\phi$, $v_n = V \cos 2n\phi$ where $\phi = \tan^{-1} \sqrt{m/M}$ and $V$ is the speed of $M$ before the first collision, and it is clear that, as $n$ increases, the general behaviour of $u_n$ and $v_n$ follows the above qualitative discussion. In particular, if $\phi$ is such that $2k\phi = \pi/2$ for some integer $k$, then after $k$ collisions $M$ will be brought to rest with all its energy having been transferred to $m$.

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90.58 Some surprising results associated with permutations

Introduction

In this note we consider a permutation of $1, 2, 3, \ldots, n$. We divide the possible $n!$ permutations into different classes. We show that cardinalities of these classes have many surprising relations and we prove these relations.
Notation

By \( A = (a_1, a_2, \ldots, a_n) \) we denote a permutation. Let \( i \) be the first integer \( a_1 \). We say that \( j \) is the top of the permutation if \( a_k = j \) and \( a_1 < a_2 < \ldots < a_{k-1} < a_k = j > a_{k+1} \). In the following, we shall consider \( n \geq 2 \) as \( n = 1 \) is a single permutation.

Illustration

\((i, j)\)s for some permutations are illustrated below:

\[
\begin{align*}
(2, 4, 6, 1, 3, 7, 5) & \quad i = 2 \quad j = 6 \\
(4, 1, 2, 7, 3, 5, 6) & \quad i = 4 \quad j = 4 \\
(1, 2, 3, 4, 5, 6, 7) & \quad i = 1 \quad j = 7
\end{align*}
\]

By \( C_n(i, j) \) we denote the class of permutations where \( i \) is the first integer and \( j \) is the top. By \( T_n(i, j) \) we denote the number of permutations belonging to class \( C_n(i, j) \). The following relations are obvious.

\[
\begin{align*}
\sum_{j=1}^{n} T_n(i, j) &= (n - 1)! & \quad (1) \\
T_n(i, j) &= 0 \text{ if } i > j. & \quad (2) \\
T_n(i, i) &= (n - 2)! (i - 1) \text{ for } n \geq 2 \text{ and } i > 1. & \quad (3)
\end{align*}
\]

Theorem: Let \( n > 2 \) then, for \( 1 \leq i < j < n \) and \( 1 < i < j \leq n \),

\[
T_n(i, j) = (i - 1) \sum_{k=2}^{j-i+1} \binom{j-i-1}{k-2} (n-k-1)! + (j-i-1) \sum_{k=2}^{j-i} \binom{j-i-2}{k-2} (n-k-1)! + 1 \quad (4)
\]

and

\[
T_n(1, n) = (n - 2) \sum_{k=2}^{n-1} \binom{n-3}{k-2} (n-k-1)! + 1. \quad (5)
\]

Proof: We first prove (4).

Let the top = \( j \) be in the \( k \)th place in the permutation. Then there are two possibilities

\[
a_{k+1} < i \quad \text{or} \quad 1 < a_{k+1} < j.
\]

Limits of \( k \) can be easily obtained.

In the first case, \( a_{k+1} \) can be one of \( 1, 2, 3, \ldots, i - 1 \), the integers between \( i \) and \( j \) can be chosen in \( \binom{j-i-1}{k-2} \) ways and the remaining integers can be arranged in \( (n-k-1)! \) ways. Similarly, in the second case, \( a_{k+1} \) can be chosen in \( (j-i-1)! \) ways and integers between \( i \) and \( j \) can be chosen in \( \binom{j-i-1}{k-2} \) ways and remaining integers can be chosen in \( (n-k-1)! \) ways. This proves (4).

In the case of \( T_n(1, n) \), if \( k < n \), the above argument holds good but for \( k = n \), there is one permutation, \( (1, 2, \ldots, n) \) and hence (5) is true.
Some of the $T_n(i, j)$ were evaluated for different $n$ and written in tabular form. Two tables, $T_6(i, j)$ and $T_7(i, j)$ are given below.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<td>0</td>
<td>0</td>
<td>16</td>
<td>33</td>
<td>65</td>
<td></td>
</tr>
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<td>0</td>
<td>24</td>
<td>6</td>
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<td>27</td>
<td>49</td>
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</tr>
<tr>
<td>3</td>
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<td>0</td>
<td>48</td>
<td>12</td>
<td>22</td>
<td>38</td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>0</td>
<td>72</td>
<td>18</td>
<td>30</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>96</td>
<td>24</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>120</td>
<td></td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>0</td>
<td>24</td>
<td>60</td>
<td>114</td>
<td>196</td>
<td>326</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 1: $T_6(i, j)$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
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<tbody>
<tr>
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<td>0</td>
<td>24</td>
<td>60</td>
<td>114</td>
<td>196</td>
<td>326</td>
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<tr>
<td>2</td>
<td>0</td>
<td>120</td>
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<td>261</td>
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<td>3</td>
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<td>84</td>
<td>436</td>
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<td>0</td>
<td>0</td>
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<td>114</td>
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</tr>
<tr>
<td>7</td>
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<td>0</td>
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<td>0</td>
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<td>total</td>
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<td>288</td>
<td>522</td>
<td>848</td>
<td>1305</td>
<td>1957</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 2: $T_7(i, j)$

The tables of $T_n(i, j)$ lead us to some surprising observations.

**Observation 1:** We note that, for every $j > 1$ and $n > 2$,

$$ T_{n+1}(1, j) = \sum_{i=1}^{j-1} T_n(i, j - 1). $$

The set $C_{n+1}(1, j)$ may be mapped bijectively to the disjoint union

$$ \bigcup_{i=1}^{j-1} C_n(i, j - 1) $$

as

$$ (1, a_2, a_3, \ldots, a_{k-1}, a_k (= j), a_{k+1}, \ldots, a_{n+1}) $$


\((a_2 - 1, a_3 - 1, \ldots, a_k - 1, j - 1, a_{k+1} - 1, \ldots, a_{n+1} - 1)\).

This image is clearly a member of \(C_n(i, j - 1)\) for some \(i\) with \(1 \leq i \leq j - 1\). Hence

\[ T_{n+1}(1, j) = \sum_{i=1}^{j-1} T_n(i, j - 1) \]

and this is the same as \(\sum_{i=1}^{n} T_n(i, j - 1)\) since the additional terms are all zero.

**Observation 2:** \(S(n, k) = T_n(i + 1, i + k + 1) - T_n(i, i + k)\) depends upon \(n\) and \(k\) but does not depend on \(i\). This can be proved using the theorem. However it is a surprising thing to note. A table of \(S(n, k)\) is presented below.

<table>
<thead>
<tr>
<th>(n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>6</td>
<td>24</td>
<td>6</td>
<td>8</td>
<td>11</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>24</td>
<td>30</td>
<td>38</td>
<td>49</td>
<td>65</td>
</tr>
</tbody>
</table>

Table 1: \(S(n, k)\) for \(k \leq 5\) and \(3 \leq n \leq 7\)

We note that \(S(n, 0) = (n - 2)!\) and \(S(n, 1) = (n - 3)!\).

In addition, the following observation is somewhat surprising. For \(2 \leq k < n - 1\)

\[ S(n, k) = S(n, k - 1) + S(n - 1, k - 1). \]  (3)

This observation can be proved using the theorem, but the algebra is somewhat involved. A combinatorial proof of this observation is interesting. The similarity between (7) and recurrence relations between binomial coefficients is also worth noting.

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