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ON THE SUPERPOSTITION OF FUNCTIONS IN CARLEMAN CLASSES

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In this paper we deal with classes of infinitely differentiable functions known in the literature as Carleman classes. Our main result is a characterisation of those Carleman classes that are closed under superposition. This result enables us to give a complete solution to a problem that has been considered by Gevrey, Cartan and Bang.

Let $M = \{M_n\}$ be a sequence of positive numbers and let $C_M(I)$ denote the Carleman class of functions $f \in C^\infty(I)$ which satisfy the following inequalities

$$\left\|f^{(n)}\right\|_\infty \leqslant A\lambda^n M_n,\ n \geqslant 0,\ A = A(f),\ \lambda = \lambda(f)$$

where I is a linear interval.

A class $C_M(I)$ is said to be *stable under superposition* if for every $g \in C_M(J)$ and $f \in C_M(I)$, where $I \supset \overline{g(J)}$, the composite function $f \circ g$ is also in the class C_M. The problem of finding conditions on M in order that the class C_M be stable under superposition was first considered by Gevrey [3] for the particular Carleman classes C_M where $M = \{(n!)^\alpha\}(\alpha > 1)$. For the general Carleman classes Cartan [2] showed that if the sequence $A = \{A_n\}$ where $A_n = \{(M_n/n!)^{1/n}\}$ is increasing then C_M is stable under superposition. The question as to whether the converse of this result is true remains unsolved. However if we suppose that the class C_M is differentiable, in the sense that $f \in C_M$ implies $f' \in C_M$, we are able to give a complete solution to this problem by showing that in this case a condition weaker than that of Cartan is both necessary and sufficient. We recall that the class $C_M(I)$ is *inverse-closed* if for every $f \in C_M(I)$ such that $f(x) \neq 0$, $f^{-1} \in C_M(I)$. It was Malliavin [4] who first gave a sufficient condition in order that $C_M(I)$ be inverse-closed, by showing that if the sequence A is almost increasing, that is $(\exists K > 0,\ s.t\ \forall n \leqslant m,\ A_n \leqslant KA_m)$, then the class C_M is inverse-closed. Rudin [6] proved that if C_M is a non-quasianalytic class of 2π-periodic functions then the converse of Malliavin's result is also true. To avoid the trivial cases we will suppose that $\lim\limits_{n\to\infty} M_n^{1/n} = \infty$ and hence the class $C_M(\mathbf{R}) \equiv C_{M^c}(\mathbf{R})$ where $\{\log M_n^c\}$ is the largest convex minorant of $\{\log M_n\}$ (see Madelbrojt [5]). If $\varliminf\limits_{n\to\infty} M_n^{1/n} = 0$,

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then $C_M(\mathbf{R})$ is reduced to the class of constant functions (see Madelbrojt [5]) which is obviously inverse-closed and if $\varliminf_{n\to\infty} M_n^{1/n} < +\infty$, then $C_M(\mathbf{R}) \equiv C_{\{1\}}(\mathbf{R})$ which is not inverse-closed, since if $f(x) = \sin x$, $f \in C_{\{1\}}(\mathbf{R})$ but $f \circ f \notin C_{\{1\}}(\mathbf{R})$ (see Bang [1]). Without loss of generality we will therefore suppose that $M = M^c$. Using the techniques developed in [7] we are now in the position to prove the following

THEOREM 1. *The following assertions are equivalent*

(i) *the sequence A is almost increasing;*

(ii) *if $f \in C_M(R)$ and f is analytic in a domain containing the closure of the range of g then $f \circ g \in C_M(\mathbf{R})$;*

(iii) *$C_M(\mathbf{R})$ is inverse-closed.*

PROOF: That (i) implies (ii) follows directly from the formula of Faà Di Bruno, namely

$$(1) \qquad (f\circ g)^{(n)}(x) = \sum \frac{n!}{k_1!k_2!\dots k_n} f^{(k)}[g(x)]\left(\frac{g'(x)}{1!}\right)^{k_1}\dots\left(\frac{g^{(n)}(x)}{n!}\right)^{k_n}$$

where the summation is over all the n-tuples $(k_1, k_2, \dots, k_n)$ such that $k_1 + k_2 + \dots + k_n = k$ and $k_1 + 2k_2 + \dots + nk_n = n$.
Trivially (ii) implies (iii). We now show that (iii) implies (i). Let

$$(2) \qquad g(x) = \sum_{v=1}^{\infty} \frac{1}{2^v} \frac{e^{ik_v x}}{T_M(k_v)}$$

where (see Madelbrojt [5])

$$T_M(r) = \sup_{n>0} \frac{r^n}{M_n}$$

and

$$M_n = \sup_{r>0} \frac{r^n}{T_M(r)}.$$

It can be easily seen that

$$\forall x \in \mathbf{R} \qquad \left|g^{(n)}(x)\right| \leqslant M_n,\ n \geqslant 0,$$

and that

$$g^{(n)}(0) = i^n s_n$$

where

$$s_n \geqslant \frac{1}{2^n} M_n.$$

Choose $f(x) = 1/(\lambda - x)$, where $\lambda > M_0$. Since $\lambda - g \in C_M(\mathbf{R})$ and $C_M(\mathbf{R})$ is inverse closed, it follows that $(f \circ g) = (\lambda - g)^{-1} \in C_M(\mathbf{R})$. Now choosing $k_s = k$, $k_j = 0$ for $j \neq s$, $n = ks$ in the formula (1) applied to the composite function $f \circ g$ at the point $x = 0$ we get

$$\left(\frac{M_s}{s!}\right)^{1/s} \leqslant K\left(\frac{M_n}{n!}\right)^{1/n}.$$

If n is not a multiple of s, let $sm < n < s(m+1)$ and so using the fact that $\{M_n^{1/n}\}$ is increasing we obtain

$$\frac{M_n^{1/n}}{n} \geqslant \frac{M_{sm}^{1/sm}}{s(m+1)} \geqslant \frac{M_{sm}^{1/sm}}{sm} \cdot \frac{sm}{s(m+1)} \geqslant \frac{1}{2}\frac{M_s^{1/s}}{s}.$$

Thus for $s \leqslant n$

$$\left(\frac{M_s}{s!}\right)^{1/s} \leqslant K\left(\frac{M_n}{n!}\right)^{1/n}$$

where K is independent of s and n, that is A is almost increasing. ∎

As remarked earlier Cartan [2] has shown that if A is increasing then the class C_M is stable under superposition. We now show that this result remains true if we suppose more generally that the sequence $A' = \{(M_n/n!)^{1/n-1}\}$ is almost increasing. In fact we have the following

THEOREM 2. *If the sequence A' is almost increasing then the class $C_M(I)$ is stable under superposition.*

PROOF: Let $g \in C_M(J)$ and $f \in C_M(I)$, where $I \supset \overline{g(J)}$, then applying formula (1) we get

$$\left|(f \circ g)^{(n)}(x)\right| \leqslant C\mu^n \sum \frac{n!}{k_1! \dots k_n!} M_k \left(\frac{M_1}{1!}\right)^{k_1} \dots \left(\frac{M_n}{n!}\right)^{k_n}. \tag{3}$$

But, since A' is almost increasing, we have

$$\left(\frac{M_2}{2!}\right)^{k_2} \dots \left(\frac{M_n}{n!}\right)^{k_n} \leqslant K^{n-k}\left(\frac{M_n}{n!}\right)^{(k_2+2k_3+\dots+(n-1)k_n)/(n-1)}$$

and

$$M_k \leqslant k! K^{k-1}\left(\frac{M_n}{n!}\right)^{(k-1)/(n-1)}.$$

And so from (3) it follows that

$$\left|(f \circ g)^{(n)}(x)\right| \leqslant B_1 \lambda^n M_n \sum \frac{k!}{k_1! \dots k_n!}.$$

Now using the identity

$$\sum \frac{k!}{k_1! \dots k_n!} = 2^{n-1}$$

we obtain

$$\left|(f \circ g)^{(n)}(x)\right| \leqslant B\lambda^n M_n.$$

∎

We now prove that the converse of Theorem 2 is true if the class $C_M(\mathbf{R})$ is differentiable. In face we prove the following

Theorem 3. *The following assertions are equivalent*

(i) *the sequence A is almost increasing;*

(ii) *the class $C_M(\mathbf{R})$ is stable under superposition, provided that A is not bounded.*

Proof: (i) implies (ii). If A is almost increasing, then

$$\forall s \leqslant n \qquad \left(\frac{M_s}{s!}\right)^{1/s} \leqslant K\left(\frac{M_n}{n!}\right)^{1/s},$$

and so

$$\left(\frac{M_s}{s!}\right)^{1/s-1} \leqslant K\left(\frac{M_n}{n!}\right)^{1/n-1} \cdot \left(\frac{M_s}{s!}\right)^{1/n(s-1)}.$$

But since the class $C_M(\mathbf{R})$ is differentiable, we have

$$\left(\frac{M_s}{s!}\right)^{1/n(s-1)} \leqslant K_1,$$

and so the sequence $\{(M_n/n!)^{1/n-1}\}$ is almost increasing. Now by Theorem 2, we get that $C_M(\mathbf{R})$ is stable under superposition. Thus (i) implies (ii). Conversely, if $C_M(\mathbf{R})$ is stable under superposition and A is not bounded, then $C_M(\mathbf{R})$ is stable under composition with analytic functions and so it is inverse-closed. It follows by Theorem 1 that the sequence $\{(M_n/n!)^{1/n}\}$ is almost increasing. Thus (ii) implied (i). Now, someone may ask if it is possible to find a Carleman class for which our result applies but Cartan's does not, and the answer is positive. Let $M = \{M_n\}$ be the following sequence

$$M_n^{1/n} = n + 1.$$

Then it is clear that M is log-convex, the class $C_M(\mathbf{R})$ is differentiable, the associated sequence $A = (M_n/n!)^{1/n}$ is almost increasing but not increasing. ∎

We now show that if the sequence A is bounded, then, in general, $C_M(\mathbf{R})$ is not stable under superposition. In fact we have the following result.

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THEOREM 4. *If the sequence A is decreasing to 0, then there exist two functions $g \in C_M(\mathbf{R})$ and $f \in C_M(J)$ where $J \supset \overline{g(\mathbf{R})}$ such that $f \circ g \notin C_M$.*

PROOF: Since the sequence A is decreasing to 0, this enables us to construct a function f belonging to $C_M(\mathbf{R})$ such that $f^{(n)}(0) = M_n$, for all $n \geqslant 0$. In fact since A is decreasing, then there exists a constant K such that

$$\forall s \leqslant n \qquad \frac{M_n}{n!} \leqslant K^{n+1}\frac{M_s}{s!} \tag{4}$$

Puting $s = 0$ in (4) we get

$$M_n \leqslant K^{n+1} n!$$

and so it follows that the series

$$f(x) = \sum_{n=0}^{\infty} \frac{M_n}{n!} x^n$$

converges for $|x| < 1/K$, and hence $f^{(k)}(0) = M_k$. We have also

$$\left|f^{(k)}(x)\right| \leqslant \sum_{n=0}^{\infty} \left|\frac{x^{n-k} M_n}{(n-k)}\right| \leqslant \sum_{n=0}^{\infty} \left|x^{n-k}\right| K^{n+1} \binom{n}{k} M_k.$$

But since $\binom{n}{k} \leqslant 2^n$ we have

$$\left|f^{(k)}(x)\right| \leqslant A\lambda^k M_k.$$

Thus f belongs to $C_M(\mathbf{R})$ in a neighbourhood of the origin. But since $C_M(\mathbf{R})$ is analytic, $f \in C_M(\mathbf{R})$. Choose now

$$g(x) = \sum_{v=1}^{\infty} \frac{e^{ik_v x}}{T_M(k_v)}.$$

We have already shown that $g \in C_M(\mathbf{R})$ and that

$$g^{(n)}(0) = i^n s_n$$

where

$$s_n \geqslant \frac{1}{2^n} M_n.$$

Suppose now that the composite function $f \circ g \in C_M(\mathbf{R})$. Then applying formula (1) to $f \circ g$ at the point $x = 0$ and $\forall j \neq s$, choosing $k_s = k$, $k_j = 0$ where $n = ks$ we have

$$\frac{n! M_k}{2^n k!} \left(\frac{M_s}{s!}\right)^k \leqslant B\mu^n M_n.$$

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Thus

$$\left(\frac{M_k}{k!}\right)\left(\frac{M_s}{s!}\right)^k \leqslant B2\mu^n\left(\frac{M_n}{n!}\right). \tag{5}$$

Now, using the fact that the sequence A is decreasing, and taking the nth root of both sides, we get

$$\left(\frac{M_s}{s!}\right)^{1/s} \leqslant B^{1/ks}2\mu\left(\frac{M_n}{n!}\right)^{(1/ks)s-(1/s)}.$$

If now we let k tend to infinity we have

$$M_s = 0, \forall s \geqslant 0$$

that is the class $C_M(\mathbb{R})$ is trivial, which is not the case and so $f \circ g \notin C_M(\mathbb{R})$. If n is not a multiple of s we proceed as in Theorem 1 and the proof of the Theorem is complete. ∎

References

[1] T. Bang, *On quasi-analytisk funktioner* (Nyt Norclisk Forlag, Copenhagen, 1946).

[2] H. Cartan, *Sur les classes de fonctions définies par des inégalités sur leurs dérivées successives* (Actualités Scientifiques et industrielles numéro 897, 1940).

[3] M. Gevrey, *Oeuvres complètes* (Editions du Centre National de la recherche scientifique, Paris, 1970).

[4] P. Malliavin, 'Calcul symbolique et sous algèbres de $L^1(G)$', *Bull. Soc. Math. France* **87** (1959), 181–190.

[5] S. Mandelbrojt, *Series adhérentes, régularisation des suites, applications* (Gauthier-Villars, Paris, 1952).

[6] W. Rudin, 'Division in algebras of infinitely differentiable functions', *J. Math. Mech* **11** (1962), 797–809.

[7] J.A. Siddiqi and M. Ider, 'Symbolic Calculus in analytic Carleman classes', *Proc. Amer. Math. Soc.* **99** (1987), 347–350.

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