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A REMARK ON PEIRCE'S LAW

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Before stating the purpose, we explain some propositional logics treated in this paper. The logical symbols we use are: implication $\rightarrow$, conjunction $\wedge$, disjunction $\vee$, and the propositional constant $\curlywedge$ denoting contradiction. The axioms for the intuitionistic propositional logic (denoted by $\boldsymbol{LJS}$) are:

(I) $\quad p \rightarrow (q \rightarrow p), \quad (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)),$

(C) $\quad (p \wedge q) \rightarrow p, \quad (p \wedge q) \rightarrow q, \quad (r \rightarrow p) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow (p \wedge q))),$

(D) $\quad p \rightarrow (p \vee q), \quad q \rightarrow (p \vee q), \quad (p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r)),$

(F) $\quad \curlywedge \rightarrow p.$

The rules of inference are *modus ponens* and *substitution*. The system characterized by the axioms (I) we call the primitive propositional logic (denoted by $\boldsymbol{LOS}$), which is the propositional part of the primitive logic $\boldsymbol{LO}$ introduced in Ono [3]. $\boldsymbol{LOS}$ is also known as the positive implicational logic. The axioms (I), (C), (D) characterize the (full) positive propositional logic (denoted by $\boldsymbol{LPS}$). Not all classically true formulas expressible in $\boldsymbol{LOS}$ are derivable from (I); they are derivable from (I) together with the axiom known as *Peirce's law*:

(P) $\quad ((p \rightarrow q) \rightarrow p) \rightarrow p.$

The axioms (I), (C), (D), (P) are sufficient for the derivation of all classically true formulas expressible in $\boldsymbol{LPS}$. Moreover, the axioms (I), (C), (D), (F), (P) characterize the classical propositional logic (denoted by $\boldsymbol{LKS}$). Indeed, all classically true propositional formulas are provable in $\boldsymbol{LKS}$. Finally, by deleting the axiom (F) from $\boldsymbol{LJS}$, we obtain Johansson's minimal propositional logic (denoted by $\boldsymbol{LMS}$). It is easy to see that the following formula

(M) $\quad ((p \rightarrow \curlywedge) \rightarrow p) \rightarrow p$

is equivalent to *the law of the excluded middle* in $\boldsymbol{LMS}$. Furthermore, it

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should be remarked that (M) is not equivalent to (P) in ***LMS***, as is well-known; in ***LJS***, however, (M) is equivalent to (P).

Now, let us consider the following formula:

$$(\mathrm{P}^*) \quad ((p \to (((q \to r) \to q) \to q)) \to p) \to p.$$

It is shown in Troelstra [4] as well as in Nagata [2] that (P*) is strictly weaker than (P) in ***LOS***. This fact suggests us a method for weakening Peirce's law, and this is really carried out in Miura and Nagata [1]. Indeed, in [1] (and also in [2] and [4]), various formulas of the type:

$$(*) \quad ((p \to A) \to p) \to p$$

are given and shown that these are strictly weaker than (P) (in ***LOS***, for example). Now, we remark that the formulas A in (*) appearing in [1], [2], and [4] are all classically true. Thus, it would be a natural course of matter to raise the following question: Is the formula (*) equivalent to Peirce's law (P) in some propositional logic if a formula A is not classically true? The answer is obviously "no." In fact, take $q \to p$ as A, then (*) is evidently provable in ***LOS***; hence, though $q \to p$ is not classically true, the formula (*) is not equivalent to (P) in any propositional logic. If we assume, however, that $p \to A$ is not classically true, then we can assert that the formula (*) is equivalent to (P) in some propositional logic, and vice versa. The purpose of this paper is to prove the following theorem.

THEOREM. *For any propositional variable p and for any formula A expressible in **LOS** (in **LPS**, or in **LJS**), the formula* (*) *is equivalent to Peirce's law* (P) *in **LOS** (in **LPS**, or in **LJS**) if and only if $p \to A$ is not classically true.*

Remark. The part "if" of Theorem does not hold for ***LMS***. In fact, (M) is not equivalent to (P) in ***LMS***, but $p \to \curlywedge$ is not classically true.

At first, we state two lemmas which are helpful to prove our theorem. One of them is a lemma proved in Tugué [5] (p. 304). Following [5], we denote by v an evaluating function of the ordinary two-valued truth table whose values are 0 (truth) and 1 (falsity). Let A be a formula and $p_1, \cdots, p_n$ be all propositional variables occurring in A. Given n-tuple $v(p_1), \cdots, v(p_n)$ of values of $p_1, \cdots, p_n$, we shall denote, as convention, the variables assigned the value 0 by $r_1, \cdots, r_u$, the rest by $s_1, \cdots, s_v$. Then, the following lemma holds for ***LOS*** (or ***LPS***).

LEMMA 1. *Let A be a formula expressible in* ***LOS*** *(or in* ***LPS****). For the given n-tuple $v(p_1), \cdots, v(p_n)$ of values of variables occurring in A,*

$$r_1, \cdots, r_u, \quad s_1 \to s_2, \quad s_2 \to s_3, \cdots, s_v \to s_1 \vdash_{\substack{\boldsymbol{LOS} \\ (\boldsymbol{LPS})}} A,$$

or

$$r_1, \cdots, r_u, \quad s_1 \to s_2, \quad s_2 \to s_3, \cdots, s_v \to s_1 \vdash_{\substack{\boldsymbol{LOS} \\ (\boldsymbol{LPS})}} A \to s_1,$$

according as $v(A) = 0$ or 1.

Lemma 1 is restricted to the negationless (or positive) propositional logics. We can extend this to the propositional logic with negation concept. That is, the following lemma holds for ***LJS***, where $v(\curlywedge) = 1$.

LEMMA 2. *For the given n-tuple $v(p_1), \cdots, v(p_n)$ of values of variables occurring in A,*

$$r_1, \cdots, r_u, \quad s_1 \to \curlywedge, \cdots, s_v \to \curlywedge \vdash_{\boldsymbol{LJS}} A,$$

or

$$r_1, \cdots, r_u, \quad s_1 \to \curlywedge, \cdots, s_v \to \curlywedge \vdash_{\boldsymbol{LJS}} A \to \curlywedge,$$

according as $v(A) = 0$ or 1.

Now, by making use of Lemma 1, we can prove the following:

LEMMA 3. *For any propositional variable p and for any formula A expressible in* ***LOS*** *(or in* ***LPS****), if $p \to A$ is not classically true, then the formula (*) is equivalent to (P) in* ***LOS*** *(or in* ***LPS****).*

Proof. It is obvious that (*) is derivable from (P) in ***LOS*** (***LPS***). So, we have only to show that (P) is derivable from (*) in ***LOS*** (***LPS***) under the assumption that $p \to A$ is not classically true. If $p \to A$ is not classically true, $v(p \to A)$ is not identically equal to 0. Let $p_1, \cdots, p_n$ be all variables occurring in A. Then, for some n-tuple $v(p_1), \cdots, v(p_n)$, $v(A) = 1$. Let us fix an n-tuple $v(p_1), \cdots, v(p_n)$ such that $v(A) = 1$. Now, consider a formula A^* obtained from A by substituting p for all variables p_i such that $v(p_i) = 0$, q for all variables p_j such that $v(p_j) = 1$. Then, A^* is a formula expressible in ***LOS*** (***LPS***) in which no variables other than p and q occur. Moreover, $v(A^*) = 1$ when $v(p) = 0$ and $v(q) = 1$. Hence, by virtue of Lemma 1, we have $p, q \to q \vdash_{\substack{\boldsymbol{LOS}\\(\boldsymbol{LPS})}} A^* \to q$. From this, $((p \to A^*) \to p) \to p \vdash_{\substack{\boldsymbol{LOS}\\(\boldsymbol{LPS})}} ((p \to q) \to p) \to p$. Since $((p \to A^*) \to p) \to p$ is derivable from (*) in ***LOS*** (***LPS***), we can conclude that (P) is derivable from (*) in ***LOS*** (***LPS***).

Similarly, the following lemma is proved by making use of Lemma 2.

LEMMA 4. *For any propositional variable p and for any formula A, if $p \to A$ is not classically true, then the formula* (*) *is equivalent to* (P) *in* ***LJS***.

Proof. Assume that $p \to A$ is not classically true. Since (*) is derivable from (P) in ***LJS*** and (P) is equivalent to (M) in ***LJS***, we have only to show that (M) is derivable from (*) in ***LJS***. We define A^* as in the proof of Lemma 3; i.e., A^* is obtained from A by substituting p for all variables p_i such that $v(p_i) = 0$, $\curlywedge$ for all variables p_j such that $v(p_j) = 1$. Then, A^* contains only one variable p, and $v(A^*) = 1$ when $v(p) = 0$. By virtue of Lemma 2, we have $p \vdash_{\boldsymbol{LJS}} A^* \to \curlywedge$. From this, $((p \to A^*) \to p) \to p \vdash_{\boldsymbol{LJS}} ((p \to \curlywedge) \to p) \to p$. Hence, (M) is derivable from (*) in ***LJS***. So, (P) is derivable from (*) in ***LJS***.

Finally, we show the converse of Lemmas 3 and 4. This is stated as follows.

LEMMA 5. *For any propositional variable p and for any formula A expressible in* ***LOS*** *(in* ***LPS***, *or in* ***LJS***), *if the formula* (*) *is equivalent to* (P) *in* ***LOS*** *(in* ***LPS***, *or in* ***LJS***), *then $p \to A$ is not classically true.*

Remark. Lemma 5 holds also for ***LMS***.

In order to prove this, we use another evaluating function v^* of the three-valued truth table whose values are 0, 1, 2. This table is defined as follows:

$$v^*(p \to q) = \begin{cases} v^*(q) & \text{if } v^*(p) < v^*(q), \\ 0 & \text{otherwise}, \end{cases}$$

$$v^*(p \wedge q) = max\,(v^*(p),\ v^*(q)),$$

$$v^*(p \vee q) = min\,(v^*(p),\ v^*(q)),$$

$$v^*(\curlywedge) = 2.$$

For this function v^*, we can easily see that

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$$v^*(((p \to q) \to p) \to p) = \begin{cases} 1 & \text{if } v^*(p) = 1 \text{ and } v^*(q) = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for a formula A, if $v^*(p \to A)$ is never equal to 2, then $v^*(((p \to A) \to p) \to p)$ is identically equal to 0. Using these facts, we can prove Lemma 5.

Proof of Lemma 5. Assume that the formula (*) (*i.e.* $((p \to A) \to p) \to p)$ is equivalent to (P) in ***LOS*** (***LPS***, ***LJS***). We wish to show that $p \to A$ is not classically true. Suppose that $p \to A$ is classically true. Then, $v^*(p \to A)$ is never equal to 2. Hence, $v^*(((p \to A) \to p) \to p)$ is identically equal to 0. Therefore, (P) is not derivable from (*) in ***LOS*** (***LPS***, ***LJS***). This contradicts to our assumption. Accordingly, we can conclude that $p \to A$ is not classically true.

Our theorem is immediate from Lemmas 3, 4, and 5.

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