COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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For fixed $k \geqq 2$, let V_k denote the class of normalized analytic functions

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

such that $z \in E = \{z; |z| < 1\}$ are regular and have $f'(0) = 1, f'(z) \neq 0$, and

$$\int_0^{2\pi} \left| \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \right|_{z=re^{i\theta}} d\theta \leqq k\pi. \tag{1}$$

Let S_k be the subclass of V_k whose members $f(z)$ are univalent in E. It was pointed out by Paatero (**4**) that V_k coincides with S_k whenever $2 \leqq k \leqq 4$. Later Rényi (**5**) showed that in this case, $f(z) \in S_k$ is also convex in one direction in E. In (**6**) I showed that the Bieberbach conjecture

$$|a_n| \leqq n, \qquad n = 2, 3, \ldots,$$

holds for functions convex in one direction. If $f \in V_k$ and $n = 2, 3$, the sharp results

$$|a_2| \leqq \tfrac{1}{2}k, \qquad |a_3| \leqq \tfrac{1}{6}(k^2 + 2), \tag{2}$$

due to Pick (see **3**, p. 5) and Lehto (**3**), respectively, are known. If $f \in S_k$, $2 \leqq k \leqq 4$, then, as was shown by Schiffer and Tammi (**8**),

$$|a_4| \leqq (1/24)(k^3 + 8k). \tag{3}$$

Equalities are attained in (2) and (3) for the extremal function

$$f(z) = \frac{1}{\epsilon k}\left[\left(\frac{1 + \epsilon z}{1 - \epsilon z}\right)^{\frac{1}{2}k} - 1\right], \qquad |\epsilon| = 1. \tag{4}$$

Lehto (**3**) has also shown that if $f(z) \in V_k$, then as $k \to \infty$, we have:

$$\max_{V_k} |a_n(f)| \sim \frac{k^{n-1}}{n!},$$

where $a_n(f) = (1/n!)f^{(n)}(0)$. W. Kirwan has informed the author orally that he has recently obtained the inequalities

$$|a_n| \leqq c(k)n^{\frac{1}{2}k-1}, \qquad n = 2, 3, \ldots,$$

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for $f \in V_k$ with $c(k) = e2^{\frac{1}{2}k-2}$. Here $c(k) \to \infty$ as $k \to \infty$. This fact and the extremal function (4) show that

$$\max_{V_k} |a_n(f)| = O(n^{\frac{1}{2}k-1}) \quad \text{as } n \to \infty.$$

In this paper we use a quite different method of attack, interesting in itself, from that of Kirwan, obtaining his result with the additional improvement that $c(k) \to 0$ as $k \to \infty$, $f \in V_k$. If $f \in S_k$, $2 \leqq k < \infty$, for each fixed k this method also furnishes a numerical bound, independent of n, for the difference $||a_{n+1}| - |a_n||$, $n = 1, 2, 3, \ldots$. That some bound, independent of n, exists follows from the result of Hayman (**2**), but an estimate for its numerical value for the class S_k has not been known except when $2 \leqq k \leqq 4$. In this case, since $f(z) \in S_k$ is also convex in one direction, the inequalities

$$(5) \qquad -3 + (2/n) \leqq |a_n| - |a_{n-1}| \leqq 2 - (1/n), \qquad n = 2, 3, \ldots,$$

obtained earlier (**7**) apply.

We prove the following theorems.

THEOREM 1. *Let $f(z) \in V_k$, $2 \leqq k < \infty$. Let $x \in E$ and*

$$F(z) = \frac{f\left(\dfrac{x+z}{1+\bar{x}z}\right) - f(x)}{f'(x)(1-|x|^2)}.$$

Then $F(z) \in V_k$ and

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leqq \frac{k|z|}{1-|z|^2}.$$

COROLLARY. *If $f(z) \in V_k$, $2 \leqq k < \infty$, then $f(z)$ maps $|z| < \frac{1}{2}(k - (k^2-4)^{\frac{1}{2}})$ onto a convex domain. The estimate is sharp. Moreover, if $f(z) = z + \sum_2^\infty a_n z^n$, then $|a_n| < k^{n-1}$, $n = 2, 3, \ldots$.*

THEOREM 2. *Let $f(z) \in V_k$, $2 \leqq k < \infty$. Then*

$$|a_n| < (k^2+k)\left(\frac{2n}{3}\right)^{\frac{1}{2}k-1}, \qquad n = 2, 3, \ldots,$$

$$\limsup_{n\to\infty} \frac{|a_n|}{n^{\frac{1}{2}k-1}} \leqq \frac{(k^2+k)}{16} \cdot \left(\frac{4e}{k+4}\right)^{\frac{1}{2}(k+4)}.$$

THEOREM 3. *Let $f(z) \in S_k$, $2 \leqq k < \infty$. Then*

$$||a_{n+1}| - |a_n|| < 2(\tfrac{1}{3}e)^3(k^2+k), \qquad n = 1, 2, \ldots.$$

Proof of Theorem 1. Let $f(z) \in V_k$, $2 \leqq k < \infty$. Let ρ be a real number in the interval $(0, 1)$ and let x be a complex number, $|x| < 1$. Let $F_\rho(z)$ be defined by the equation

$$F_\rho(z) = \frac{f(\rho\zeta) - f(\rho x)}{\rho f'(\rho x)(1-|x|^2)}, \qquad \zeta = \frac{x+z}{1+\bar{x}z}.$$

$F_\rho(z)$ is regular for $|z| \leqq 1$, $F_\rho'(0) = 1$ and $F_\rho'(z) \neq 0$ for $|z| \leqq 1$. A calculation yields:

$$1 + z\frac{F_\rho''(z)}{F_\rho'(z)} = \left\{1 + \rho\zeta\frac{f''(\rho\zeta)}{f'(\rho\zeta)}\right\}\frac{(1-|x|^2)z}{(x+z)(1+\bar{x}z)} + \frac{x - \bar{x}z^2}{(x+z)(1+\bar{x}z)}.$$

Let

$$z = e^{i\theta},\quad \frac{x + e^{i\theta}}{1 + \bar{x}e^{i\theta}} = e^{i\phi},\quad \frac{1-|x|^2}{|x+e^{i\theta}|^2}\,d\theta = d\phi.$$

Then

$$\mathrm{Re}\left\{1 + e^{i\theta}\frac{F_\rho''(e^{i\theta})}{F_\rho'(e^{i\theta})}\right\}d\theta = \mathrm{Re}\left\{1 + \rho e^{i\phi}\frac{f''(\rho e^{i\phi})}{f'(\rho e^{i\phi})}\right\}d\phi,$$

$$\int_0^{2\pi}\left|\mathrm{Re}\left\{1 + e^{i\theta}\frac{F_\rho''(e^{i\theta})}{F_\rho'(e^{i\theta})}\right\}\right|d\theta = \int_0^{2\pi}\left|\mathrm{Re}\left\{1 + \rho e^{i\phi}\frac{f''(\rho e^{i\phi})}{f'(\rho e^{i\phi})}\right\}\right|d\phi \leqq k\pi.$$

Since the integral

$$\int_0^{2\pi}\left|\mathrm{Re}\left\{1 + re^{i\theta}\frac{F_\rho''(re^{i\theta})}{F_\rho'(re^{i\theta})}\right\}\right|d\theta$$

is an increasing function of r, it is bounded by $k\pi$ for $0 \leqq r < 1$. Let $F(z) = \lim_{\rho\to 1} F_\rho(z)$. It follows that

$$\int_0^{2\pi}\left|\mathrm{Re}\left\{1 + re^{i\theta}\frac{F''(re^{i\theta})}{F'(re^{i\theta})}\right\}\right|d\theta \leqq k\pi,\qquad 0 \leqq r < 1,$$

therefore $F(z) \in V_k$.

The function $F(z)$ has $|\frac{1}{2}F''(0)| \leqq \frac{1}{2}k$ by (2). Hence

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| = \frac{|z|}{1-|z|^2}\,|F''(0)| \leqq \frac{k|z|}{1-|z|^2}. \tag{6}$$

From (6) we obtain

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqq \frac{1 - k|z| + |z|^2}{1-|z|^2} \geqq 0 \quad \text{for } |z| \leqq R = \frac{k - (k^2-4)^{\frac{1}{2}}}{2},$$

with equality holding for the extremal function (4). We conclude that if $f(z) \in V_k$, then $f(z)$ maps $|z| \leqq R$ onto a convex domain. When $k = 4$, $f(z)$ is schlicht in E, and R reduces to the well-known radius of convexity $2 - \sqrt{3}$.

Since $f(RZ)/R = \sum_1^\infty a_n R^{n-1}z^n$ is convex for $|z| < 1$, we have $|a_n|R^{n-1} \leqq 1$ which implies that $|a_n| \leqq (\frac{1}{2}(k + (k^2-4)^{\frac{1}{2}}))^{n-1} < k^{n-1}$, $n = 2, 3, \ldots$. This completes the proof of Theorem 1 and the Corollary.

Proofs of Theorems 2 *and* 3. Let $f(z) \in V_k$. We may assume for convenience that $f(z)$ is regular on $|z| = 1$ since otherwise we could consider the function $f(\rho z)/\rho$, $0 < \rho < 1$, and let $\rho \to 1$ at the end of the proof. Since $f'(z) \neq 0$ in E, we may write, when $\zeta = e^{i\phi}$,

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left[1 + \zeta\frac{f''(\zeta)}{f'(\zeta)}\right]\frac{\zeta + z}{\zeta - z}\,d\phi.$$

For $z = 0$ we have

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right] d\phi.$$

Hence

$$\frac{f''(z)}{f'(z)} = \frac{1}{\pi}\int_0^{2\pi} \operatorname{Re}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right]\frac{d\phi}{\zeta - z}. \tag{7}$$

A differentiation of (7) yields

$$\frac{f'''(z)}{f'(z)} = \left(\frac{f''(z)}{f'(z)}\right)^2 + \frac{1}{\pi}\int_0^{2\pi} \operatorname{Re}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right]\frac{d\phi}{(\zeta - z)^2}. \tag{8}$$

Put $z = re^{i\theta}$ in (8) and integrate with respect to θ. Then

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'''(z)}{f'(z)}\right| d\theta &\leqq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f''(z)}{f'(z)}\right|^2 d\theta \\
&\quad + \frac{1}{\pi}\int_0^{2\pi}\left|\operatorname{Re}\left[1 + \zeta\frac{f''(\zeta)}{f'(\zeta)}\right]\right|\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|\zeta - z|^2}\,d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f''(z)}{f'(z)}\right|^2 d\theta \\
&\quad + \frac{1}{1 - r^2}\cdot\frac{1}{\pi}\int_0^{2\pi}\left|\operatorname{Re}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right]\right| d\phi \\
&\leqq \sum_{n=0}^{\infty} |d_n|^2 r^{2n} + \frac{k}{1 - r^2},
\end{aligned}$$

where

$$\frac{f''(z)}{f'(z)} = \sum_0^{\infty} d_n z^n = \frac{1}{\pi}\int_0^{2\pi}\left[\operatorname{Re}\left\{1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}\right]\left(\sum_0^{\infty}\frac{z^n}{\zeta^{n+1}}\right) d\phi,$$

$$\begin{gathered}
d_n = \frac{1}{\pi}\int_0^{2\pi}\operatorname{Re}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right]\frac{d\phi}{\zeta^{n+1}}, \\
|d_n| \leqq \frac{1}{\pi}\int_0^{2\pi}\left|\operatorname{Re}\left[1 + \frac{\zeta f''(\zeta)}{f'(\zeta)}\right]\right| d\phi \leqq k, \\
\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f'''(z)}{f'(z)}\right| d\theta \leqq \sum_0^{\infty} |d_n|^2 r^{2n} + \frac{k}{1 - r^2} \leqq \frac{(k^2 + k)}{1 - r^2}.
\end{gathered} \tag{9}$$

For $f(z) = z + \sum_2^{\infty} a_n z^n \in V_k$ we have

$$\begin{aligned}
n(n-1)(n-2)|a_n| &\leqq \frac{1}{2\pi r^{n-3}}\int_0^{2\pi} |f'''(re^{i\theta})|\, d\theta \\
&= \frac{1}{2\pi r^{n-3}}\int_0^{2\pi} |f'(re^{i\theta})|\left|\frac{f'''(re^{i\theta})}{f'(re^{i\theta})}\right| d\theta.
\end{aligned} \tag{10}$$

An integration of the inequality (6) yields the known inequalities (see **3**)

$$\frac{(1-r)^{\frac{1}{2}k-1}}{(1+r)^{\frac{1}{2}k+1}} \leqq |f'(re^{i\theta})| \leqq \frac{(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}}. \tag{11}$$

For $z = re^{i\theta}$, (10) and (11) yield:

$$\begin{aligned} n(n-1)(n-2)|a_n| &\leqq \frac{(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}} \cdot \frac{1}{2\pi r^{n-3}} \int_0^{2\pi} \left|\frac{f'''(z)}{f'(z)}\right| d\theta \\ &\leqq \frac{(1+r)^{\frac{1}{2}k-1}}{(1-r)^{\frac{1}{2}k+1}} \cdot \frac{1}{r^{n-3}} \cdot \frac{(k^2+k)}{1-r^2} \\ &= \frac{(k^2+k)(1+r)^{\frac{1}{2}k-2}}{r^{n-3}} \cdot (1-r)^{-\frac{1}{2}k-2}. \end{aligned} \tag{12}$$

Let $r = 1 - 3/n$, $n > 3$, in (12). Then

$$|a_n| \leqq \frac{(k^2+k)}{27} e^3 \left(2 - \frac{3}{n}\right)^{\frac{1}{2}k-2} \cdot \frac{n^2}{(n-1)(n-2)} \left(\frac{n}{3}\right)^{\frac{1}{2}k-1} < (k^2+k)\left(\frac{2n}{3}\right)^{\frac{1}{2}k-1}.$$

The inequalities (2) show that the inequalities

$$|a_n| < (k^2+k)\left(\frac{2n}{3}\right)^{\frac{1}{2}k-1}, \qquad n > 3,$$

also hold when $n = 2$ or 3.

If in (12) we take $r = 1 - (k+4)/2n$, $n > \frac{1}{2}(k+4)$, we deduce similarly that

$$|a_n| \leqq (k^2+k)\left(\frac{e}{k+4}\right)^2 \left(\frac{4e}{k+4}\right)^{\frac{1}{2}k} \left(1 + O\left(\frac{1}{n}\right)\right) n^{\frac{1}{2}k-1},$$

$$\limsup_{n\to\infty} \frac{|a_n|}{n^{\frac{1}{2}k-1}} \leqq \left(\frac{k^2+k}{16}\right)\left(\frac{4e}{k+4}\right)^{\frac{1}{2}(k+4)}, \tag{13}$$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \frac{|a_n|}{n^{\frac{1}{2}k-1}} = 0.$$

This completes the proof of Theorem 2.

We turn next to the proof of Theorem 3. Let $f(z) \in S_k$, $2 \leqq k < \infty$. Let z_1 be a point on $|z| = r$, where $\max_{|z|=r} |f(z)| = |f(z_1)|$. Since $f(z)$ is schlicht in E, we have the inequality of Golusin (**1**), namely

$$|(z - z_1)f'(z)| \leqq \frac{2|z|}{(1-|z|)^2}. \tag{14}$$

Furthermore we have

$$(z - z_1)f'''(z) = -6a_3z_1 - \sum_{n=3}^{\infty} [n(n^2-1)a_{n+1}z_1 - n(n-1)(n-2)a_n]z^{n-2}.$$

From (9) and (14) we have

$$n(n-1)|(n+1)a_{n+1}z_1-(n-2)a_n| \leqq \frac{1}{2\pi r^{n-2}}\int_0^{2\pi} |(z-z_1)f'(z)|\left|\frac{f'''(z)}{f'(z)}\right| d\theta$$

$$\leqq \frac{1}{r^{n-2}}\cdot\frac{2r}{(1-r)^2}\cdot\frac{(k^2+k)}{1-r^2}=\frac{2(k^2+k)}{r^{n-3}(1+r)}\cdot(1-r)^{-3}.$$

We pick $|z_1| = r = (n-2)/(n+1)$, $n > 2$. Then

$$n(n-1)(n-2)||a_{n+1}|-|a_n|| \leqq n(n-1)|(n+1)a_{n+1}z_1-(n-2)a_n|$$

$$\leqq \frac{2(k^2+k)}{\left(\dfrac{2n-1}{n+1}\right)}\cdot\left(1+\frac{3}{n-2}\right)^{\frac{1}{3}(n-2)\cdot 3}\cdot\left(\frac{n-2}{n+1}\right)\left(\frac{n+1}{3}\right)^3$$

$$<\frac{2}{27}(k^2+k)e^3\left(\frac{n-2}{2n-1}\right)(n+1)^3,$$

$$||a_{n+1}|-|a_n|| \leqq 2(k^2+k)\left(\frac{e}{3}\right)^3\frac{(n+1)^3}{n(n-1)(2n-1)}<2\left(\frac{e}{3}\right)^3(k^2+k)$$

for $n > 6$. The inequalities of Theorem 3 are obviously satisfied for $n \geqq 1$ whenever $2 \leqq k \leqq 4$ because of the inequalities (5). If $k > 4$, then $2(\frac{1}{3}e)^3(k^2+k) > 29.7$. For the range $1 \leqq n \leqq 6$, the inequalities of Theorem 3 are still valid since $|a_n| < en$, $n = 2, 3, \ldots$, whenever $f(z) \in S_k$. This completes the proof of Theorem 3.

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