BULL. AUSTRAL. MATH. SOC. 28A60, 28A32
VOL. 18 (1978), 267-285. (54D35, 54D60, 60B10)

Measure theoretic techniques in topology and mappings of replete and measure replete spaces

George Bachman and Alan Sultan

We prove in this paper several results on lattice related measures and images of such measures under mappings. If we apply these to various areas of point set topology we obtain as corollaries many known and new results on sequential compactness, repleteness, and measure repleteness - areas of recent considerable interest to mathematicians.

1. Introduction

In recent papers, we were concerned with general measure extension procedures and their applications to various areas of analysis and point set topology. The results, which were proved for measures on pavings, enabled one to obtain simultaneously many different types of results from a unified point of view (see, in particular, [5, 6, 7, 25, 26]). In this paper we study similar questions but with particular emphasis on the concepts of outer measure and μ^*-measurability. In this manner we are able to obtain many new theorems and many new nontrivial applications to the areas of compactness, repleteness, and measure repleteness. Specifically after defining terms in Section 2, we prove in Section 3 a general theorem on weak convergence of measures, generalizing theorems of Alexandroff [3], Moran [23], and Varadarajan [27]. When this theorem is applied to the specific case of 2-valued measures, we obtain many new theorems on different types of sequential compactness and countably compact lattices.

Received 10 January 1978.

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

In the next section we deal with mappings of measures and apply our results to obtain new theorems on mappings of measure replete spaces, a concept studied by several authors, including Moran [21, 22, 23], Varadarajan [27], Mosiman and Wheeler [24], Kirk [18, 19, 20], Gardner [11], Haydon [14], and others in some very special cases. We generalize this even further to the concept of fully measure replete. In the final section we apply the concept of μ^*-measurability and obtain a substantial generalization of one of the main theorems of [6]. Again we are able to apply this to obtain many new theorems on repleteness of a different character. The techniques presented are very general and have applications above and beyond those presented here.

2. Definitions and notations

Our definitions and notations will be the same as those in [6] and we assume that the reader is familiar with the basic results of that paper, in particular with Section 2 of that paper. We keep our blanket assumption that $\emptyset$ and X are elements of all lattices involved, and that all measures are bounded and nonnegative. We will need the following additional notations and concepts: if $\mu \in \mathrm{MR}(\sigma, L)$, then by the *support* of μ denoted by $S(\mu)$, we mean $\bigcap\{A \in L : \mu(A) = \mu(X)\}$. L is called *measure-replete* if and only if every $\mu \in \mathrm{MR}(\sigma, L)$ has non empty support. Since L is replete if and only if every $\mu \in \mathrm{IR}(\sigma, L)$ has non empty support, that is, is fixed, we see that L-measure replete implies L-replete: however, the converse in general does not hold. If $L = Z$ for some Tychonoff space X, then L is measure replete if and only if X is *measure compact* in the sense of [22, p. 495] and this is true if and only if $\mathrm{MR}(\sigma, L) = \mathrm{MR}(\tau, L)$ (see [21, p. 634]). If $L = F$ a similar statement holds (see [11, p. 96]). It is not difficult to see that a similar results holds if L is any delta lattice.

The *vague* or *weak topology* on $\mathrm{MR}(L)$ is that characterized by the following convergence of nets: $\mu_\alpha \to \mu$ if and only if $\int f d_{\mu_\alpha} \to \int f d_\mu$ for all $f \in C_b(L)$, where $\mu_\alpha, \mu \in \mathrm{MR}(L)$. If L is delta normal, then the vague topology on $\mathrm{IR}(L)$ coincides with the Wallman-Frink topology on $\mathrm{IR}(L)$, having, as a base for the closed sets, sets of the form $W(A) = \{\mu \in \mathrm{IR}(L) : \mu(A) = 1\}$. This is a fairly easy consequence of the

Portmanteau Theorem (see [26] for details). We also note that if L is separating and disjunctive as well as delta normal, then the vague topology on X, when X is identified with the collection of measures concentrated at a point, coincides with the topology on X having as a base for the closed sets the lattice L, that is, the $\tau(L)$ topology. (Again see [26].) If X is a Tychonoff space we denote by βX the Stone-Čech compactification of X, and by υX, the real compactification of X. In general, when we are dealing with several different lattices, we will, for clarity, subscript a lattice by the set from which it comes. Thus for example F_X will denote the lattice of closed sets of X, while F_Y denotes the lattice of closed sets of Y. If L is a lattice of subsets of X we will denote by L^* the collection of subsets of X which are μ^*-measurable with respect to every $\mu \in IR(\sigma, L)$. Finally we will need one result for applications: if X is an analytic space (the continuous image of a complete separable metric space) then $\sigma(F) \subset s(F)$ (see [16], p. 115).

3. Measure and sequential compactness

We prove in this section some results on weak or vague convergence of measures which generalize theorems of Moran [23], Varadarajan [27], and Alexandroff [3]. When our results are applied to the special case of 2-valued regular measures, we obtain as corollaries some new results on sequential compactness and countably compact lattices generalizing several known results.

THEOREM 3.1. *Suppose* $\mu_n \in MR(\sigma, L)$, $n = 1, 2, 3, \ldots$, *where* L *is a delta lattice which is normal and countably paracompact. Then if* $\mu_n \to \mu$ *in the vague topology,* $\mu \in MR(\sigma, L)$.

Proof. Let $\upsilon_n = \mu_n$ restricted to $Z(L)$, and $\upsilon = \mu$ restricted to $Z(L)$. Since f is L-continuous, $f^{-1}(C) \in Z(L)$ for every closed set $C \subset R$ (the real line). It follows from the definition of the integral given in [2], p. 576, that $\int f d\upsilon_n = \int f d\mu_n$ and that $\int f d\upsilon = \int f d\mu$. Thus $\upsilon_n \to \upsilon$ weakly. Since the lattice $Z(L)$ is complement generated (that is, completely normal in the sense of [1, 2, 3]), it follows from [3],

p. 209, Theorem 3, that υ is σ-smooth and thus by Theorem 4.3 of [6] that μ is σ-smooth, since μ is the unique L-regular extension of υ to $A(L)$.

One immediate corollary of the above theorem is:

COROLLARY 3.2. *If* L *is a separating disjunctive delta lattice which is normal and countably paracompact, then if* $\mathrm{IR}(L)$ *is sequentially compact,* $\mathrm{IR}(\sigma, L)$ *is sequentially compact.*

Proof. If $\mu_n \to \mu$ then $\mu \in \mathrm{MR}(\sigma, L)$ by Theorem 3.1. Since $\mathrm{IR}(L)$ is sequentially closed in $\mathrm{MR}(L)$ (by a proof similar to the proof of Theorem 11, p. 187 of [27]) we have that $\mu \in \mathrm{IR}(\sigma, L)$.

Taking $L = Z_X$ in the case that X is Tychonoff and noting that $\mathrm{IR}(L) = \beta X$ ([27], p. 212), we obtain:

COROLLARY 3.3. *If* βX *is sequentially compact, then* υX *is sequentially compact.*

THEOREM 3.4. *If* $\mathrm{IR}(\sigma, L)$ *is sequentially compact where* L *is a separating disjunctive delta normal lattice which is countabily paracompact, then* L *is countably compact.*

Proof. Suppose L is not countably compact. Then there is a sequence of sets $B_n \downarrow \emptyset$ such that $B_n \in L$ and each $B_n \neq \emptyset$. Choose an $x_n \in B_n$ for each n , and consider $\{\mu_{x_n}\}$, the sequence of measures concentrated at the points x_n . Then we may assume by hypothesis that some subsequence also denoted by $\{\mu_{x_n}\}$ converges weakly to a $\mu \in \mathrm{IR}(\sigma, L)$, by the previous theorem. We have that $\mu_{x_n}(B_k) = 1$ for all $n \geq k$; hence, by Theorem 2, p. 180 of [3], $\limsup \mu_{x_n}(B_k) \leq \mu(B_k) = 1$ for all k . Since μ is σ-smooth, $\mu(\cap B_k) = 1$ contradicting the fact that $\cap B_k = \emptyset$. Thus L is countably compact.

COROLLARY 3.5. *If* X *is a Tychonoff space and* υX *is sequentially compact, then* X *is pseudocompact.*

Proof. Take $L = Z$ in Theorem 3.4 and use the fact that Z is

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

countably compact if and only if X is pseudocompact ([27], Theorem 16, p. 170).

It is possible to apply the above techniques to get some other topological and measure theoretic corollaries. For this we must prove a lemma.

LEMMA 3.6. *If* $\mu_\alpha \to \mu$ *in the vague topology, where* $\mu_\alpha, \mu \in MR(\sigma, L_1)$, *then if* L_1 *semiseparates* L_2 , *and* L_1 *is delta normal, then*

(1) $\limsup \mu_\alpha^*(L_2) \le \mu^*(L_2)$ *for all* $L_2 \in L_2$,

(2) $\liminf \mu_{*\alpha}(L_2') \ge \mu_*(L_2')$ *for all* $L_2 \in L_2$.

(Here μ^* represents the outer measure and μ_* the inner measure associated with μ .)

Proof. We only prove (1) since the proof of (2) is similar. Since $\mu_\alpha \to \mu$, $\limsup \mu_\alpha(L_1) \le \mu(L_1)$ for all $L_1 \in L_1$. Since L_1 is a delta lattice, μ_α^* is L_1-regular and thus $\mu_\alpha^*(L_2) = \inf \mu_\alpha(L_1')$ where $L_1' \supset L_2$, and $L_1 \in L_1$. But L_1 semiseparates L_2 , and we therefore have that

$$\mu_\alpha^*(L_2) = \inf \mu_\alpha(L_1) \quad \text{where } L_1 \in L_1 \text{ and } L_1 \supset L_2 . \tag{*}$$

Thus $\mu_\alpha^*(L_2) \le \mu_\alpha(L_1)$ for any $L_1 \supset L_2$ where $L_1 \in L_1$, and thus $\limsup \mu_\alpha^*(L_2) \le \limsup \mu_\alpha(L_1) \le \mu(L_1)$ for any $L_1 \in L_1$ containing L_2 . Taking the infimum over such L_1 we get

$$\limsup \mu_\alpha^*(L_2) \le \inf \mu(L_1) = \mu^*(L_2) ,$$

the last inequality following from (*) above.

THEOREM 3.7. *If* L *is a separating disjunctive delta lattice of subsets of* X *which is normal and countably paracompact, and if* L *semi-separates* $\tau(L)$, *then* X *with the vague topology is sequentially compact if* $IR(\sigma, L)$ *with the vague topology is sequentially compact.*

Proof. Suppose $\{x_n\}$ is a sequence of distinct points in X .

Consider $\{\mu_{x_n}\} = A$. By hypothesis we may suppose $\{\mu_{x_n}\}$ converges to a $\mu \in \mathrm{IR}(\sigma, L)$. To complete the proof we need only show that $\{x_n\}$ has a limit point $x \in X$, for then μ_x is a limit point of A and therefore $\mu = \mu_x$ since $\mathrm{IR}(\sigma, L)$ is Hausdorff. Suppose A does not have a limit point in X . Then A and all its subsets are closed; hence elements of $\tau(L)$, since the $\tau(L)$ topology and weak topology coincide. Choose any x_i in A and fix it. Let $B_i = \{x_j\}_{j \neq i}$. Then $B_i \in \tau(L)$, and thus there exists a $B \in L$ such that $B \supset B_i$ and $x_i \notin B$. Since $\mu(B) \geq \limsup \mu_{x_n}(B) = 1$, we have $\mu(B) = 1$. Thus $\mu(B') = 0$ and hence $\mu^*(\{x_i\}) = 0$. But $\mu^*(A) \geq \limsup \mu^*_{x_n}(A) = 1$ by the previous lemma, and this is a contradiction since $\mu^*(A) \leq \sum \mu^*\{x_i\} = 0$.

We give some corollaries of this.

COROLLARY 3.8. *If X is a Tychonoff space and if Z semiseparates F , then if υX is sequentially compact, then X is sequentially compact.*

Proof. Take $L = Z$ and make use of the fact that $\mathrm{IR}(\sigma, L) = \upsilon X$ ([27], p. 215).

COROLLARY 3.9. *If βX is sequentially compact and Z semiseparates F , then X is sequentially compact.*

Proof. Using Corollary 3.3 it follows that υX is sequentially compact and the result follows from the previous corollary.

REMARK 3.10. Corollary 3.9 is known if Z separates F , that is, if X is a normal space. The proof is not difficult (see [28], p. 148, Theorem 8.32). However, it is also known that the result is false if X is just a Tychonoff space. Thus, our result, which is new, represents an intermediate result.

4. Measure replete spaces

In this section we prove some theorems on the images of measures under certain mappings. We apply the general results to get as corollaries several known and new results on measurecompactness and its generalizations.

In the next section we modify some of the proofs here and apply them to 2-valued regular measures getting a substantial generalization of Theorem 5.4 of [6], one of the main theorems for 2-valued measures in that paper. Measure compact spaces are abundant. Any paracompact space every closed discrete subspace of which is measurecompact, is measurecompact. This is proved in [27]. Generalizations of this have appeared in [23] and [25], and Gardner [11] has given theorems on when a space is F-measure replete. Measure compactness does not behave as well as realcompactness. Measure compactness is not closed under arbitrary intersections, in contrast to realcompactness. While realcompactness is productive, measurecompactness is not even finitely productive, although in locally compact spaces it is countably productive, as Kirk has shown in [19]. The first theorem we prove is an analogue of Theorem 5.7 of [6] for arbitrary regular measures. However, we first need a lemma.

LEMMA 4.1. *Let* $T : X \to Y$ *and let* $\mu \in M(\sigma, L)$ *, where* L *is a delta lattice. If* $T^{-1}\{y\}$ *is* L*-Lindelöf for each* $y \in Y$ *and* $H = \{A_\alpha \in L : \mu(A_\alpha) = \mu(X)\}$ *, then* $T(\cap A_\alpha) = \cap T(A_\alpha)$*, where* $A_\alpha \in H$ *.*

Proof. Suppose $y \in \cap T(A_\alpha)$; then $T^{-1}\{y\} \cap A_\alpha \neq \emptyset$ for all $A_\alpha \in H$. Let $B = T^{-1}\{y\} \cap H$. Then B is a $T^{-1}\{y\} \cap L$ filter with the countable intersection property and since $T^{-1}\{y\}$ is L-Lindelöf, $\cap T^{-1}\{y\} \cap A_\alpha \neq \emptyset$. If $z \in T^{-1}\{y\}$ and if $z \in \cap A_\alpha$, then $T(z) = y$. Thus $\cap T(A_\alpha) \subset T(\cap A_\alpha)$ and since the reverse inclusion is trivial, the lemma is proved.

THEOREM 4.2. *Suppose* $T : X \to Y$ *is* $L_1 - L_2$ *continuous and* $L_1 - \tau(L_2)$ *closed, where* L_1 *and* L_2 *are delta lattices of subsets of* X *and* Y *respectively, and where* $\sigma(L_2) \subset s(L_2)$ *. Then if* $T^{-1}\{y\}$ *is* L_1*-Lindelöf for each* $y \in Y$ *, then* L_2 *is measure replete implies* L_1 *is measure replete.*

Proof. Suppose $\mu \in MR(\sigma, L_1)$. Define $\nu = \mu T^{-1}$. Then

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

$\upsilon \in MR(\sigma, L_2)$ by Theorem 2.3 of [6]. By hypothesis, $S(\upsilon) \neq \emptyset$. Let $H = \{A_\alpha \in L_1 : \mu(A_\alpha) = \mu(X)\}$. For each $A_\alpha \in H$, $T(A_\alpha) = \cap L_\alpha$ where $L_\alpha \in L_2$. Clearly for any such L_α, $\upsilon(L_\alpha) = \mu(X) = \upsilon(Y)$. Thus $S(\upsilon) \subset L_\alpha$ for all α, and hence $S(\upsilon) \subset \cap T(A_\alpha)$. By the previous lemma, $\cap A_\alpha \neq \emptyset$ and L_1 is measure replete.

COROLLARY 4.3. *If* $T : X \to Y$ *is a continuous* Z *map (that is,* $T(Z) \in F_Y$ *for each* $Z \in Z_X$ *) where* X *and* Y *are Tychonoff spaces and* Y *is measure compact, then if* $T^{-1}\{y\}$ *is Lindelöf for each* $y \in Y$, *then* X *is measure compact.*

Proof. Take $L_1 = Z_X$ and $L_2 = Z_Y$

If in the above corollary we take $X \subset Y$, where X is closed, and T to be the inclusion map, then we have:

COROLLARY 4.4 (Kirk [18]). *If* X *is closed in a Tychonoff measure compact space* Y *then* X *is measurecompact.*

COROLLARY 4.5 (Kirk [18]). *The product of a measure compact space* A *and a compact space* B *is measure compact.*

Proof. In Corollary 4.3 take $X = A \times B$, $Y = A$, $L_1 = Z_X$, $L_2 = Z_Y$, and T to be the projection map from X to Y. Since B is compact, T is closed, as is well known. $T^{-1}\{y\}$ is trivially Lindelöf, since it is compact.

If in Theorem 4.2 we take T to be $L_1 - L_2$ continuous, $L_1 - L_2$ closed and onto, then we may relax the condition that $\sigma(L_2) \subset s(L_2)$, since the measure υ constructed in the proof there is necessarily L_2 - regular. This is essentially due to the remark made in the proof of Theorem 5.1 (a) of [6]. Thus we have:

COROLLARY 4.6. *If* $T : X \to Y$ *is a continuous closed surjection such that* $T^{-1}\{y\}$ *is Lindelöf, then* F_Y*-measure replete implies* F_X*-measure replete.*

Other corollaries are easy to obtain, and we turn to the question of the preservation of measure repleteness under direct images.

THEOREM 4.7. *Let L_1 and L_3 be lattices of subsets of X with $L_1 \subset L_3 \subset \tau(L_1)$, where L_3 is a delta normal lattice and either L_1 is delta and $\sigma(L_1) \subset s(L_1)$, or L_1 semiseparates L_3. Let $L_2 \subset L_4$ be lattices of subsets of Y such that L_4 is an L_2 countably paracompact lattice, and both L_2 and L_4 are delta normal, where L_2 semiseparates L_4. Then if T is an $L_3 - L_4$ continuous closed surjection such that L_3 is $T^{-1}(L_4)$ countably paracompact, then L_1 measure replete implies L_2 measure replete.*

Proof. Since $L_1 \subset L_3 \subset \tau(L_1)$, we have, as is easily seen by a simple restriction argument, that L_3 is measure replete. Now let $\upsilon \in \mathrm{MR}(\sigma, L_2)$. By Theorem 4.3 of [*6*], υ may be extended to a $\upsilon_1 \in \mathrm{MR}(\sigma, L_4)$. Define μ_1 on $A\left[T^{-1}(L_4)\right]$ by $\mu_1 T^{-1}(B) = \upsilon_1(B)$. $\mu_1 \in \mathrm{MR}\left[\sigma, T^{-1}(L_4)\right]$ and extends to a $\mu \in \mathrm{MR}(\sigma, L_3)$ again by Theorem 4.3 of [*6*]. Since $S(\mu) \neq \emptyset$ and $\upsilon_1 = \mu T^{-1}$, $S(\upsilon_1) \neq \emptyset$ and hence $S(\upsilon) \neq \emptyset$ since $S(\upsilon) \supset S(\upsilon_1)$. Thus L_2 is measure replete.

COROLLARY 4.8. *Let X be a normal Tychonoff space and T be a perfect map from X onto a countably paracompact Tychonoff space Y. Then X measure compact implies Y is measure compact.*

Proof. In the theorem take $L_1 = Z_X$, $L_2 = Z_Y$, $L_3 = F_X$, $L_4 = F_Y$. Note that since normality is preserved under continuous closed surjections, Y is a countably paracompact space, being countably paracompact and normal (see [*4*], p. 248), and hence F_Y is a countably paracompact Z_Y lattice. Clearly L_2 separates L_4 and hence semiseparates L_4, and since T is perfect, L_3 is $T^{-1}(L_4)$ countably paracompact, by Lemma 5.3 of [*6*].

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

If in Theorem 4.7 one takes $L_1 = L_3 = Z_X$ and $L_2 = L_4 = Z_Y$, and notes that for open perfect mappings $T(Z_1) \subset Z_2$, one has:

COROLLARY 4.9. *If X and Y are Tychonoff spaces and if $T : X \to Y$ is an open perfect map, then X is measurecompact implies Y is measure-compact.*

Clearly the following is also true.

COROLLARY 4.10. *If $T : X \to Y$ is a perfect map from the countably paracompact normal space X onto Y, then F_X is measure replete implies F_Y is measure replete.*

The following theorem is an analogue of Theorem 7.1 of [6] for measure repleteness. Two special cases have been given in [11], Theorems 3.7 and 7.4, and the proof of the following is virtually identical to the proofs given in [11].

THEOREM 4.11. *If X is the union of a sequence of relatively measure replete subspaces where L is a delta lattice then L is measure replete.*

One can generalize the notion of L measure replete even further, namely, call L *fully measure replete* if and only if every $\mu \in M(\sigma, L)$ has non empty L-support. Here by L-support we mean $\cap\{L \in L : \mu(L) = \mu(X)\}$. Many of the known theorems about support and measure repleteness may now be carried over to this situation. For example we mention that if L is delta, L is fully measure replete if and only if $M(\sigma, L) = M(\tau, L)$. (The proof follows the proof of Theorem 2.1 of [21].) The following analogue of Theorem 4.2 also clearly holds. The same proof works, and we only note that we are able to relax the conditions that $\sigma(L_2) \subset s(L_2)$ and that L_2 is a delta lattice since we do not need the measure υ in that proof to be L_2-regular.

THEOREM 4.12. *Suppose $T : X \to Y$ is $L_1 - L_2$ continuous and $L_1 - \tau(L_2)$ closed where L_1 is a delta lattice. Suppose $T^{-1}\{y\}$ is L_1-Lindelöf for each $y \in Y$. Then L_2 is fully measure replete implies L_1 is fully measure replete.*

COROLLARY 4.13. *If X and Y are topological spaces, and $T : X \to Y$ is continuous and closed, then if $T^{-1}\{y\}$ is Lindelöf for each $y \in Y$, then F_Y is fully measure replete implies F_X is fully measure replete.*

Proof. Take $L_1 = F_X$ and $L_2 = F_Y$.

We note that the analogue of Corollary 4.3 holds for fully measure replete; however, this gives us no new information, since if $L = Z$ then L is fully measure replete if and only if L is measure replete, clearly, since in this case $MR(\sigma, L) = M(\sigma, L)$.

We have noted that if $L_1 \subset L_2 \subset \tau(L_1)$, then L_1 fully measure replete implies L_2 fully measure replete. However if $L_1 \subset L_2$ and L_2 is fully measure replete, then it need not follow that L_1 is fully measure replete. However we do have the following:

LEMMA 4.14. *If $L_1 \subset L_2$ and L_2 is delta normal, then if $\mu \in M(\sigma, L_1)$ and L_2 is L_1 countably paracompact, then μ extends to a $\upsilon \in M(\sigma, L_2)$. Thus L_2 fully measure replete implies L_1 fully measure replete.*

Proof. We first observe that if $\upsilon \in M(L_2)$ there exists a $\rho \in MR(L_2)$ such that $\upsilon \le \rho$ on L_2. To see this, form the linear functional Φ on $C_b(L_2)$ given by $\Phi(f) = \int f d\upsilon$ where $f \in C_b(L_2)$. By Theorem 2.4 of [6], $\Phi(f) = \int f d\rho$, where $\rho \in MR(L_2)$, and furthermore $\rho(A) = \inf \Phi(f)$, where $f \in C_b(L_2)$ and $f \ge \chi_A$. Clearly $\upsilon(A)$ is less than or equal to the above infimum for $A \in L_2$; thus $\upsilon(A) \le \rho(A)$. To prove the theorem now suppose that $\mu \in M(\sigma, L_1)$. Then by [8, Theorem 11, p. 264], for example, there exists a $\upsilon \in M(L_2)$ which extends μ. By the above there exists a $\rho \ge \upsilon$ on L_2, where $\rho \in MR(L_2)$. If we can show that ρ is σ-smooth on L_2, then by regularity ρ is σ-smooth on $\sigma(L_2)$ and therefore $\rho \in M(\sigma, L_2)$. Thus by hypothesis $S(\rho)$ will be nonempty, and since $S(\mu) \supset S(\rho)$, $S(\mu)$ will be nonempty, and it will follow that L_1 is fully measure replete. We proceed therefore to show that ρ is σ-smooth on L_2 . Suppose then that $A_n \downarrow \emptyset$, where $A_n \in L_2$. Then by hypothesis there exist $B_n \in L_1$ where $A_n \subset B_n' \downarrow \emptyset$. Thus $\rho(A_n) \le \rho(B_n')$. However $\rho(B_n') \le \mu(B_n')$ since $\rho \ge \mu$ on L_1 and $\rho(X) = \mu(X)$. Thus since μ is σ-smooth $\rho(B_n')$, hence $\rho(A_n) \downarrow 0$, and the proof is complete.

THEOREM 4.15. *Let L_1 and L_3 be lattices of subsets of X with $L_1 \subset L_3 \subset \tau(L_1)$, where L_3 is a delta normal lattice. Let $L_2 \subset L_4$ be lattices of subsets of Y such that L_4 is L_2 countably paracompact and L_4 is delta normal. If $T : X \to Y$ is an $L_3 - L_4$ continuous closed surjection such that L_3 is $T^{-1}(L_4)$ countably paracompact, then if L_1 is fully measure replete, L_2 is fully measure replete.*

Proof. The proof given in Theorem 4.7 works here.

We remark that using a proof somewhat similar to the proof of Lemma 4.14 we may show that if L is countably paracompact and delta normal, then L is fully measure replete if and only if L is measure replete.

5. Mappings of replete spaces

In this section we modify some of the results proved in the previous section to 2-valued measures. We get in particular a substantial generalization of one of the main theorems of [6] for 2-valued measures. That theorem had numerous applications to the area of repleteness and we are able to obtain here new results not obtainable by that theorem. The first lemma is true for arbitrary measures.

LEMMA 5.1. *If L is a delta lattice, $\mu \in \mathrm{MR}(\sigma, L \cap S)$, and $S \subset X$, then if $\sigma(L) \subset s(L)$ and if υ is defined on $\sigma(L)$ by $\upsilon(B) = \mu(B \cap S)$, where $B \in \sigma(L)$, then $\upsilon \in \mathrm{MR}(\sigma, L)$ and for any subset E of S , $\upsilon^*(E) = \mu^*(E)$. In particular, if $\mu \in \mathrm{IR}(\sigma, L \cap E)$, then $\upsilon^*(S) = 1$.*

Proof. That $\upsilon \in \mathrm{MR}(\sigma, L)$ is clear. We have $\mu^*(E) = \inf \mu(B)$, where $B \in \sigma(L) \cap S$ and $B \supset E$. But $B = B^* \cap S$ where $B^* \in \sigma(L)$. Since $E \subset S$, $B \supset E$ if and only if $B^* \supset E$. Thus

$$\mu^*(E) = \inf \mu(B) = \inf \mu(B^* \cap S) = \inf \upsilon(B^*) = \upsilon^*(E) ,$$

where in this chain of equalities $B \in \sigma(L)$, $B = B^* \cap S$, and $B^* \supset E$.

The following lemma is very important in matters of repleteness.

LEMMA 5.2. *If L is a separating disjunctive delta lattice and $\upsilon \in \mathrm{IR}(\sigma, L)$, then if $\{A_\alpha\}$ is a family of υ-thick relatively L-replete subsets of X, then $\cap A_\alpha \neq \emptyset$ and υ is fixed at some point $p \in \cap A_\alpha$.*

Proof. Since A_α is υ-thick in X for each α, we may project υ onto each A_α. If the projection is υ_α, then by construction $\upsilon_\alpha(B) = \upsilon(B^*)$, where $B \in \sigma(L \cap A_\alpha)$, $B^* \in \sigma(L)$, and $B^* \cap A_\alpha = B$. As is well known, υ_α is well defined, and since $\upsilon_\alpha \in \mathrm{IR}(\sigma, L \cap A_\alpha)$, each υ_α is fixed at a unique point $p_\alpha \in A_\alpha$. Thus υ must be concentrated at each p_α. But L is separating and disjunctive and therefore υ can only be concentrated at a single point. It follows that all of the p_α are the same point p and that $p \in \cap A_\alpha$.

Before proceeding with the main result, we give some immediate applications of the above lemma.

COROLLARY 5.3. *If L is a separating disjunctive delta lattice of subsets of X, with $\sigma(L) \subset s(L)$, and, if $\{A_\alpha\}$ is a family of relatively L-replete subsets of X, then if $E = \cap A_\alpha$, E is relatively L replete. If in addition there is a lattice of subsets of E, L_E, such that $L \cap E \subset L_E \subset \tau(\sigma(L \cap E))$, then L_E is also replete.*

Proof. If $E = \emptyset$, there is nothing to prove. Suppose then that $E \neq \emptyset$. Suppose that $\mu \in \mathrm{IR}(\sigma, L \cap E)$ and that υ is defined as in Lemma 5.1. If $\upsilon^*(A_\alpha) = 0$ for any α then $\upsilon^*(E) = 0$ contradicting Lemma 5.1. Thus $\upsilon^*(A_\alpha) = 1$ for all A_α. It follows from the preceding lemma that υ is fixed at a unique point $p \in E$. Hence μ is fixed at

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

p . The remainder of the proof is simple, for if $\rho \in \mathrm{IR}(\sigma, L_E)$ and ρ_1 its restriction to $L \cap E$, then $\rho_1 \in \mathrm{IR}(\sigma, L \cap E)$, since

$$\sigma(L \cap E) = \sigma(L) \cap E \subset s(L) \cap E \subset s(L \cap E) .$$

Thus ρ_1 is concentrated at p and so is its unique $L \cap E$ regular extension ρ_2 to $\sigma(L \cap E)$. Since $S(\rho_2) = S(\rho_1) \neq \emptyset$ and since $L_E \subset \tau(\sigma(L \cap E))$, $S(\rho) \neq \emptyset$ and L_E is replete.

The above corollary generalizes Theorem 5.3 of [*12*]. As an application we have the following:

COROLLARY 5.4. *If X is analytic, then the arbitrary intersection of F-replete (that is α-complete* [9], [*10*]*) spaces is F-replete.*

Proof. Take $L = F$ in Corollary 5.3.

We are now ready to present the main theorem of this section. This theorem generalizes considerably Theorem 5.4 of [*6*].

THEOREM 5.5. *If $T : X \to Y$ is $L_1^* - L_2$ continuous, where L_1 is a delta lattice and L_2 is a separating disjunctive delta lattice with $\sigma(L_2) \subset s(L_2)$, then if $L_3 \supset L_1$ is a lattice of subsets of X such that*

(a) $L_3 \subset \tau(L_1^*)$ and

(b) every $\mu \in \mathrm{IR}(\sigma, L_3)$ when restricted to $\sigma(L_1)$ is in $\mathrm{IR}(\sigma, L_1)$, and if

(1) *$T^{-1}\{y\}$ is relatively L_1 replete for each $y \in Y$ and*

(2) *$T(L_1)$ is relatively L_2 replete for each $L_1 \in L_1$,*

it follows that L_3 is replete.

Proof. Suppose $\mu \in \mathrm{IR}(\sigma, L_3)$ and μ_1 its restriction to $\sigma(L_1)$, and μ_1^* its extension to L_1^* (which is L_1 regular since L_1 is a delta lattice). Let $H = \{L_1 \in L_1 \mid \mu_1(L_1) = 1\}$. Define $\upsilon = \mu_1 T^{-1}$ on $\sigma(L_2)$.

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

Clearly $\upsilon \in \mathrm{IR}(\sigma, L_2)$. If $L_\alpha \in H$, then $\upsilon^*(T(L_\alpha)) = \inf \upsilon(B)$, where $B \in \sigma(L_2)$ and $B \supset T(L_\alpha)$. Thus $\upsilon^*(T(L_\alpha)) \geq \mu_1(L_\alpha) = 1$, and therefore $T(L_\alpha)$ is υ-thick in Y . It follows from Lemma 5.2 that $\cap T(L_\alpha) \neq \emptyset$, and thus by Lemma 5.1 of [6] applied to μ_1 , $\cap H \neq \emptyset$. That is $S(\mu_1) \neq \emptyset$. Thus $S(\mu_1^*) \neq \emptyset$, and since $L_3 \subset \tau(L_1^*)$, $S(\mu) \neq \emptyset$.

One notes the following differences between the theorem above and Theorem 5.4 of [6]. Firstly, we do not require that T be $L_1 - \tau(\sigma(L_2))$ closed. Secondly, we require no repleteness assumptions on Y . Finally we weaken the condition that $L_3 \subset \tau(L_1)$. It is not yet clear that this theorem generalizes Theorem 5.4 of [6]. This will follow after we prove a lemma.

LEMMA 5.6. *If* $S \in \tau(L^*)$ *, where* L *is a separating disjunctive delta lattice, and if* $\sigma(L) \subset s(L)$ *, then if* L *is replete,* S *is relatively* L *replete.*

Proof. If $\mu \in \mathrm{IR}(\sigma, L \cap S)$ and υ is constructed as in Lemma 5.1, then υ is concentrated at some unique point p . If $p \notin S \in \tau(L^*)$, then there is a $B \in L^*$ such that $B \supset S$ and $p \notin B$. Thus $\upsilon^*(B) = 0$, whence $\upsilon^*(S) = 0$, contradicting Lemma 5.1. Thus $p \in S$ and μ is concentrated at p .

It is now easy to see that Theorem 5.4 of [6] is a corollary of Theorem 5.5 above. For if T is $L_1 - \tau(\sigma(L_2))$ closed, indeed, even if T is $L_1 - \tau(L_2^*)$ closed, then if L_2 is replete, $T(L_1)$ is automatically relatively L_2-replete by the previous lemma. It follows that Theorems 5.5 and 5.7 of [6] generalize: namely we may replace the phrases "T is $L_1 - L_2$ continuous" and "T is $L_1 - \tau(\sigma(L_2))$ closed" respectively by "T is $L_1^* - L_2$ continuous" and "T is $L_1 - \tau(L_2^*)$ closed" respectively, and those theorems will still be true.

We give some immediate corollaries. The first corollary is a generalization of a theorem of Isiwata [17, Theorem 5.3].

COROLLARY 5.7. *If* $T : X \to Y$ *is continuous, where* X *and* Y *are*

https://doi.org/10.1017/S0004972700008078 Published online by Cambridge University Press

Tychonoff spaces, then if $T^{-1}\{y\}$ *is Z-embedded and realcompact for each* $y \in Y$ *, and if either*

(a) $T(Z)$ *is Z-embedded and realcompact for each* $Z \in \mathcal{Z}_X$ *, or*

(b) Y *is realcompact and* $T(Z) \in \tau(\sigma(\mathcal{Z}_Y))$ *for each* $Z \in \mathcal{Z}_X$ *,*

then X *is realcompact.*

Proof. One need only take $\mathcal{L}_1 = \mathcal{L}_3 = \mathcal{Z}_X$ and $\mathcal{L}_2 = \mathcal{Z}_Y$. The hypotheses in *(a)* and *(b)* imply that $T(L_1)$ is relatively $\mathcal{L}_2$-replete for each $L_1 \in \mathcal{L}_1$. The remaining parts of the hypothesis are readily verified.

The next corollary generalizes a well known theorem of Varadarajan [27, Theorem 4, p. 217].

COROLLARY 5.8. *If* Y *is realcompact and* $E \in \tau(\mathcal{Z}_Y^*)$ *, where* $E \subset Y$ *, then* E *is realcompact.*

Proof. Take $X = E$, and T to be the inclusion map. Take $\mathcal{L}_1 = \mathcal{Z}_Y \cap E$, $\mathcal{L}_2 = \mathcal{Z}_Y$, and $\mathcal{L}_3 = \mathcal{Z}_E$. Then *(a)*, *(b)*, and (1) of the theorem are trivial. (2) follows from the fact that $T(L_1) \in \tau(\mathcal{Z}_Y^*)$ for each $L_1 \in \mathcal{L}_1$, since $E \in \tau(\mathcal{Z}_Y^*)$.

The next corollary also seems new.

COROLLARY 5.9. *If* $T : X \to Y$ *is Baire measurable, where* X *and* Y *are Tychonoff spaces, and the image of every Baire set of* X *is an intersection of Baire sets of* Y *, then if* $T^{-1}\{y\}$ *is Z-embedded and realcompact for each* $y \in Y$ *, then* Y *realcompact implies* X *is realcompact.*

Proof. In the theorem take $\mathcal{L}_1 = \mathcal{L}_3 = \sigma(\mathcal{Z}_X)$ and $\mathcal{L}_2 = \sigma(\mathcal{Z}_Y)$. That $T^{-1}\{y\}$ is relatively $\mathcal{L}_1$ replete for each $y \in Y$ follows from the fact that $\mathcal{L}_1 \cap T^{-1}\{y\}$ gives the Baire sets of $T^{-1}\{y\}$, and since Baire replete is equivalent to realcompact as noted by Hewitt [15]. That $T(L_1)$ is relatively $\mathcal{L}_2$ replete for each $L_1 \in \mathcal{L}_1$ follows from Lemma 5.6.

Again one may go on combining these theorems with those in [6], [7], to get other theorems. However this is easy and will not be done here. We again remark however that the theorems given here allow one to work with sets in L^* , an, in general, larger class than $\sigma(L)$, and allow us to eliminate repleteness assumptions on Y completely.

References

[1] А.Д. Александров [A.D. Alexandroff], "Аддитивные функции множества в абстрактных пространсрвах" [Additive set-functions in abstract spaces], *Rec. Math.* [*Mat. Sbornik*] *N.S.* 8 (50) (1940), 307-348.

[2] А.Д. Александров [A.D. Alexandroff], "Аддитивные функции множества в абстрактных пространствах" [Additive set-functions in abstract spaces], *Rec. Math.* [*Mat. Sbornik*] *N.S.* 9 (51) (1941), 563-628.

[3] A.D. Alexandroff, "Additive set-functions in abstract spaces", *Rec. Math.* [*Mat. Sbornik*] *N.S.* 13 (55) (1943), 169-238.

[4] Richard A. Alò and Harvey L. Shapiro, *Normal topological spaces* (Cambridge Tracts in Mathematics, 65. Cambridge University Press, London, New York, 1974).

[5] George Bachman and Ronald Cohen, "Regular lattice measures and repleteness", *Comm. Pure Appl. Math.* 26 (1973), 587-599.

[6] George Bachman and Alan Sultan, "Regular lattice measures: mappings and spaces", *Pacific J. Math.* 67 (1976), 291-321.

[7] George Bachman and Alan Sultan, "Extensions of regular lattice measures with topological applications", *J. Math. Anal. Appl.* 57 (1977), 539-559.

[8] Garrett Birkhoff, *Lattice theory*, 3rd edition (American Mathematical Society Colloquium Publications, 25. American Mathematical Society, Providence, Rhode Island, 1967. Reprinted 1973).

[9] Nancy Dykes, "Mappings of realcompact spaces", *Pacific J. Math.* 31 (1969), 347-358.

[10] Nancy Dykes, "Generalizations of realcompact spaces", *Pacific J. Math.* 33 (1970), 571-581.

[11] R.J. Gardner, "The regularity of Borel measures and Borel measure-compactness", *Proc. London Math. Soc.* (3) **30** (1975), 95-113.

[12] Hugh Gordon, "Rings of functions determined by zero-sets", *Pacific J. Math.* **36** (1971), 133-157.

[13] Paul R. Halmos, *Measure theory* (Van Nostrand, Toronto, New York, London, 1950).

[14] Richard Haydon, "On compactness in spaces of measures and measure-compact spaces", *Proc. London Math. Soc.* (3) **29** (1974), 1-16.

[15] Edwin Hewitt, "Linear functionals on spaces of continuous functions", *Fund. Math.* **37** (1950), 161-189.

[16] J. Hoffmann-Jørgensen, *The theory of analytic spaces* (Various Publication Series, 10. Matematisk Institut, Aarhus Universitet, Denmark, 1970).

[17] Takesi Isiwata, "Mappings and spaces", *Pacific J. Math.* **20** (1967), 455-480.

[18] R.B. Kirk, "Measures in topological spaces and B-compactness", *K. Nederl. Akad. Wetensch. Proc. Ser. A* **72** = *Indag. Math.* **31** (1969), 172-183.

[19] R.B. Kirk, "Locally compact, B-compact spaces", *K. Nederl. Akad. Wetensch. Proc. Ser. A* **72** = *Indag. Math.* **31** (1969), 333-344.

[20] R.B. Kirk and J.A. Crenshaw, "A generalized topological measure theory", *Trans. Amer. Math. Soc.* **207** (1975), 189-217.

[21] W. Moran, "The additivity of measures on completely regular spaces", *J. London Math. Soc.* **43** (1968), 633-639.

[22] W. Moran, "Measures and mappings on topological spaces", *Proc. London Math. Soc.* (3) **19** (1969), 493-508.

[23] W. Moran, "Measures on metacompact spaces", *Proc. London Math. Soc.* (3) **20** (1970), 507-524.

[24] Steven E. Mosiman and Robert F. Wheeler, "The strict topology in a completely regular setting: relations to topological measure theory", *Canad. J. Math.* **24** (1972), 873-890.

[25] Alan Sultan, "A general measure extension procedure", *Proc. Amer. Math. Soc.* (to appear).

[26] Alan Sultan, "Measure compactification and representation", *Canad. J. Math.* (to appear).

[27] V.S. Varadarajan, "Measures on topological spaces", *Amer. Math. Soc. Transl.* (2) 48 (1965), 161-228.

[28] Albert Wilansky, *Topology for analysis* (Ginn, Waltham, Massachusetts; Toronto; London; 1970).

Department of Mathematics,
Polytechnic Institute of New York,
Brooklyn,
New York,
USA;

Department of Mathematics,
Queens College,
City University of New York,
Flushing,
New York,
USA.