FIXED POINT THEOREM AND NONLINEAR ERGODIC THEOREM FOR NONEXPANSIVE SEMIGROUPS WITHOUT CONVEXITY

WATARU TAKAHASHI

ABSTRACT. We first prove a nonlinear ergodic theorem for nonexpansive semigroups without convexity in a Hilbert space. Further we prove a fixed point theorem for non-expansive semigroups without convexity which generalizes simultaneously fixed point theorems for left amenable semigroups and left reversible semigroups.

1. Introduction. Let \( H \) be a real Hilbert space with norm \( \| \cdot \| \) and inner product \( \langle \cdot , \cdot \rangle \) and let \( C \) be a nonempty subset of \( H \). A mapping \( T : C \to C \) is said to be \textit{Lipschitzian} if there exists a nonnegative number \( k \) such that

\[
\|Tx - Ty\| \leq k\|x - y\| \quad \text{for every } x, y \in C,
\]

and \textit{nonexpansive} in the case \( k = 1 \). If \( S \) is a semitopological semigroup and \( S = \{T_s : s \in S\} \) is a continuous representation of \( S \) as Lipschitzian mappings of \( C \) into itself, it is called a \textit{Lipschitzian semigroup} on \( C \). A Lipschitzian semigroup \( S = \{T_s : s \in S\} \) with Lipschitzian constants \( k_s, s \in S \) is called a \textit{nonexpansive semigroup} if \( k_s = 1 \) for every \( s \in S \). When \( C \) is closed and convex, there are many fixed point theorems and nonlinear ergodic theorems for nonlinear semigroups in a Hilbert space; for example, see [1–8, 10–12, 14–17]. Recently, Mizoguchi and Takahashi [10] proved a fixed point theorem which generalizes some results of them by introducing the notion of submean. And also Ishihara [7] proved a fixed point theorem for left reversible Lipschitzian semigroups without convexity.

In this paper, we first prove a nonlinear ergodic theorem for nonexpansive semigroups without convexity in a Hilbert space. This is a generalization of Rodé’s result [14]. Further by the method of [10, 15–17], we prove a fixed point theorem without convexity which generalizes simultaneously fixed point theorems for left amenable semigroups and left reversible semigroups. This is a generalization of results of Lau [8], Takahashi [15] and Ishihara [7].

2. Nonlinear ergodic theorem. Throughout this paper, let \( S \) be a semitopological semigroup, \textit{i.e.}, a semigroup with a Hausdorff topology such that for each \( s \in S \) the mappings \( t \to t \cdot s \) and \( t \to s \cdot t \) of \( S \) into itself are continuous. Let \( B(S) \) be the Banach

Key words and phrases: Fixed point, nonexpansive mapping, invariant mean, nonlinear ergodic theorem

Received by the editors September 4, 1990.

AMS subject classification: Primary: 47A35, 47H09.


880
space of all bounded real valued functions on $S$ with supremum norm and let $X$ be a subspace of $B(S)$ containing constants. Then, an element $\mu$ of $X^*$ (the dual space of $X$) is called a mean on $X$ if $\|\mu\| = \mu(1) = 1$. We know that $\mu \in X^*$ is a mean on $X$ if and only if

$$\inf \{ f(s) : s \in S \} \leq \mu(f) \leq \sup \{ f(s) : s \in S \}$$

for every $f \in X$. Let $\mu$ be a mean on $X$ and $f \in X$. Then, according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ given by

$$(l_s f)(t) = f(st)$$

and

$$(r_s f)(t) = f(ts)$$

for all $t \in S$. Let $X$ be a subspace of $B(S)$ containing constants which is $l_s$-invariant ($r_s$-invariant), i.e., $l_s(X) \subset X$ ($r_s(X) \subset X$) for each $s \in S$. Then a mean $\mu$ on $X$ is said to be left invariant (right invariant) if

$$\mu(f) = \mu(l_s f) \quad \left( \mu(f) = \mu(r_s f) \right)$$

for all $f \in X$ and $s \in S$. An invariant mean is a left and right invariant mean.

Let $C$ be a nonempty subset of $H$. Then a family $S = \{ T_s : s \in S \}$ of mappings of $C$ into itself is called a Lipschitzian semigroup on $C$ if it satisfies the following:

1. $T_{st} x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;
2. for each $x \in C$, the mapping $s \rightarrow T_s x$ is continuous on $S$;
3. for each $s \in S$, $T_s$ is a Lipschitzian mapping of $C$ into itself, i.e., there is $k_s \geq 0$ such that

$$\| T_s x - T_s y \| \leq k_s \| x - y \|$$

for all $x, y \in C$. A Lipschitzian semigroup $S = \{ T_t : t \in S \}$ on $C$ is said to be nonexpansive if $k_s = 1$ for every $s \in S$. For a Lipschitzian semigroup $S = \{ T_s : s \in S \}$ on $C$, we denote by $F(S)$ the set of common fixed points of $T_s, s \in S$.

Let $C(S)$ be the Banach space of all bounded continuous real-valued functions on $S$ and let $RUC(S)$ be the space of all bounded right uniformly continuous functions on $S$, i.e., all $f \in C(S)$ such that the mapping $s \rightarrow r_s f$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under $l_s$ and $r_s$; see [9] for details. When $S = \{ T_s : s \in S \}$ is a nonexpansive semigroup on $C$ such that $\{ T_s x : s \in S \}$ is bounded for some $x \in C$, then we know from [8] that for each $u \in C$ and $v \in H$, the functions $f(t) = \| T_t u - v \|^2$ and $g(t) = \langle T_t u, v \rangle$ are in $RUC(S)$. Let $\mu$ be a mean on $RUC(S)$. Then since for each $y$ in $H$, the real valued function $t \rightarrow \langle T_t x, y \rangle$ is in $RUC(S)$, we define the value $\mu_t \langle T_t x, y \rangle$ of $\mu$ at this function. By linearity of $\mu$ and of the inner product, this is linear in $y$; moreover, since

$$|\mu_t \langle T_t x, y \rangle| \leq \| \mu \| \cdot \sup_t \| \langle T_t x, y \rangle \| \leq (\sup_t \| T_t x \|) \cdot \| y \|,$$

it is continuous in $y$. So, by the Riesz theorem, there exists an $x_0 \in H$ such that

$$\mu_t \langle T_t x, y \rangle = \langle x_0, y \rangle$$

for every $y \in H$. 

---

*Downloaded from https://www.cambridge.org/core. 14 Jul 2021 at 08:25:48, subject to the Cambridge Core terms of use.*
We write such an $x_0$ by $T_\mu x$. Before proving a nonlinear ergodic theorem for nonexpansive semigroups without convexity, we give a definition concerning means. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $\text{RUC}(S)$. Then $\{\mu_\alpha : \alpha \in A\}$ is said to be asymptotically invariant if for each $f \in \text{RUC}(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

**THEOREM 1.** Let $C$ be a nonempty subset of a Hilbert space $H$ and let $S$ be a semi-topological semigroup such that $\text{RUC}(S)$ has an invariant mean. Let $S = \{T_t : t \in S\}$ be a nonexpansive semigroup on $C$ such that $\{T_t x : t \in S\}$ is bounded and $\cap_{s \in S} \text{co}\{T_s x : t \in S\} \subset C$ for some $x \in C$. Then, $F(S) \neq \emptyset$. Further, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on $\text{RUC}(S)$, the net $T_{\mu_\alpha} x, \alpha \in A$ converges weakly to an element $x_0 \in F(S)$.

**PROOF.** Let $\mu$ be an invariant mean on $\text{RUC}(S)$. Then, we know that there exists an $x_0 \in H$ such that $\mu_t \langle T_t x, y \rangle = \langle x_0, y \rangle$ for every $y \in H$. For such an $x_0$, we can prove $x_0 \in \cap_{s \in S} \text{co}\{T_s x : t \in S\}$. If not, we have $x_0 \notin \text{co}\{T_s x : t \in S\}$ for some $s \in S$. By the separation theorem, there exists a $y_0$ in $H$ such that

$$\langle x_0, y_0 \rangle < \inf \{\langle z, y_0 \rangle : z \in \text{co}\{T_s x : t \in S\} \}.$$ 

So we have

$$\inf_{t \in S} \langle T_t x, y_0 \rangle \leq \mu_t \langle T_t x, y_0 \rangle = \mu_t \langle T_t x, y_0 \rangle = \langle x_0, y_0 \rangle < \inf \{\langle z, y_0 \rangle : z \in \text{co}\{T_s x : t \in S\} \} \leq \inf_{t \in S} \langle T_t x, y_0 \rangle.$$ 

This is a contradiction. Therefore we have

$$x_0 \in \cap_{s \in S} \text{co}\{T_s x : t \in S\}$$ 

and hence $x_0 \in C$. On the other hand, since, for each $y$ in $H$, the real valued function $t \to \|T_t x - y\|^2$ is in $\text{RUC}(S)$, we can also define the value $\mu_t \|T_t x - y\|^2$ of $\mu$ at this function. Let

$$r = \inf \{\mu_t \|T_t x - y\|^2 : y \in H\}$$ 

and

$$M = \{z \in H : \mu_t \|T_t x - z\|^2 = r\}.$$ 

Then, since, for each $y \in H$ and $t \in S$

$$\|x_0 - y\|^2 = \|T_t x - y\|^2 - \|T_t x - x_0\|^2 - 2 \langle T_t x - x_0, x_0 - y \rangle,$$

we have

$$\|x_0 - y\|^2 = \mu_t (\|T_t x - y\|^2 - \|T_t x - x_0\|^2 - 2 \langle T_t x - x_0, x_0 - y \rangle)
= \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - x_0\|^2 - 2 \langle x_0 - x_0, x_0 - y \rangle
= \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - x_0\|^2 \geq 0.$$
This implies that the set $M$ consists of a single point $x_0$. Now, we can show $x_0 \in F(S)$. In fact, for each $s \in S$,
\[
\mu_t \|T_t x - T_s x_0\|^2 = \mu_t \|T_t x - T_s x_0\|^2 \\
= \mu_t \|T_t T_s x - T_s x_0\|^2 \\
\leq \mu_t \|T_t x - x_0\|^2 = r.
\]
Hence, $T_s x_0 = x_0$ for every $s \in S$.

Since $\mu$ is an invariant mean on RUC(S), from [16], we know
\[
\mu_t \|T_t x - z\|^2 \leq \inf_{s} \sup_{t} \|T_{ts} x - z\|^2
\]
for every $z \in H$. On the other hand, since for $z \in F(S)$ and $a, s \in S$,
\[
\inf_{u} \sup_{t} \|T_{tu} x - z\|^2 \leq \sup_{t} \|T_{tas} x - z\|^2 \\
= \sup_{t} \|T_{tu} T_s x - T_{tas} z\|^2 \\
\leq \sup_{t} \|T_s x - z\|^2 = \|T_s x - z\|^2,
\]
it follows that
\[
\inf_{u} \sup_{t} \|T_{tu} x - z\|^2 \leq \mu_s \|T_s x - z\|^2.
\]
Therefore, for each $z \in F(S)$, we have
\[
\mu_t \|T_t x - z\|^2 = \inf_{s} \sup_{t} \|T_{ts} x - z\|^2.
\]
This implies that the point $x_0$ is independent of $\mu$, that is, $T_{\mu} x = x_0$ for each invariant mean $\mu$. Last, we show that $T_{\mu_{\alpha}} x$ converges weakly to $x_0$.

Let $\mu$ be a cluster point of the net $\mu_{\alpha}$, $\alpha \in A$ in the weak* topology. Then $\mu$ is an invariant mean. In fact, since it is obvious that $\mu$ is a mean, we show that $\mu$ is left invariant. For each $\varepsilon > 0, f \in RUC(S)$ and $s \in S$, there exists $\alpha_0 \in A$ such that
\[
|\mu_{\alpha}(f) - \mu_{\alpha}(l_{\alpha} f)| \leq \frac{\varepsilon}{3}
\]
for all $\alpha \geq \alpha_0$. Since $\mu$ is a cluster point of the net $\mu_{\alpha}$, $\alpha \in A$, we can choose $\alpha_1 (\geq \alpha_0)$ such that
\[
|\mu_{\alpha_1}(f) - f| \leq \frac{\varepsilon}{3}
\]
and
\[
|\mu_{\alpha_1}(l_{\alpha} f) - \mu(l_{\alpha} f)| \leq \frac{\varepsilon}{3}.
\]
Hence, we have
\[
|\mu(f) - \mu(l_{\alpha} f)| \leq |\mu(f) - \mu_{\alpha_1}(f)| + |\mu_{\alpha_1}(f) - \mu_{\alpha_1}(l_{\alpha} f)| \\
+ |\mu_{\alpha_1}(l_{\alpha} f) - \mu(l_{\alpha} f)| \\
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
Since $\varepsilon > 0$ is arbitrary, we have
\[ \mu(f) = \mu(l_f) \]
for every $f \in \text{RUC}(S)$ and $s \in S$. This implies that $\mu$ is left invariant. Similarly, $\mu$ is right invariant.

Let $\{T_{\mu_{\alpha}}x\}$ be a subnet of $\{T_{\mu_{\alpha}}x\}$ such that $T_{\mu_{\alpha}}x$ converges weakly to some $z$ in $H$. Then since a cluster point $\lambda$ of the net $\{\mu_{\alpha}\}$ is also a cluster point of the net $\{\mu_{\alpha}\}$, $\lambda$ is an invariant mean. So, we have $z = T_{\lambda}x = x_0$. This implies that $T_{\mu_0}x$ converges weakly to $x_0 \in F(S)$.

3. Fixed point theorem. In this section, we prove a fixed point theorem for nonlinear semigroups without convexity. Let $X$ be a subspace of $B(S)$ containing constants. Then, according to Mizoguchi and Takahashi [10], a real valued function $\mu$ on $X$ is called a submean on $X$ if the following conditions are satisfied:

1. $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
2. $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
3. for $f, g \in X, f \leq g$ implies $\mu(f) \leq \mu(g)$;
4. $\mu(c) = c$ for every constant $c$.

For a submean $\mu$ on $X$ and $f \in X$, according to time and circumstances, we also use $\mu_t(f(t))$ instead of $\mu(f)$.

**Lemma** [10]. Let $S$ be a semitopological semigroup, let $X$ be a subspace of $B(S)$ containing constants and let $\mu$ be a submean on $X$. Let $\{x_t : t \in S\}$ be a bounded subset of a Hilbert space $H$ and let $D$ be a closed convex subset of $H$. Suppose that for each $x \in D$, the real-valued function $f$ on $S$ defined by
\[ f(t) = \|x_t - x\|^2 \text{ for all } t \in S \]
belongs to $X$. If
\[ g(x) = \mu_t\|x_t - x\|^2 \text{ for all } x \in D \]
and
\[ r = \inf\{g(x) : x \in D\}, \]
then there exists a unique element $z \in D$ such that $g(z) = r$. Further the following inequality holds:
\[ r + \|z - x\|^2 \leq g(x) \text{ for every } x \in D. \]

Let $X$ be a subspace of $B(S)$ containing constants which is $l_S$-invariant, i.e., $l_s(X) \subset X$ for each $s \in S$. Then a submean $\mu$ on $X$ is said to be left invariant if $\mu(f) = \mu(l_f)$ for all $s \in S$ and $f \in X$. Now, we can prove a fixed point theorem for nonlinear semigroups without convexity in a Hilbert space.
THEOREM 2. Let $C$ be a nonempty subset of a Hilbert space $H$ and let $X$ be a $l_S$-invariant subspace of $B(S)$ containing constants which has a left invariant submean $\mu$ on $X$. Let $S = \{T_s : s \in S\}$ be a Lipschitzian semigroup on $C$ with Lipschitzian constants $k_s, s \in S$. Suppose that $\{T_s x : s \in S\}$ is bounded and $\bigcap_{s \in S} \overline{cô\{T_s x : t \in S\}} \subset C$ for some $x \in C$. If for each $u \in C$ and $v \in H$, the real valued function $f$ on $S$ defined by

$$f(t) = \|T_t u - v\|^2 \text{ for all } t \in S$$

and the function $h$ on $S$ defined by

$$h(t) = k^2_t \text{ for all } t \in S$$

belong to $X$ and $\mu_s(k^2_s) \leq 1$, then there exists an element $z \in C$ such that $T_s z = z$ for all $s \in S$.

PROOF. Define a real valued function $g$ on $H$ by

$$g(y) = \mu_t \|T_t x - y\|^2 \text{ for each } y \in H.$$  

If $r = \inf \{g(y) : y \in H\}$, then by Lemma there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leq g(y) \text{ for every } y \in H.$$ 

For each $s \in S$, let $Q_s$ be the metric projection of $H$ onto $\overline{cô\{T_s x : t \in S\}}$. Then by Phelps [13], $Q_s$ is nonexpansive and for each $t \in S$,

$$\|T_s x - Q_s z\|^2 = \|Q_s T_s x - Q_s z\|^2 \leq \|T_s x - z\|^2.$$ 

So, we have

$$\mu_t \|T_t x - Q_s z\|^2 = \mu_t \|T_s x - Q_s z\|^2 \leq \mu_t \|T_s x - z\|^2$$ 

and thus $Q_s z = z$. This implies

$$z \in \overline{cô\{T_s x : t \in S\}} \text{ for all } s \in S$$

and hence

$$z \in \bigcap_{s \in S} \overline{cô\{T_s x : t \in S\}} \subset C.$$ 

Since by Lemma

$$\|z - y\|^2 \leq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2 \text{ for all } y \in H,$$

putting $y = T_s z$ for each $s \in S$, we have

$$\|z - T_s z\|^2 \leq \mu_t \|T_t x - T_s z\|^2 - \mu_t \|T_t x - z\|^2$$ 

$$= \mu_t \|T_s x - T_s z\|^2 - \mu_t \|T_t x - z\|^2$$ 

$$\leq (k^2_s - 1) \mu_t \|T_t x - z\|^2.$$
and hence
\[ \mu_s \| z - T_sz \|^2 \leq (\mu_s(k_s^2) - 1) \mu_t \| T_t x - z \|^2 \leq 0. \]
This implies \( \mu_s \| z - T_sz \|^2 = 0 \). Since for every \( a, s \in S \),
\[ \| z - T_az \|^2 \leq 2\| z - T_sz \|^2 + 2\| T_sz - T_az \|^2, \]
we have
\[ \| z - T_az \|^2 \leq 2\mu_s \| z - T_sz \|^2 + 2\mu_s \| T_sz - T_az \|^2 \]
\[ = 2\mu_s \| T_sz - T_az \|^2 = 2\mu_s \| T_\alpha z - T_az \|^2 \]
\[ \leq 2k_s^2 \mu_s \| T_\alpha z - z \|^2 = 0. \]
Therefore, we have \( T_\alpha z = z \) for every \( s \in S \).

As a direct consequence of Theorem 2, we obtain the result \( F(S) \neq \emptyset \) in Theorem 1. Further, we can prove the following fixed point theorem. A semitopological semigroup \( S \) is left reversible if any two closed right ideals of \( S \) have nonvoid intersection. In this case, \( (S, \leq) \) is a directed system when the binary relation “\( \leq \)” on \( S \) is defined by \( a \leq b \) if and only if \( \{a\} \cup \overline{aS} \supset \{b\} \cup \overline{bS} \).

**COROLLARY [7].** Let \( C \) be a nonempty subset of a Hilbert space \( H \) and let \( S \) be a left reversible semigroup. Let \( S = \{T_t : t \in S\} \) be a Lipschitzian semigroup on \( C \) such that \( \{k_s : s \in S\} \) is bounded and \( \lim_{s \to 0} \sup k_s \leq 1 \). If \( \{T_t x : t \in S\} \) is bounded and \( \bigcap_{s \in S} \overline{T_s x : t \in S} \subset C \) for some \( x \in C \). Then there exists \( z \in C \) such that \( T_sz = z \) for every \( s \in S \).

**PROOF.** Defining a real valued function \( \mu \) on \( B(S) \) by
\[ \mu(f) = \lim_{s \to 0} \sup f(s) \] for every \( f \in B(S) \),
\( \mu \) is a left invariant submean on \( B(S) \). Since \( \lim_{s \to 0} \sup k_s \leq 1 \) implies \( \lim_{s \to 0} \sup k_s^2 \leq 1 \), by using Theorem 2, the proof is complete.

We may comment on the relationship between the hypothesis of Theorem 1: “\( RUC(S) \) has an invariant mean” and Corollary [7]: “\( S \) is left reversible”. As well known, they do not imply each other in general. But if \( RUC(S) \) has sufficiently many functions to separate closed sets, then “\( RUC(S) \) has an invariant mean” would imply “\( S \) is left and right reversible”.

**REFERENCES**


*Department of Information Science*
*Tokyo Institute of Technology*
*Oh-Okayama, Meguro-ku*
*Tokyo 152, Japan*