Über einige neuere Fortschritte der additiven Zahlentheorie. By E. LANDAU. Pp. 94. 6s. 1937. Cambridge Tracts in Mathematics and Mathematical Physics, 35. (Cambridge)

This tract, which is a development of lectures delivered by Professor Landau in Cambridge in 1935 at the invitation of the Faculty Board of Mathematics, gives a connected account of some of the remarkable advances which have been made in the additive theory of numbers since 1930. The lectures were in English, but the tract is in German and enjoys the distinction of being the first of the Cambridge tracts to be issued in a language other than English.

The subject matter of the tract centres round three famous conjectures of Waring, Goldbach, and Gauss, and will be best appreciated by a reader familiar with the chapters on Waring’s problem, Goldbach’s problem, and the theory of binary quadratic forms in the author’s Vorlesungen über Zahlentheorie. The tract is, however, so written that only the classical elements of the theory of numbers are assumed. The key references to original sources are given in the Introduction; from these the reader can trace back the detailed history of the various topics for himself.

In Waring’s problem the fundamental number is denoted, after Hardy and Littlewood, by $G(k)$. This is the smallest number $s$ such that every sufficiently large positive integer can be expressed as a sum of $s$ positive or zero $k$th powers. Its existence for each positive integer $k$ was conjectured by Waring in 1770 and proved by Hilbert in 1909. But the first explicit estimate $G(k) \leq \tilde{G}(k)$ in terms of an elementary function $\tilde{G}(k)$ was found by Hardy and Littlewood with the help of their powerful analytical method, which reduces the study of an arithmetical function $r(n)$ (in this case the number of representations of $n$ as a sum of $s$ non-negative $k$th powers) to the study of the “generating function” $\sum r(n)x^n$ near its circle of convergence. The best estimate $\tilde{G}(k)$ ultimately obtained by Hardy and Littlewood was exponentially large for large $k$. In 1934 Vinogradoff proved that $G(k)=O(k \log k)$ when $k \to \infty$.

The basis of this remarkable advance is an elementary lemma giving (in suitable circumstances) a non-trivial estimate of a finite trigonometrical sum of the type

$$\sum_{m,n} e^{m\theta n} \quad (\theta \text{ real}),$$

in which $m$ and $n$ range independently over arbitrary sets of integers (not necessarily consecutive). The ingenious method by which double sums of this type are made to play a part in Waring’s problem is expounded, with notable simplifications due to Heilbronn, in Chapter 1. Whether $G(k)$ is actually of order $k$, as Hardy and Littlewood conjectured in 1925, is still undecided.

Chapter 2 contains a simplified proof of Schnirelmann’s theorem (1930) that every integer $n > 1$ can be expressed as a sum of not more than $A$ primes, where $A$ is an absolute constant (independent of $n$). This is a step in the direction of Goldbach’s conjecture of 1742 that every even integer can be expressed as a sum of two primes. Intermediate between these two assertions stands the result to which Hardy and Littlewood were led by the application of their analytical method, that every sufficiently large odd integer can be expressed as a sum of three primes. But this was obtained by them only on the assumption of an unproved hypothesis concerning the zeros of Dirichlet’s $L$-functions. Schnirelmann’s theorem was the first of its kind to be rigorously proved.

The proof of this theorem introduced the notion of “density”, and suggested a deeper study of this notion for its own sake. The density $\alpha$ of a set
\( \alpha A \) of (distinct) positive integers \( \alpha \) is defined as the lower bound of \( A(x)/x \) for \( x = 1, 2, 3, \ldots \), where \( A(x) \) is the number of \( \alpha \) not exceeding \( x \). The sum \( S = \alpha A + B \) of two sets \( \alpha A \) and \( B \) is defined as the set of distinct integers of the form \( a, b \) or \( a + b \), where \( a \) and \( b \) are typical members of \( \alpha A \) and \( B \) respectively. What can we say about the density \( \sigma \) of \( S \) in terms of the densities \( \alpha, \beta \) of \( \alpha A, B \)?

The proof of Schnirelmann's theorem uses two simple general propositions: (i) \( 1 - \sigma \leq (1 - \alpha)(1 - \beta) \), that is, \( \sigma \geq \alpha + \beta - \alpha \beta \); (ii) if \( \alpha + \beta \geq 1 \), then \( \sigma = 1 \) (so that \( S \) includes all positive integers). It has been conjectured that \( \sigma \geq \alpha + \beta \) when \( \alpha + \beta < 1 \). This is an unsolved problem, but important contributions to it were made by Khintchine in 1932 and by Besicovitch in 1935, and are described in Chapter 4. Khintchine's theorem is that the conjecture is true when \( \alpha = \beta \). The proof of this is described by Professor Landau as "elementar und doch ein sehr kompliziertes grosses Kunstwerk". Chapter 3 deals with some special cases in which \( \sigma > \alpha + \beta \). The sets

\[
(\mathcal{P}) \quad 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots,
\]

\[
(\alpha A_k) \quad 1^k, 2^k, 3^k, 4^k, \ldots \quad (k \geq 1),
\]

\[
(\mathcal{G}_a) \quad 1, a, a^2, a^3, \ldots \quad (a > 1),
\]

all have density zero, but \( \mathcal{P} + \mathcal{P}, \mathcal{P} + \alpha A_k, \mathcal{P} + \mathcal{G}_a \) all have positive density. The first of these results was the crucial step in the proof of Schnirelmann's theorem, and the other two were proved by Romanoff in 1934. The proofs are "elementary", but employ the elaborate form of the "sieve of Eratosthenes" devised by Viggo Brun in 1920. There is a further group of theorems, introduced by Khintchine in 1933 and extended by Erdős, to the effect that, for certain special \( B \) with \( \beta = 0 \) and any \( \alpha A \) with \( 0 < \alpha < 1 \), we have \( \sigma > \alpha \) and indeed \( \sigma \geq \alpha + B\alpha(1 - \alpha) \) with a positive \( B \) depending only on \( B \).

Examples of such sets \( B \) are \( \alpha A_k \) and \( \mathcal{P} \).

An appendix is devoted to a proof of Siegel's theorem on the class-number \( h(d) \) of binary quadratic forms of negative discriminant \( d \). A conjecture of Gauss (1801) suggests that \( h(d) \to \infty \) when the discriminant \( d \) tends to \(-\infty \). In 1934 Heilbronn succeeded in proving this by bridging the gap between results of Hecke on the one hand and of Deuring and Mordell on the other which showed that the conclusion followed from hypotheses of opposite kinds concerning the zeros of Dirichlet's \( L \)-functions and of the Riemann zeta-function respectively. In 1935 Siegel adapted Heilbronn's method to the proof of a more general theorem, which in the case of negative discriminants asserts that \( h(d) \geq C' \| d \|^{-\epsilon} \) for any \( \epsilon > 0 \) and some positive \( C' \) depending only on \( \epsilon \).

The subject is still advancing, and important new results have been published since the tract appeared. Thus Davenport and Heilbronn have shown that the set \( S = \mathcal{P} + \alpha A_k \), proved by Romanoff to be of positive density, includes "almost all" positive integers, that is to say that \( S(x)/x \to 1 \) when \( x \to \infty \). But the crowning event is Vinogradoff's proof of the theorem (obtained by Hardy and Littlewood only on the assumption of an unproved hypothesis) that every sufficiently large odd integer can be expressed as a sum of three primes. This theorem, if it had been available, might have found a very natural place in the tract; for the result supersedes Schnirelmann's theorem, and the proof combines in a remarkable way results and methods from various parts of the theory. The analytical method of Hardy and Littlewood, a novel adaptation of the "sieve of Eratosthenes", Vinogradoff's lemma on trigonometrical sums, and Siegel's theorem on the class-number are all brought into play. If it was a source of disappointment to Professor Landau that he should have concluded his tract too early to include a reference to this sensational discovery, he may well have derived consolation from the knowledge of the
part which he himself has played, by his own writings and by personal en­
couragement given to others, in stimulating the intense activity to which his tract bears witness. The news of his death on February 19 comes as a great shock to workers in the theory of numbers, who will appreciate all the more keenly the magnitude of the debt which they owe to him—an indebtedness greatly increased by a publication which makes available such a wealth of material in such a compact and readable form.

A. E. I.


Woodger’s book is, as he legitimately claims, an entirely original experiment. It is the first attempt to put some aspects of biological theory into axiomatic form. The main system which is proposed and worked out in some detail involves ten undefined signs. Of these two, \( P \) meaning "part of", and \( T \) "before in Time", are general concepts, while the other eight are more specifically biological: org for "organised unities", \( U \) for "division or fusion", cell for "cells", \( m \) and \( f \) for "male and female", \( wh \) for "whole organisms", \( Env \) for the "environment" and \( genet \) for the class of "genetic properties". By manipulating these few concepts, Woodger builds up more complex notions which suffice to express some biological laws, particularly those dealing with heredity.

From the logistic point of view, probably the most interesting feature of the system is the extensive use made of the class of relations known as hierarchies, which are defined as follows:

\[ \text{hier} = DfR(R \subseteq \text{cls. } E \cap B \cdot R). \]

These are clearly branching systems, and in the biological examples the two most important cases are when \( R \subseteq 1 \to 2 \) as for cells, or \( R \subseteq 1 \to \text{cls. for sexually reproducing organisms.} \) From the first case it is simple to define biologically appropriate concepts, such as that of the level (generation) of a cell in a hierarchy derived from a single beginner cell. The consequences of the second relation have not been so fully worked out, but look as though they would provide some interesting possibilities which might be useful, for instance, in the theory of inbreeding.

The main importance of the book, however, is as a contribution to biological theory, and it is from this point of view that it must be judged. Standing as it does completely alone as the first contribution to axiomatic biology, its value can probably not yet be correctly assessed. It has clearly the status of a forerunner, but its ultimate importance will depend on whether the method which it introduces can be developed into a valuable part of biology. There is no doubt that the particular fields which Woodger has chosen to expound are the most suitable for this type of approach. The elementary rules of heredity have the simplicity and clarity of a theorem of mathematics, but they are almost the only part of biology of which this is true. Even here it is only some aspects of the behaviour of the fundamental entities, genes, chromosomes, etc., which are simple and clearcut. The axiomatic approach quite rightly begins by cloaking the difficulties under its undefined symbols; for instance, the whole problem of biological organisation lies beneath Woodger’s symbols \( org \) and \( wh \), the question of the relation between genes as counters handed from generation to generation and as determinants of the course of development is implicit in the concept \( genet \). These difficulties, however, are the real test, and the axiomatic method will not be of major importance unless it can say something about them.