ON CERTAIN EXTENSIONS OF FUNCTION RINGS

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1. Introduction. The present note is concerned with the existence and properties of certain types of extensions of Banach algebras which allow a faithful representation as the normed ring $\mathbf{C}(E)$ of all bounded continuous real functions on some topological space E. These Banach algebras can be characterized intrinsically in various ways **(1)**; they will be called function rings here. A function ring $\mathbf{E}$ will be called a normal extension of a function ring $\mathbf{C}$ if $\mathbf{E}$ is directly indecomposable, contains $\mathbf{C}$ as a Banach subalgebra and possesses a group G of automorphisms for which $\mathbf{C}$ is the ring of invariants, that is, the set of all elements fixed under G. G will then be called a group of automorphisms of $\mathbf{E}$ over $\mathbf{C}$. If $\mathbf{E}$ is a normal extension of $\mathbf{C}$ with precisely one group of automorphisms over $\mathbf{C}$, which is then the invariance group of $\mathbf{C}$ in $\mathbf{E}$, then $\mathbf{E}$ will be called a Galois extension of $\mathbf{C}$. Such an extension will be called finite if its group is finite.

The discussions below prove, for any directly indecomposable function ring, the existence of normal extensions with arbitrarily prescribed group and give a characterization of all finite Galois extensions of a function ring $\mathbf{C} = \mathbf{C}(E)$, E completely regular, which fully decompose all maximal ideals in C, in terms of regular quasi-covering spaces of E. In the special case, for instance, where E is normal and the union of finitely many simply connected open sets, the result is that any extension $\mathbf{E}$ of the considered type is a $\mathbf{C}(X)$ given by a regular covering space (X, ϕ, E) whose Poincaré group induces isomorphically the group of $\mathbf{E}$ over $\mathbf{C}$.

The proofs for these statements are obtained through arguments relating to fibre spaces which will then be interpreted for function rings via the well-known relations of the automorphisms and subrings of such rings to the underlying spaces or their Stone-Čech extensions. As it seems advisable, for the sake of clarity, to treat these two different aspects of this subject quite separately, all the topological material needed here will be presented first, whereas the transition to function rings will be left to the latter part of the paper. This transition will be based on the following familiar facts:

(1) Any function ring $\mathbf{C}$ is the $\mathbf{C}(S)$ of a unique compact S, its maximal ideal space, and the automorphisms of $\mathbf{C}$ are all induced by space automorphisms of S.

(2) If $\mathbf{C} = \mathbf{C}(E)$ with non-compact completely regular E, then $S = \beta E$, the Stone-Čech compactification of E.

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(3) If the directly indecomposable function ring **E** is an extension of **C** and X and E are the respective maximal ideal spaces, then X is mapped continuously onto E by $\phi: \mathfrak{M} \to \mathfrak{M} \cap \mathbf{C}$, $\mathfrak{M}$ the maximal ideals of **E**.

2. Automorphisms of fibre spaces. A fibre space is a triple (X, ϕ, E) consisting of two spaces X and E together with a continuous mapping $\phi: X \to E$. Although usually not required, $\phi X = E$ will always be assumed here. The base of (X, ϕ, E) is E, the fibre above $x \in E$ is the set $\phi^{-1}x$, and the group $G(\phi)$ of (X, ϕ, E) is the group of all space automorphisms of X which transform each fibre into itself.

If X is any space and s an automorphism of X, then s induces an automorphism $\bar{s}$ in $\mathbf{C}(X)$ by means of the formula $(\bar{s}f)x = f(s^{-1}x)$. Thus, in a fibre space (X, ϕ, E) the group $G(\phi)$ induces a group $\bar{G}(\phi)$ of automorphisms in $\mathbf{C}(X)$; $s \to \bar{s}$ is a homomorphism and may, but need not be an isomorphism.

For any fibre space (X, ϕ, E) the functions $f \in \mathbf{C}(E)$ determine functions $f^* = f\phi$ on X. By a known theorem (**4**, ch. I, §9) these are precisely those $g \in \mathbf{C}(X)$ which are constant on each fibre. The transition $f \to f^*$ imbeds isomorphically the function ring $\mathbf{C}(E)$ into the function ring $\mathbf{C}(X)$; the imbedded ring will be denoted by $\mathbf{C}(E)^*$.

The first topological fact needed later on concerns the existence of certain fibre spaces. It will be assumed that any space occurring contains more than one point.

LEMMA 1. *If E is a connected completely regular space and G any given group, then there exists a fibre space (X, ϕ, E) with connected completely regular X such that G is (isomorphic to) a subgroup of $G(\phi)$ acting transitively on each fibre.*

Proof. Regard G as a discrete space. Then the product space $Y = G \times E$ has G as a group of automorphisms if the action of $s \in G$ be defined by $s(t, x) = (st, x)$. Now, take a fixed $a \in E$, identify all $(s, a) \in Y$, call the resulting quotient space X and let θ be the natural mapping $Y \to X$. Since E is connected, the sets $E_s = \{s\} \times E \subseteq Y$ and $\theta E_s \subseteq X$ are also connected, and therefore $X = \bigcup \theta E_s$ is connected because $\bigcap \theta E_s$ is non-void.

The continuous mapping $(s, x) \to x$ induces a continuous mapping $\phi: X \to E$ with $\phi X = E$. Also, any $s \in G$ induces a mapping s_* of X onto itself by $s_*\theta(t, x) = \theta(st, x)$, continuous because $(t, x) \to \theta(st, x)$ is a continuous mapping of Y onto X, constant on the set of all (t, a), $t \in G$. Furthermore, since $(st)_* = s_*t_*$, any s_* has a continuous inverse and is thus an automorphism of X. Finally, the mapping $s \to s_*$ is an isomorphism since $s_*\theta(t, x) = \theta(st, x) \neq \theta(t, x)$ for all $x \neq a$ if s is not the unit transformation. Obviously, these s_* map each fibre of (X, ϕ, E) onto itself and their totality acts transitively on each fibre.

It remains to prove that X is completely regular. For this it is sufficient to show that it can be imbedded in some compact space. Let K be a compact space containing Y, which exists by the complete regularity of E, and identify in K all the points of the closure of the set of all (s, a), $s \in G$. The quotient space L thus obtained is again compact and contains X as a subspace (**4**, ch. I, §9). This completes the proof.

Remark. Obviously, any permutation p of G gives rise to an automorphism of Y and hence to an automorphism p_* of X. If $E - \{a\}$ is connected, then the group $G(\phi)$ of (X, ϕ, E) consists precisely of these p_* since the set of all points in θE_s other than $\theta(s, a)$ which an element of $G(\phi)$ maps into the same θE_t is open-closed in $\theta E_s - \{\theta(s, a)\}$.

Lemma 1 will be employed in connection with the following statements about a fibre space (X, ϕ, E) and its group $G = G(\phi)$.

LEMMA 2. *If* $\mathbf{C}(E)$ *separates the points of* E, *then any automorphism* s *of* X *whose* $\bar{s}$ *leaves each* $g \in \mathbf{C}(E)^*$ *fixed belongs to* G. *If* $\mathbf{C}(X)$ *separates the points of* X, *then* $\bar{G}$ *is isomorphic to* G *under* $s \to \bar{s}$. *If* $H \subseteq G$ *acts transitively on each fibre, then* $\mathbf{C}(E)^*$ *is the ring of invariants of* $\bar{H}$. *If* $H \subseteq G$ *does not act transitively on each fibre and if the functions on the orbit space* X/H *separate points, then the ring of invariants of* $\bar{H}$ *is greater than* $\mathbf{C}(E)^*$.

Proof. Suppose s is an automorphism of X not belonging to G, that is, $\phi a \neq \phi(s^{-1}a)$ for some $a \in X$. Then, if $f \in \mathbf{C}(E)$ separates these two points one obviously has $\bar{s}f^* \neq f^*$ and thus $\bar{s}$ does not leave all $g \in \mathbf{C}(E)^*$ fixed. Next, consider, for any $s \in G$ which is not the unit transformation, an $a \in X$ such that $s^{-1}a \neq a$ which exists since G acts effectively by definition. Then, if $f \in \mathbf{C}(X)$ separates $s^{-1}a$ from a one has $\bar{s}f \neq f$ and thus $\bar{s}$ is not the unit transformation, either. Further, if $f \in \mathbf{C}(X)$ satisfies $\bar{s}f = f$ for all $s \in H$ where H acts transitively on each fibre, then f is constant on each fibre and hence belongs to $\mathbf{C}(E)^*$. Finally, if H does not act transitively on each fibre one has $X/H \neq E$ and there exist distinct points a and b in X/H which have the same image c under the mapping $X/H \to E$ induced by ϕ. A function $g \in \mathbf{C}(X/H)$ separating a from b then gives a $g^* \in \mathbf{C}(X)$ with $\bar{s}g^* = g^*$ for all $s \in H$ which is, however, not constant on the fibre above c.

3. Quasi-covering spaces. By a quasi-covering space is meant a fibre space (X, ϕ, E) with the property that each $x \in E$ has an open neighbourhood V whose $\phi^{-1}V$ is the union of disjoint open V' on which $\phi | V'$, the restriction of ϕ to V', is a homeomorphism onto V. An open set V with this property will be called evenly covered by (X, ϕ, E). To emphasize the role of E, (X, ϕ, E) will also be called a quasi-covering space of E. If E should be locally connected and X connected, this reduces to the usual definition of a covering space (**3**, ch. II, §VI).

A quasi-covering space (X, ϕ, E) will be called regular if its group $G(\phi)$ acts transitively on each fibre. Then, if X is connected, any $s \in G(\phi)$ other than the unit transformation has no fixed points. For if V' is the part of $\phi^{-1}V$ belonging to $x' \in \phi^{-1}x$, V assumed to be an evenly covered open neighbourhood of x, and $sx' = x'$, then $sV' \cap V'$ is a neighbourhood of x' consisting of points fixed under s, whereas if $sx' = x'' \neq x'$ and V'' belongs to x'', then $s^{-1}(sV' \cap V'')$ is a neighbourhood of x' consisting of points moved by s; hence the sets of the two different kinds of points are both open and if X is connected, one of them must be void and the other X.

From the special type of Galois extensions of function rings to be considered below there arise fibre spaces (X, ϕ, E) with completely regular X and E and finite $G = G(\phi)$ such that (i) each fibre consists of as many points as G has elements and (ii) $\bar{G}$ has $\mathbf{C}(E)^*$ as its ring of invariants. In this situation one has

LEMMA 3. *(X, ϕ, E) is a regular quasi-covering space.*

Proof. First, the orbit space X/G is again completely regular. This follows from the fact that X/G is a subspace of the compact $\beta X/G'$, G' the continuous extension of G to βX, which in turn is a consequence of a known theorem on quotient spaces (**4**, ch. I, §9). Now, the functions on X/G certainly separate points, and as the ring of invariants of $\bar{G}$ is $\mathbf{C}(E)^*$, Lemma 2 implies that G acts transitively on each fibre. An immediate consequence of this and the hypothesis on the number of points in each fibre is that no $s \in G$ other than the unit has any fixed points. Further it follows that the mapping ϕ is open since $E = X/G$ and for any open $O \subseteq X$ $O^* = \cup sO$, $s \in G$, is again open.

Now, suppose there exists a $c \in X$ on none of whose neighbourhoods V $\phi|V$ is one-to-one. This means there are distinct points x_V and y_V in each V and an $s_V \in G$ such that $s_V x_V = y_V$. Then for some $s \in G$ the collection of those V for which $s_V = s$ must be cofinal in the neighbourhood filter of c, that is, any neighbourhood U of c contains some V with $s_V = s$. Call these neighbourhoods W. Since the two (Moore-Smith) sequences x_W and y_W both converge to c, one has $sc = s(\lim x_W) = \lim sx_W = \lim y_W = c$. However, $sc = c$ only holds for the unit of G which contradicts the assumption that all x_V and y_V are distinct. Therefore, any $x \in X$ has a neighbourhood V which is mapped one-to-one. V can be taken as open, and as ϕ is continuous and open $\phi|V$ is a homoemorphism. Finally, $\phi|sV$ is also a homeomorphism for each $s \in G$, $U = \phi V$ is an open neighbourhood of $\phi x \in E$ and $\phi^{-1}U = \cup sV$. This completes the proof.

The next item to be considered is the construction of a quasi-covering space of a completely regular E from a quasi-covering space of its βE. In this, the following notion will be employed. If (Y, ψ, W) is a fibre space and E a dense subspace of W, then the fibre space (X, ϕ, E), where $X = \psi^{-1}E$ and $\phi = \psi|X$, will be called the restriction of (Y, ψ, W) to the base E. The group $G(\psi)$, since it transforms X into itself, induces a subgroup of $G(\phi)$ by restriction to X. Obviously, if $G(\psi)$ acts transitively on each fibre, then $G(\phi)$ does too.

LEMMA 4. *If* $(Y, \psi, \beta E)$ *is a regular quasi-covering space with compact* Y, *then its restriction* (X, ϕ, E) *to the base* E *is again a regular quasi-covering space, with* $\beta X = Y$ *and* $G(\phi)$ *induced by* $G(\psi)$.

Proof. Let U be an evenly covered open neighbourhood in βE of $x \in E$ and U' the disjoint open sets into which $\psi^{-1}U$ then splits. For the neighbourhood $V = U \cap E$ of x in E one then has $\phi^{-1}V = \psi^{-1}V = (\psi^{-1}U) \cap X$, and this is the union of the disjoint open $U' \cap X$ in X each of which is mapped homeomorphically onto V by ϕ. Hence (X, ϕ, E) is a quasi-covering space and thus, of course, a regular one.

To obtain $\beta X = Y$, use will be made of completely regular filters. A filter $\mathfrak{A}$ on a space S is called completely regular if for any $A \in \mathfrak{A}$ there exsits some $f \in \mathbf{C}(S)$ such that $0 \leqslant f \leqslant 1$, $f^{-1}\{0\} \in \mathfrak{A}$ and $f = 1$ outside A. If K is a compact space containing S densely, then each $z \in K - S$ determines a filter $\mathfrak{T}(z)$ on S, its trace filter, consisting of the sets $V \cap S$ where V ranges over the neighbourhoods of z in K. These $\mathfrak{T}(z)$ are completely regular filters and $K = \beta S$ holds exactly if they are all maximal completely regular **(4**, ch. 9, §1**)**. In this case, the $\mathfrak{T}(z)$ will be precisely the non-convergent maximal completely regular filters on S, the convergent ones just being the neighbourhood filters of the points of S.

Now, let U be an open neighbourhood of $u \in Y - X$ such that $\psi|U$ is one-to-one. Because $U \cap X$ and $(\psi U) \cap E$ correspond to each other, the trace filter $\mathfrak{T}(u)$ of u on X is mapped by ϕ onto the trace filter $\mathfrak{T}(\psi u)$ of ψu on E. Hence, for any completely regular filter $\mathfrak{R} \supseteq \mathfrak{T}(u)$ on X one has $\phi\mathfrak{R} \supseteq \mathfrak{T}(\psi u)$. However, $\phi\mathfrak{R}$ is again completely regular: for any $A \in \phi\mathfrak{R}$ take a $B \in \mathfrak{R}$ such that its closure in X lies in $U \cap X$ and $\phi B \subseteq A$, and then some $f \in \mathbf{C}(X)$ with $0 \leqslant f \leqslant 1$, $f^{-1}\{0\} \in \mathfrak{R}$ and $f = 1$ outside B. With this f, define g on E by

$$gx = \begin{cases} 1 \text{ if } x \notin (\psi U) \cap E \\ fz \text{ if } x = \phi z,\ x \in (\psi U) \cap E. \end{cases}$$

This g is continuous, has value 1 outside ϕB and hence outside A, and satisfies $0 \leqslant g \leqslant 1$ and $g^{-1}\{0\} \in \phi\mathfrak{R}$. Thus, $\phi\mathfrak{R}$ is completely regular. But now, $\mathfrak{T}(\psi u)$ is maximal completely regular because the extension of E considered is βE; therefore, $\phi\mathfrak{R} = \mathfrak{T}(\psi u) = \phi\mathfrak{T}(u)$ and as $\mathfrak{T}(u)$ contains a set on which ϕ is one-to-one, this gives $\mathfrak{R} = \mathfrak{T}(u)$. Since $\mathfrak{R} \supseteq \mathfrak{T}(u)$ was arbitrary completely regular, $\mathfrak{T}(u)$ is hereby seen to be maximal and this shows $\beta X = Y$.

Finally, any $s \in G(\phi)$ can be uniquely extended to an automorphism s' of $Y = \beta X$ and from $s'u = s'(\lim \mathfrak{T}(u)) = \lim s\mathfrak{T}(u)$ for any $u \in Y - X$ one obtains $\psi(s'u) = \psi u$, that is, $s' \in G(\psi)$. Hence, $G(\phi)$ is the restriction of $G(\psi)$ to X, and this completes the proof.

There is a partial converse to Lemma 4. If (X, ϕ, E) is a fibre space with completely regular X and E, then ϕ has a unique continuous extension ψ: $\beta X \rightarrow \beta E$ with $\psi(\beta X) = \beta E$ **(7)**. The fibre space $(\beta X, \psi, \beta E)$ will be called the extension of (X, ϕ, E) to the base βE. Each $s \in G(\phi)$ has an extension to an automorphism of βX which, by the same argument as in the last paragraph, belongs to $G(\psi)$ such that $G(\phi)$ induces a subgroup of $G(\psi)$. A good deal more can be said if further conditions are assumed for (X, ϕ, E).

LEMMA 5. *If (X, ϕ, E) is a finite regular quasi-covering space with connected X such that each non-convergent maximal completely regular filter on E contains an evenly covered open set, then its extension $(\beta X, \psi, \beta E)$ to the base βE is again a regular quasi-covering space and $G(\psi)$ is induced by $G(\phi)$.*

Proof. If G' denotes the continuous extension of $G = G(\phi)$ to βX, then since $G' \subseteq G(\psi)$, ψ induces a continuous mapping of the compact orbit space $\beta X/G'$ onto βE such that E, which is also a subspace of $\beta X/G'$ (**4**, ch. I, §9) remains pointwise fixed. Therefore $\beta X/G' = \beta E$ by the maximality of βE and thus G' acts transitively on each fibre of $(\beta X, \psi, \beta E)$. Next, no $s' \in G'$ other than the unit has any fixed point. Such a point would have to be a $u \in \beta X - X$ and if $s'u = u$, $s' \in G'$ and not the unit, it would have arbitrarily small neighbourhoods U with $s'U = U$, since s' is of finite order, and hence its trace filter $\mathfrak{T}(u)$ on X would have a basis of sets V with $sV = V$. However, by continuity one has $\mathfrak{T}(\psi u) \subseteq \phi\mathfrak{T}(u)$, and by hypothesis $\mathfrak{T}(\psi u)$ contains an evenly covered open W. Then, $\phi^{-1}W \in \mathfrak{T}(u)$ where $\phi^{-1}W = \bigcup W_k$ with finitely many disjoint open W_k such that $\phi|W_k$ is a homeomorphism; it follows that $W_k \in \mathfrak{T}(u)$ for some k (**2**), but of course there is no $V \subseteq W_k$ with $sV = V$. This contradiction proves $s'u \neq u$.

One now obtains, by the same argument as in the proof of Lemma 3, that $(\beta X, \psi, \beta E)$ is a quasi-covering space; since $G' \subseteq G(\psi)$ is transitive on each fibre this implies regularity, and since X is connected one has $G' = G(\psi)$.

Remark. If (X, ϕ, E) satisfies the hypothesis in Lemma 5, then, by this lemma, E is the union of finitely many evenly covered sets since βE is compact. Conversely, this condition implies the hypothesis in Lemma 5, at least if E is a normal space since for such E any finite open covering of E is induced by one of βE. In particular, therefore, if E is normal and the union of finitely many simply connected open sets (in the sense of Chevalley) then, for any finite regular covering space (X, ϕ, E) the extension $(\beta X, \psi, \beta E)$ is a regular quasi-covering space, with $G(\psi)$ induced by $G(\phi)$. More generally, the same holds if E, not necessarily normal, has any compact extension K such that each $u \in K - E$ has an open neighbourhood for which $V = U \cap E$ is simply connected, for then any maximal completely regular filter on E will converge to some such u and hence contain the corresponding V which in turn will be evenly covered by any quasi-covering space (X, ϕ, E). Similarly, the above statement concerning $(\beta X, \psi, \beta E)$ is true if E is a finite dimensional separable metric space, for according to (**6**) X then contains a finite number of open W_i such that $\phi|W_i$ is a homeomorphism and $E = \bigcup \phi W_i$, and thus E is normal and the union of finitely many evenly covered open sets, the latter because $\phi^{-1}\phi W_i = \bigcup sW_i$, $s \in G(\phi)$.

4. Extensions of function rings. Let $\mathbf{C}$ denote any directly indecomposable function ring and G an arbitrary group.

PROPOSITION 1. *There exists a normal extension* $\mathbf{E}$ *of* $\mathbf{C}$ *which has* G *as a group of automorphisms over* $\mathbf{C}$.

Proof. If E is the maximal ideal space of $\mathbf{C}$ and (X, ϕ, E) the fibre space of Lemma 1, then, since E and X are both completely regular, Lemma 2 shows that $\mathbf{E} = \mathbf{C}(X)$ has the required properties if one identifies $\mathbf{C}$ with $\mathbf{C}(E)^*$.

An immediate consequence of Proposition 1 is that any directly indecomposable function ring $\mathbf{C}$ possesses directly indecomposable extensions $\mathbf{E}$ such that each element of $\mathbf{E}$ is algebraic over $\mathbf{C}$, for to obtain such extensions one only has to take a normal extension of C with some finite group. In other words, there is no such thing as algebraic closure in the class of all directly indecomposable function rings.

Another observation that can be made here is that if a directly indecomposable function ring $\mathbf{C}$ contains a maximal ideal which is not the sum of two closed ideals whose intersection is the zero ideal, then for any natural number n there exists a normal extension $\mathbf{E}$ of $\mathbf{C}$ such that the invariance group of $\mathbf{C}$ in $\mathbf{E}$ is isomorphic to the symmetric group S_n of n objects. First, the condition for the maximal ideals stated means that the maximal ideal space E of $\mathbf{C}$ contains a point a such that $E - \{a\}$ is connected. Then the remark following the proof of Lemma 1 shows that for (X, ϕ, E), constructed with any group of order n, one has $G(\phi) \cong S_n$ and since X is compact here, (1) in §1 implies that $\mathbf{C}(E)^* \cong \mathbf{C}$ has $\bar{G}(\phi) \cong S_n$ as its invariance group in $\mathbf{E} = \mathbf{C}(X)$.

The extensions obtained from Lemma 1 are normal but not Galois. Examples for the latter arise from regular quasi-covering spaces (X, ϕ, E) with connected X. There, $G(\phi)$ is the only group $H \subseteq G(\phi)$ acting transitively on each fibre and if X (and thus E) is compact or, for instance, completely regular and satisfying the first axiom of countability, then by Lemma 2, by (1) or (2) in §1 and by (5) $\bar{G}(\phi)$ is the invariance group of $\mathbf{C}(E)^*$ in $\mathbf{C}(X)$ and no proper subgroup of $\bar{G}(\phi)$ has $\mathbf{C}(E)^*$ as its ring of invariants. In other words, $\mathbf{C}(X)$ is then a Galois extension of $\mathbf{C}(E)^*$ with group $\bar{G}(\phi)$ which is, furthermore, isomorphic to $G(\phi)$.

If a Galois extension $\mathbf{E}$ of $\mathbf{C} = \mathbf{C}(E)$ is given, in the manner just described, by some regular quasi-covering space (X, ϕ, E) it will be called the Galois extension of $\mathbf{C}$ associated with (X, ϕ, E). The finite Galois extensions of this type have a certain property which can best be formulated by means of the following concept, borrowed from the ideal theory of algebraic number fields: a finite Galois extension $\mathbf{E}$ of $\mathbf{C}$ is said to decompose fully the maximal ideal $\mathfrak{m}$ in $\mathbf{C}$ if there are exactly as many maximal ideals $\mathfrak{M} \supseteq \mathfrak{m}$ in $\mathbf{E}$ as the group of $\mathbf{E}$ over $\mathbf{C}$ has elements. It is clear that all Galois extensions of $\mathbf{C}(E)$ associated with finite regular quasi-covering spaces (X, ϕ, E) fully decompose each fixed maximal ideal of $\mathbf{C}(E)$. However, not all finite Galois extensions of $\mathbf{C}(E)$

with this latter property are associated with regular quasi-covering spaces of E. Take, for instance, E to be an open annular region in the plane, (X, ψ, E) its regular covering space with group of order 2, and $\mathfrak{T}_1$ and $\mathfrak{T}_2$ two maximal completely regular filters on Y above the same maximal completely regular filter on E. Now, let $\mathbf{E}$ be the ring of all $f \in \mathbf{C}(Y)$ with $\lim f\mathfrak{T}_1 = \lim f\mathfrak{T}_2$. $\mathbf{E}$ is then a Galois extension of $\mathbf{C} = \mathbf{C}(E)^*$, its group also being of order 2. However, it is not associated with any regular quasi-covering space of E since this would have to have group of order 2, and (Y, ψ, E) is the only such quasi-covering space whereas $\mathbf{E} \neq \mathbf{C}(Y)$.

This observation suggests that in order to describe at least a class of finite Galois extensions of $\mathbf{C}(E)$ associated with regular quasi-covering spaces of E by means of simple algebraic conditions one has to assume full decomposition for all maximal ideals. With this, the following characterization can be obtained.

PROPOSITION 2. *Let E be a connected completely regular space. Then, a Galois extension $\mathbf{E}$ of $\mathbf{C} = \mathbf{C}(E)$ has finite group and fully decomposes each maximal ideal in $\mathbf{C}$ if and only if it is associated with a finite regular quasi-covering space (X, ϕ, E), with connected X, in which each non-convergent maximal completely regular filter on E contains an evenly covered open set.*

Proof. In the case of compact E, in which there are no non-convergent maximal completely regular filters, this is obvious from **(3; 1)** and Lemma 3. If E is not compact, one can first use the proposition for βE since $\mathbf{C}(E) = \mathbf{C}(\beta E)$. Hence, any extension $\mathbf{E}$ of the kind stated is associated with a regular quasi-covering space $(Y, \psi, \beta E)$ to which Lemma 4 can be applied; this gives a regular quasi-covering space (X, ϕ, E) with which $\mathbf{E}$ is also associated, and because of its origin and $Y = \beta X$, (X, ϕ, E) has the additional propreties concerning the maximal completely regular filters. Also, X is connected since $Y = \beta X$ is. Conversely, if (X, ϕ, E) satisfies all the conditions listed, then Lemma 5 shows that each maximal ideal in $\mathbf{C}(E)^*$ is contained in precisely as many maximal ideals of $\mathbf{C}(X)$ as $G(\phi)$ has elements. This means the extension of $\mathbf{C} = \mathbf{C}(E)$ associated with (X, ϕ, E) fully decomposes each maximal ideal of $\mathbf{C}$.

The condition for (X, ϕ, E) concerning the maximal completely regular filters on E prevents Proposition 2 from giving a complete description of the Galois extensions of $\mathbf{C}(E)$ which are associated with finite regular quasi-covering spaces of E. However, the remark following Lemma 5 shows that for certain types of spaces this condition is redundant. Hence one has:

COROLLARY 1. *If E possesses a compact extension K such that each $u \in K - E$ has an open neighbourhood U for which $U \cap E$ is simply connected or if E is a finite dimensional separable metric space, then the extensions of $\mathbf{C} = \mathbf{C}(E)$ associated with finite regular quasi-covering spaces of E are precisely the finite Galois extensions of $\mathbf{C}$ which fully decompose each maximal ideal of $\mathbf{C}$.*

In Proposition 2, attention is paid to the way in which the function ring $\mathbf{C}$ is concretely represented on some space. Independent of such representation of $\mathbf{C}$ one has:

COROLLARY 2. *If the finite Galois extension* $\mathbf{E}$ *of* $\mathbf{C}$ *fully decomposes each maximal ideal of* $\mathbf{C}$, *then its group* G *permutes the maximal ideals of* $\mathbf{E}$ *above each maximal ideal of* $\mathbf{C}$ *transitively and* $\mathbf{E}$ *is finitely generated over* $\mathbf{C}$.

Proof. $\mathbf{E}$ is associated with a regular quasi-covering space (X, ϕ, E) where X and E are the maximal ideal spaces of $\mathbf{E}$ and $\mathbf{C}$. The first part follows immediately from this. For the second part, one observes first that E is the union of finitely many evenly covered V_i and the parts V_{ik} into which the $\phi^{-1}V_i$ split form an open covering of X. Then, let $f_{ik} \in \mathbf{C}(X)$ be a decomposition of the unity, subordinate to this open covering, that is, $O \leqslant f_{ik} \leqslant 1$, $f_{ik} = 0$ outside V_{ik} and $\sum f_{ik} = 1$ (**4**, ch. IX, §4) and take h_{ik} such that $h_{ik}^2 = f_{ik}$. Now, for any $g \in \mathbf{C}(X)$ put $g_{ik} = \sum \bar{s}(gh_{ik})$, $s \in G(\phi)$; one then has $g_{ik}h_{ik} = gh_{ik}^2$ and hence $g = \sum g_{ik}h_{ik}$ where $g_{ik} \in \mathbf{C}(E)^*$. Thus, the h_{ik} generate $\mathbf{C}(X)$ over $\mathbf{C}(\mathrm{E})^*$.

5. Concluding remarks. In view of Corollary 1 of Proposition 2 one might ask whether the restriction on (X, ϕ, E) in this proposition is not, in fact, always redundant, in which case one would have a simple algebraic characterization of all Galois extensions of a function ring $\mathbf{C}(E)$ which are associated with finite quasi-covering spaces of E. This question is equivalent with the following problem: if G is a finite group of automorphisms of a connected completely regular X such that no $s \in G$ other than the unit has any fixed points, does the continuous extension of G to βX have the same property? In order to show that the answer is positive it would be sufficient to prove, for any such X and G, the existence of some compact extension of E onto which G can be continuously extended without losing its particular property; however, whilst this can be done for various special types of spaces, we do not know whether it is possible in general.

Quite apart from this problem, a certain "external" characterization of the Galois extensions of $\mathbf{C} = \mathbf{C}(E)$ which are associated with finite regular quasi-covering spaces of E can be given in the following way: the methods in §3 readily show that any Galois extension $\mathbf{E}$ of $\mathbf{C}$ which has finite group G and decomposes fully each fixed maximal ideal of $\mathbf{C}$ can be represented as the ring of (some) functions on a regular quasi-covering space (X, ϕ, E) such that G is induced isomorphically by $G(\phi)$. Therefore, a finite Galois extension $\mathbf{E}$ of $\mathbf{C}$ is associated with a regular quasi-covering space if and only if it fully decomposes each fixed maximal ideal of $\mathbf{C}$ and is not contained in any larger Galois extension of $\mathbf{C}$ with the same property whose group isomorphically induces the group of $\mathbf{E}$.

Another question arising from Proposition 2 is whether a similar treatment be possible in the case of infinite groups. For this the concept of full decomposition of a maximal ideal has to be extended first, and it is natural to do this by calling a maximal ideal $\mathfrak{m}$ in $\mathbf{C}$ fully decomposed by the Galois extension $\mathbf{E}$ of $\mathbf{C}$ with the group G if for any maximal ideal $\mathfrak{M} \supseteq \mathfrak{m}$ in E $s\mathfrak{M}$, $s \in G$, ranges over all maximal ideals above in such that $s\mathfrak{M} \neq t\mathfrak{M}$ if $s \neq t$. However, a Galois extension $\mathbf{E}$ of $\mathbf{C} = \mathbf{C}(E)$ associated with a regular quasi-covering space (X, ϕ, E) whose $G(\phi)$ is infinite cannot even fully decompose any fixed maximal ideal of $\mathbf{C}$, for each fibre $\phi^{-1}x$, $x \in E$, of the extension $(\beta X, \psi, \beta E)$ must contain points from $\beta X - X$ since otherwise $\phi^{-1}x = \psi^{-1}x$, which would mean $\phi^{-1}x$ is closed in βX besides being discrete and thus finite. Therefore, the concept of full decomposition of maximal ideals is useless for the description of the infinite Galois extensions of a function ring $\mathbf{C}(E)$ which are associated with regular quasi-covering spaces of E, whereas, on the other hand, such extensions do exist for suitable E.

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