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Zariski dense surface subgroups in $SL(n, \mathbb{Q})$ with odd n

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Abstract

For odd n we construct a path $\rho: \Pi_1(S) \to SL(n\mathbb{R})$ of discrete, faithful, and Zariski dense representations of a surface group such that $\rho_t(\Pi_1(S)) \subset SL(n, \mathbb{Q})$ for every $t \in \mathbb{Q}$.

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1. *Introduction*

Constructing Zariski dense surface subgroups in $SL(n, \mathbb{R})$ has attracted attention as a step to finding *thin groups*, these are infinite index subgroups of a lattice in $SL(n, \mathbb{R})$ which are Zariski dense. Finding thin subgroups inside lattices in a variety of Lie groups has been a topic of significant interest in recent years, in part from the connections thin groups have to expanders and the affine sieve of Bourgain, Gamburd and Sarnak [**BGS10, Sar14**].

Though thin subgroups are in a sense generic [**Fuc14, FR17**], finding particular specimens of thin surface subgroups in a given lattice remains a difficult task. In this direction, in 2011 Long, Reid and Thistlethwaite [**LRT11**] produced the first infinite family of nonconjugate thin surface groups in $SL(3, \mathbb{Z})$. Their approach relies on parametrising a family of representations ρ_t of the triangle group $\Delta(3, 3, 4)$ in the *Hitchin component*, so that for every $t \in \mathbb{Z}$ the subgroup $\rho_t(\Delta(3, 3, 4))$ is in $SL(3, \mathbb{Q})$ and has integral traces. By results of Bass [**Bas80**] these two properties together with $\rho_t(\Delta(3, 3, 4))$ being non-solvable and finitely generated guarantee that it is conjugate to a subgroup of $SL(3, \mathbb{Z})$. In 2018 Long and Thistlethwaite [**LT18**] used a similar approach to obtain an infinite family of non-conjugate Zariski dense surface subgroups in $SL(4, \mathbb{Z})$ and $SL(5, \mathbb{Z})$.

Ballas and Long [**BL18**] in turn used the idea of "bending" a representation of the fundamental group of a hyperbolic n-manifold $\pi_1(N)$ along an embedded totally geodesic and separating hypersurface to obtain thin groups in $SL(n+1, \mathbb{R})$ which are isomorphic to $\pi_1(N)$. The aim of this paper is to combine the aforementioned approaches to construct a family of Zariski dense rational surface group representations by bending orbifold representations. Our main result is the following:

THEOREM 1. *For every surface S finitely covering the orbifold $\mathcal{O}_{3,3,3,3}$ and every odd $n > 1$ there exists a path of discrete, faithful and irreducible representations $\rho_t : \pi_1(S) \to SL(n, \mathbb{R})$, so that:*

(i) $\rho_0(\pi_1(S)) < SL(n, \mathbb{Z})$;

(ii) ρ_t *is Zariski dense for every* $t > 0$*; and*

(iii) $\rho_t(\pi_1(S)) < SL(n, \mathbb{Q})$ *for every* $t \in \mathbb{Q}$.

Every representation ρ_t in Theorem 1 is a surface Hitchin representation. Several of its properties are derived from the seminal work of Labourie [**Lab06**] on Anosov representations, the classification of Zariski closures of surface Hitchin representations by Guichard [**Gui**], and the recent introduction of orbifold Hitchin representations by Alessandrini, Lee and Schaffhauser [**ALS19**]. We provide an overview of these results in Sections 2 and 3. The discussion in these sections applies to all n, with the assumption of odd n coming into play later in Section 4. At the end of Section 3 we also prove the following criterion for Zariski density, which will be subsequently used to discard Zariski closures.

PROPOSITION 2. *Let* $\rho : \pi_1(\mathcal{O}) \to PSL(n, \mathbb{R})$ *be an orbifold Hitchin representation such that*:

if $n = 2k$ *is even then* $\rho(\pi_1(\mathcal{O}))$ *is not conjugate to a subgroup of* $PSp(2k, \mathbb{R})$ *or,*
if $n = 2k + 1$ *is odd then* $\rho(\pi_1(\mathcal{O}))$ *is not conjugate to a subgroup of* $PSO(k, k + 1)$.

Then $\rho(H)$ *is Zariski dense in* $PSL(n, \mathbb{R})$ *for every finite index subgroup* H *of* $\pi_1(\mathcal{O})$.

In Section 4 we give a general construction to obtain a path of representations as in Theorem 1. This is based on bending the fundamental group $\pi_1(\mathcal{O})$ of a hyperbolic 2-dimensional orbifold along a simple closed curve in $\mathcal{O}$ with infinite order as an element of $\pi_1(\mathcal{O})$. Theorem 1 then follows from applying the results in Section 2 to a suitable representation of the fundamental group of the orbifold $\mathcal{O}_{3,3,3,3}$ whose underlying topological space is S^2 and has four cone points of order 3. This final step is covered in Section 5.

Remark. During the finalisation of this project, Long and Thistlethwaite used bending to construct thin surface groups in $SL(n, \mathbb{Z})$ for every odd n [**LT20**], the even case remains open.

2. *Hitchin representations*

In this section we give a short overview of surface and orbifold Hitchin representations.

Recall that a subgroup $H < GL(n, \mathbb{R})$ is *irreducible* if the only invariant subspaces for the action of H on $\mathbb{R}^n$ are $\{0\}$ and $\mathbb{R}^n$. A representation $\rho : \Gamma \to GL(n, \mathbb{R})$ is said to be *irreducible* if the image subgroup $\rho(\Gamma)$ is irreducible, and it is *strongly irreducible* if the restriction of ρ to every finite index subgroup is irreducible. These characteristics are defined similarly for projective representations $\rho : \Gamma \to PGL(n, \mathbb{R})$

2·1. *Spaces of representations*

Let G be a Lie group and let Γ be a group with a finite presentation $\langle \alpha_1, \ldots, \alpha_k \mid r_1, \ldots, r_m \rangle$. Then every relator r_i defines a map $R_i : G^k \to G$. If we let $\mathrm{Hom}(\Gamma, G) = \cap_{i=1}^m R_i^{-1}(Id)$, then the map $\phi \mapsto (\phi(\alpha_1), \ldots, \phi(\alpha_k))$ is a bijection between the set of all group homomorphisms from Γ to G and $\mathrm{Hom}(\Gamma, G)$. We will regard $\mathrm{Hom}(\Gamma, G)$ as having the subspace topology from G^k.

Let $\mathrm{Hom}^+(\Gamma, G)$ be the subset of representations in $\mathrm{Hom}(\Gamma, G)$ which decompose as a direct sum of irreducible representations and let $\mathrm{Rep}^+(\Gamma, G) = \mathrm{Hom}^+(\Gamma, G)/G$ be the

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quotient space by the conjugation action. With the quotient topology $\mathrm{Rep}^+(\Gamma, G)$ has the structure of an algebraic variety [**BGPGW07**, section 5·2].

In the following we will frequently use the representation

$$\tilde{\omega}_n : SL(2, \mathbb{R}) \longrightarrow SL(n, \mathbb{R}) \tag{1}$$

given by the action of $SL(2, \mathbb{R})$ on the vector space $\mathcal{P}$ of homogeneous polynomials in 2 variables of degree $n-1$. It is known that the representation $\tilde{\omega}_n$ is absolutely irreducible and is, up to conjugation, the unique irreducible representation from $SL(2, \mathbb{R})$ into $SL(n, \mathbb{R})$. This representation induces a *projective representation* $\omega_n : PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ which is also irreducible and unique up to conjugation.

2·2. *Hitchin representations of surface groups*

Let S be a closed surface of genus $g > 1$. In 1988 Goldman proved that $\mathrm{Rep}^+(\pi_1(S), PSL(2, \mathbb{R}))$ has $4g-3$ connected components, two of which are diffeomorphic to $\mathbb{R}^{6g-6}$ and called these *Teichmüller spaces* [**Gol88**, theorem A] [**Hit92**, theorem 10·2]. The two Teichmüller spaces $\mathcal{T}^{\pm}(S)$ are precisely the sets of conjugacy classes by $PSL(2, \mathbb{R})$ of *Fuchsian representations*, which are discrete and faithful representations $\rho : \pi_1(S) \to PSL(2, \mathbb{R}) \equiv \mathrm{Isom}^+(\mathbb{H}^2)$.

Definition 3. For $n > 2$ a representation $r : \pi_1(S) \to PSL(n, \mathbb{R})$ is called *Fuchsian* if it can be decomposed as $r = \omega_n \circ r_0$ where $r_0 : \pi_1(S) \to PSL(2, \mathbb{R})$ is discrete and faithful, and $\omega_n : PSL(2, \mathbb{R}) \to PSL(n, \mathbb{R})$ is the unique irreducible representation introduced in Section 2·1.

Definition 4. The *Fuchsian locus* is the set of all $PSL(n, \mathbb{R})$ conjugacy classes of Fuchsian representations, namely the set $\omega_n(\mathcal{T}^{\pm}(S))$.

The space $\mathrm{Rep}^+(\pi_1(S), PSL(n, \mathbb{R}))$ has three topological connected components if n is odd and 6 if n is even [**Hit92**, theorem 10·2]. The Fuchsian locus is contained in one component in the odd case and in two components in the even case. Each of these distinguished components, called *Hitchin components*, is diffeomorphic to $\mathbb{R}^{(1-n^2)(1-g)}$. When $n > 2$ is even, both Hitchin components are related by an inner automorphism of $PSL(n, \mathbb{R})$. In the odd case, where there is only one component, we will denote the Hitchin component by $\mathrm{Hit}(\pi_1(S), PSL(n, \mathbb{R}))$.

Definition 5. Let S be a closed surface of genus greater than one. A representation $r : \pi_1(S) \to PSL(n, \mathbb{R})$ is a *surface Hitchin representation* if its $PSL(n, \mathbb{R})$-conjugacy class belongs to a Hitchin component of $\mathrm{Rep}^+(\pi_1(S), PSL(n, \mathbb{R}))$.

In [**Lab06**], Labourie introduces *Anosov representations* and proves that surface Hitchin representations are B-Anosov where B is any Borel subgroup of $PSL(n, \mathbb{R})$. This gives surface Hitchin representations essential algebraic properties, out of which we will use Theorem 7 below.

Definition 6 ([**BCL20**, section 2·2]). A matrix $A \in SL(n, \mathbb{R})$ is *purely loxodromic* if it is diagonalizable over $\mathbb{R}$ with eigenvalues of distinct modulus. If $A \in PSL(n, \mathbb{R})$ then we say A is *purely loxodromic* if any lift of A to an element of $SL(n, \mathbb{R})$ is purely loxodromic.

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THEOREM 7 ([**Lab06**, theorem 1·5, lemma 10·1]). *A surface Hitchin representation $r:\ \pi_1(S) \to PSL(n,\mathbb{R})$ is discrete, faithful and strongly irreducible. Moreover, the image of every non-trivial element of $\pi_1(S)$ under r is purely loxodromic.*

2·3. *Hitchin representations of orbifold groups*

Let $\mathcal{O}$ be a 2-dimensional closed orbifold of negative orbifold Euler characteristic $\chi(\mathcal{O})$ and let $\pi_1(\mathcal{O})$ be its orbifold fundamental group. In [**Thu78**] Thurston proves there is a connected component of the representation space $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(2,\mathbb{R}))$ that parametrizes hyperbolic structures on $\mathcal{O}$. This component is called the *Teichmüller space* of the orbifold $\mathcal{O}$, we will denote it by $\mathcal{T}(\mathcal{O})$. As with surfaces, the orbifold Teichmüller space consists of conjugacy classes of discrete and faithful representations of $\pi_1(\mathcal{O})$ into $PGL(2,\mathbb{R}) \equiv \mathrm{Isom}(\mathbb{H}^2)$, which we will call *Fuchsian representations* too. More recently, Alessandrini, Lee, and Schaffhauser used the irreducible representation ω_n to define the *Hitchin component* $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n,\mathbb{R}))$ of $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n,\mathbb{R}))$ as the unique connected component in this representation space which contains the connected Fuchsian locus $\omega_n(\mathcal{T}(\mathcal{O}))$ [**ALS19**, definition 2·3] and prove $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n,\mathbb{R}))$ is homeomorphic to an open ball [**ALS19**, theorem 1·2].

Definition 8 ([**ALS19**, definition 2·4]). Let $\mathcal{O}$ be a 2-dimensional connected closed orbifold with negative orbifold Euler characteristic. A representation $r:\ \pi_1(\mathcal{O}) \to PGL(n,\mathbb{R})$ is an *orbifold Hitchin representation* if its $PGL(n,\mathbb{R})$-conjugacy class belongs to the Hitchin component $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n,\mathbb{R}))$ of $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n,\mathbb{R}))$.

The definition of Anosov representations has been generalized by Guichard and Wienhard [**GW12**, definition 2·10] to include representations of word hyperbolic groups into semisimple Lie groups. With this more general definition, and just as their surface counterparts, orbifold Hitchin representations are also B-Anosov where B is a Borel subgroup of $PGL(n,\mathbb{R})$ [**ALS19**, proposition 2·26] and thus share some strong algebraic properties.

THEOREM 9 ([**ALS19**, theorem 1·1]). *An orbifold Hitchin representation $r:\ \pi_1(\mathcal{O}) \to PGL(n,\mathbb{R})$ is discrete, faithful and strongly irreducible. Moreover, the image of every infinite order element of $\pi_1(\mathcal{O})$ under r is purely loxodromic.*

3. *Zariski dense Hitchin representations*

In this section we focus on Zariski density of Hitchin representations and prove Corollary 15 which gives a criterion to determine when the image of a finite index subgroup of an orbifold group under a Hitchin representation is Zariski dense.

3·1. *Zariski closures of Hitchin representations*

Let G be an algebraic matrix Lie group, then G has both its standard topology as a subset of some $\mathbb{R}^N$ and the Zariski topology. If X is a subset of G then its *Zariski closure* is the closure of X in G with respect to the Zariski topology. We say a subgroup $H < G$ is *Zariski dense* in G if its Zariski closure equals G. A representation $r:\ \Gamma \to G$ is *Zariski dense* if $r(\Gamma)$ is Zariski dense in G.

The image of the irreducible representation $\omega_n :\ PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ is contained, if n is even, in a conjugate of $PSp(n,\mathbb{R})$, which is the projectivisation of the symplectic group $Sp(n,\mathbb{R})$. If $n = 2k+1$ is odd, the image of ω_n is contained in a conjugate of the orthogonal

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group $SO(k, k+1) = PSO(k, k+1)$. This implies that the images of Fuchsian representations are contained in (a conjugate of) $PSp(n, \mathbb{R})$ or in $SO(k, k+1)$ and, in particular, they are not Zariski dense. More generally, for surface Hitchin representations Guichard [**Gui**] has announced a classification of Zariski closures of their lifts. An alternative proof of this result has been given recently by Sambarino [**Sam20**, corallary 1·5] The version of this result we cite here comes from theorem 11·7 in [**BCLS15**].

THEOREM 10 ([**Gui, Sam20**]). *If $r : \pi_1(S) \to SL(n, \mathbb{R})$ is the lift of a surface Hitchin representation and H is the Zariski closure of $r(\pi_1(S))$, then*:

if $n = 2k$ is even, H is conjugate to either $\omega_n(SL(2, \mathbb{R}))$, $Sp(2k, \mathbb{R})$ or $SL(2k, \mathbb{R})$;
if $n = 2k+1$ is odd and $n \neq 7$, then H is conjugate to either $\omega_n(SL(2, \mathbb{R}))$, $SO(k, k+1)$ or $SL(2k+1, \mathbb{R})$;
if $n = 7$, then H is conjugate to either $\omega_7(SL(2, \mathbb{R}))$, G_2, $SO(3,4)$ or $SL(7, \mathbb{R})$.

3·2. *A criterion for Zariski density*

Here we prove Proposition 2 which gives us a criterion to find Zariski dense Hitchin representations.

LEMMA 11. *Let $\rho : \pi_1(\mathcal{O}) \to PSL(n, \mathbb{R})$ with n even be an orbifold Hitchin representation. Then for every $[\alpha] \in \pi_1(\mathcal{O})$ of infinite order there is a lift $A \in SL(n, \mathbb{R})$ of $\rho([\alpha])$ which has n positive distinct eigenvalues.*

Proof. First consider a Fuchsian representation $\sigma : \pi_1(\mathcal{O}) \to PSL(2, \mathbb{R})$ and $[\alpha]$ an infinite order element of $\pi_1(\mathcal{O})$. Since $\mathcal{O}$ is a hyperbolic orbifold, $\sigma([\alpha])$ is conjugate to a hyperbolic element $\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \in PSL(2, \mathbb{R})$. We can lift this element to a matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in SL(2, \mathbb{R})$ with $\lambda > 0$. Let $\tilde{\omega}_n : SL(2, \mathbb{R}) \to SL(n, \mathbb{R})$ be the unique irreducible representation in (1), then $\tilde{\omega}_n \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \in SL(n, \mathbb{R})$ has n distinct positive eigenvalues $\lambda^{n-1}, \lambda^{n-3}, \ldots, \lambda^{-(n-3)}, \lambda^{-(n-1)}$ and is a lift of $\omega_n \circ \sigma([\alpha]) \in PSL(n, \mathbb{R})$.

Now consider a Hitchin representation $\rho : \pi_1(\mathcal{O}) \to PSL(n, \mathbb{R})$. Let ρ_t be a path of Hitchin representations such that ρ_0 is Fuchsian and $\rho_1 = \rho$. This induces a path $\rho_t([\alpha]) \subset PSL(n, \mathbb{R})$. By the previous argument we may lift $\rho_t([\alpha])$ to a path $\tilde{A}_t \in SL(n, \mathbb{R})$ such that $\tilde{A}_0$ has n distinct positive eigenvalues. Since each eigenvalue of $\tilde{A}_t$ varies continuously and $\det \tilde{A}_t \neq 0$, all eigenvalues of $\tilde{A}_t$ are positive. Moreover, by Theorem 9 the absolute values of the eigenvalues of $\rho_t([\alpha])$ are distinct. This in turn implies all the eigenvalues of $\tilde{A}_t$ are distinct. Therefore $\tilde{A}_1 \in SL(n, \mathbb{R})$ is a lift of $\rho([\alpha])$ with n positive distinct eigenvalues.

To prove our criterion for Zariski density (Propositions 13 and 14) we will make use of the following theorem by Culver.

THEOREM 12 ([**Cul66**, theorem 2]). *Let C be a real square matrix. Then the equation $C = \exp(X)$ has a unique real solution X if and only if all the eigenvalues of C are positive real and no elementary divisor (Jordan block) of C belonging to any eigenvalue appears more than once.*

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PROPOSITION 13. *Let $\rho : \pi_1(\mathcal{O}) \to PSL(n, \mathbb{R})$ with n even be an orbifold Hitchin representation so that $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSp(n, \mathbb{R})$. If S is a surface finitely covering $\mathcal{O}$ then $\rho(\pi_1(S))$ is Zariski dense.*

Proof. Let S be a surface finitely covering $\mathcal{O}$ and suppose that $\rho(\pi_1(S))$ is conjugate to a subgroup of $PSp(n, \mathbb{R})$. Then there exists an alternating form $\Omega \in SL(n, \mathbb{R})$ such that $Sp(\Omega) = \{g \in SL(n, \mathbb{R}) \mid g^T \Omega g = \Omega\}$ and $\rho(\pi_1(S)) \subset PSp(\Omega) = Sp(\Omega)/\pm I$.

Let $[\alpha] \in \pi_1(\mathcal{O})$ be an infinite order element. By Lemma 11 we can lift $\rho([\alpha]) \in PSL(n, \mathbb{R})$ to a matrix $A \in SL(n, \mathbb{R})$ with n positive distinct eigenvalues. Since $\pi_1(S)$ has finite index in $\pi_1(\mathcal{O})$ there exists a $k \in \mathbb{N}$ such that $\rho([\alpha])^k \in \rho(\pi_1(S))$. Then A^k is a lift of $\rho([\alpha])^k$ and $A^k \in Sp(\Omega)$. Given that A has n positive distinct eigenvalues, by Theorem 12 there is a unique $X \in M_{n\times n}(\mathbb{R})$ such that $\exp(X) = A$. Then using that $\exp(kX) = A^k$ preserves Ω we get that

$$\begin{aligned}\exp(kX)^T \Omega \exp(kX) = \Omega &\implies \Omega^{-1} \exp(kX)^T \Omega = \exp(kX)^{-1} \\ &\implies \exp(\Omega^{-1}(kX)^T \Omega) = \Omega^{-1} \exp(kX)^T \Omega = \exp(-kX).\end{aligned}$$

Applying Theorem 12 now to $\Omega^{-1} \exp(kX)^T \Omega$ we obtain that

$$\begin{aligned}\Omega^{-1}(kX)^T \Omega = -kX &\Rightarrow -\Omega(kX)^T \Omega = -kX \\ &\Rightarrow \Omega(kX)^T \Omega = kX.\end{aligned}$$

This implies that $kX \in \mathfrak{sp}(\Omega)$ and thus $A = \exp(X) \in Sp(\Omega)$. Given that A is a lift of $\rho([\alpha])$, we have that $\rho([\alpha]) \in PSp(\Omega)$. Since $\pi_1(\mathcal{O})$ is generated by its infinite order elements we get that $\rho(\pi_1(\mathcal{O})) \subset PSp(\Omega)$, a contradiction. So it cannot be that $\rho(\pi_1(S))$ is conjugate to a subgroup of $PSp(n, \mathbb{R})$. In particular, if r is a lift of the Hitchin surface representation $\rho|_{\pi_1(S)}$ then the Zariski closure of $r(\pi_1(S))$ cannot be conjugate to a subgroup of $Sp(n, \mathbb{R})$. By Theorem 10 it must be that the Zariski closure of $r(\pi_1(S))$ is $SL(n, \mathbb{R})$. Therefore the Zariski closure of $\rho(\pi_1(S))$ is $PSL(n, \mathbb{R})$.

In the case when $n = 2k + 1$ is odd, by Theorem 10 the Zariski closure of $\rho(\pi_1(S))$ where ρ is a surface Hitchin representation is either conjugate to a subgroup of $SO(k, k + 1)$ or equals $SL(n, \mathbb{R})$. By assuming there exists a symmetric bilinear form J such that $\rho(\pi_1(S)) \subset SO(J)$ we have an analogous proof to that of Proposition 13 to get a criterion for Zariski density of surface Hitchin representations in the odd case.

PROPOSITION 14. *Let $\rho : \pi_1(\mathcal{O}) \to SL(n, \mathbb{R})$ with n odd be an orbifold Hitchin representation such that there is no real quadratic form J for which $\rho(\pi_1(\mathcal{O})) \subset SO(J)$. If S is a surface finitely covering $\mathcal{O}$ then $\rho(\pi_1(S))$ is Zariski dense.*

Given that any finite index subgroup of $\pi_1(\mathcal{O})$ contains a surface subgroup which has finite index in $\pi_1(\mathcal{O})$ we obtain the following result.

PROPOSITION 15. *Let $\rho : \pi_1(\mathcal{O}) \to PSL(n, \mathbb{R})$ be an orbifold Hitchin representation such that*:

if $n = 2k$ is even then $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSp(2k, \mathbb{R})$ or,
if $n = 2k + 1$ is odd then $\rho(\pi_1(\mathcal{O}))$ is not conjugate to a subgroup of $PSO(k, k + 1)$.

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Then for every finite index subgroup H of $\pi_1(\mathcal{O})$ the image $\rho(H)$ is Zariski dense in $PSL(n,\mathbb{R})$.

4. *Bending representations of orbifold groups*

Theorem 19 in this section gives a general construction of a path ρ_t of Zariski dense Hitchin surface representations into $SL(n,\mathbb{R})$ for odd n. By requiring that the initial representation ρ_0 has image inside $SL(n,\mathbb{Q})$ we obtain Corollary 20, in which every representation ρ_t with $t \in \mathbb{Q}$ also has image in $SL(n,\mathbb{Q})$.

4·1. *Bending representations*

Let $\mathcal{O}$ be a 2-dimensional orientable connected closed orbifold of negative orbifold Euler characteristic and $\mathcal{O}_L$, $\mathcal{O}_R$ be open connected suborbifolds with connected intersection $\mathcal{O}_L \cap \mathcal{O}_R$. Given a representation $\rho : \pi_1(\mathcal{O}) \to G$ there is a standard way of "bending" ρ by an element δ of the centraliser in G of $\rho(\pi_1(\mathcal{O}_L \cap \mathcal{O}_R))$ to obtain a representation $\rho_\delta : \pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\pi_1(\mathcal{O}_L\cap\mathcal{O}_R)} \pi_1(\mathcal{O}_R) \to G$ so that $\rho_\delta(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta\rho(\pi_1(\mathcal{O}_R))\delta^{-1}\rangle$ (see for example [**Gol87**, section 5].

From now onwards we will consider the case where there is a simple closed curve $\gamma \subset \mathcal{O}$, not parallel to a cone point, that divides $\mathcal{O}$ into two orbifolds $\mathcal{O}_L$ and $\mathcal{O}_R$ which share γ as their common boundary, so that $\pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R)$.

PROPOSITION 16. *Let $\rho : \pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R) \to SL(n,\mathbb{Q})$ be a representation for which $\rho([\gamma])$ has n distinct positive eigenvalues. Then there exists a path of representations $\rho_t : \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ with $t \geq 0$ such that*:

(i) *$\rho_0 = \rho$;*

(ii) *$\rho_t(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta_t\rho(\pi_1(\mathcal{O}_R))\delta_t^{-1}\rangle$ for some $\delta_t \in SL(n,\mathbb{R})$ which commutes with $\rho([\gamma])$; and*

(iii) *ρ_t has image in $SL(n,\mathbb{Q})$ for every $t \in \mathbb{Q}$.*

Proof. The matrix $\rho([\gamma])$ is conjugate to a diagonal matrix D with entries $\lambda_1, \ldots, \lambda_n > 0$ along its diagonal. Now for every $t > 0$ define

$$\delta_t = (t\rho([\gamma]) + I)\det(t\rho([\gamma]) + I)^{-\frac{1}{n}}. \tag{2}$$

Notice that $\det(t\rho([\gamma]) + I) = \det(tD + I) = \Pi_{k=1}^{n}(t\lambda_i + 1) > 0$, so $t\rho([\gamma]) + I$ is invertible for all t. Then each δ_t is in $SL(n,\mathbb{R})$ and we can check that δ_t commutes with $\rho([\gamma])$. Since ρ is a rational representation, whenever $t \in \mathbb{Q}$ the matrix $t\rho([\gamma]) + I$ has rational entries and non-zero determinant.

Let $\rho_t : \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ be the representation such that $\rho_t(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta_t\rho(\pi_1(\mathcal{O}_R))\delta_t^{-1}\rangle$. Notice that $\rho_0 = \rho$ and that for every $t \in \mathbb{Q}$ the representation ρ_t has image in $SL(n,\mathbb{Q})$.

4·2. *Discarding Zariski closures*

For the rest of Section 4 we focus on the case where $n = 2k + 1$ is odd. Recall that in this case $SL(n,\mathbb{R}) \equiv PSL(n,\mathbb{R})$.

LEMMA 17. *Let $\rho : \Gamma \to SL(n,\mathbb{R})$ be an irreducible representation and suppose there is a quadratic form J such that $\rho(\Gamma) \subset SO(J)$. Then J is unique up to scaling.*

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Proof. Suppose $\rho(\Gamma) < SO(J_1) \cap SO(J_2)$. Then for any $\rho(\gamma) \in \rho(\Gamma)$ we have that

$$J_1^{-1}\rho(\gamma)J_1 = \rho(\gamma)^{-T} = J_2^{-1}\rho(\gamma)J_2,$$

which implies that $\rho(\gamma)J_1J_2^{-1} = J_1J_2^{-1}\rho(\gamma)$. Since n is odd, $J_1J_2^{-1}$ has a real eigenvalue λ. Then $\mathrm{Ker}(J_1J_2^{-1} - \lambda I)$ is a non-zero invariant subspace for the irreducible representation ρ, which implies $J_1 = \lambda J_2$.

PROPOSITION 18. *Let $\rho : \pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R) \to SL(n,\mathbb{R})$ be a representation in which the restrictions $\rho|_{\pi_1(\mathcal{O}_L)}$ and $\rho|_{\pi_1(\mathcal{O}_R)}$ are irreducible and $\rho([\gamma])$ has n positive distinct eigenvalues. Suppose there is a quadratic form J such that $\rho(\pi_1(\mathcal{O})) \subset SO(J)$. Then there exists a path of representations $\rho_t : \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ such that:*

(i) *$\rho_0 = \rho$ and*

(ii) *for each $t > 0$ there is no quadratic form $\tilde{J}$ such that $\rho_t(\pi_1(\mathcal{O})) \subset SO(\tilde{J})$.*

Proof. By Proposition 16 there are $\delta_t \in SL(n,\mathbb{R})$ that commute with $\rho([\gamma])$, with which we can construct a path of representations $\rho_t : \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ such that $\rho_0 = \rho$ and $\rho_t(\pi_1(\mathcal{O})) = \langle \rho(\pi_1(\mathcal{O}_L)), \delta_t\rho(\pi_1(\mathcal{O}_R))\delta_t^{-1} \rangle$.

Now fix $t > 0$. Suppose there exists a quadratic form $\tilde{J}$ such that $\rho_t(\pi_1(\mathcal{O})) \subset SO(\tilde{J})$. Since $\rho(\pi_1(\mathcal{O})) \subset SO(J)$, in particular $\rho_t(\pi_1(\mathcal{O}_L)) = \rho_0(\pi_1(\mathcal{O}_L)) \subset SO(J) \cap SO(\tilde{J})$. The restriction $\rho_t|_{\pi_1(\mathcal{O}_L)}$ is irreducible, so by Lemma 17 J is a real multiple of $\tilde{J}$. Similarly, by construction $\rho_t(\pi_1(\mathcal{O}_R)) \subset SO(\delta_t J \delta_t^T) \cap SO(\tilde{J})$ and $\rho_t|_{\pi_1(\mathcal{O}_R)}$ is irreducible too. Thus $\delta_t J \delta_t^T$ is also a multiple of $\tilde{J}$. This implies there is a $\lambda \in \mathbb{R}$ such that $\lambda J = \delta_t J \delta_t^T$ and then $\lambda^n = \det(\delta_t)^2 = 1$. Since n is odd it must be that $\lambda = 1$ and we obtain $\delta_t \in SO(J)$. Given that

$$(t\rho([\gamma]) + I)J(t\rho([\gamma])^T + I) \;=\; t^2 J \;+\; tJ(\rho([\gamma])^T)^{-1} \;+\; tJ\rho([\gamma])^T \;+\; J,$$

having $J = \delta_t J \delta_t^T$ would imply that $\mu I = \rho([\gamma])^{-1} + \rho([\gamma])$ for some $\mu \in \mathbb{R}$. Recall that $\rho([\gamma])$ is conjugate to a diagonal matrix D whose eigenvalues are all distinct. If $\mu I = \rho([\gamma])^{-1} + \rho([\gamma])$ then by conjugating we would obtain that $\mu I = D^{-1} + D$, which is not the case given that $n > 2$.

4·3. *Representations of surface groups*

Recall we are assuming that $\mathcal{O}$ is a 2-dimensional orientable connected closed orbifold of negative orbifold Euler characteristic. Such orbifolds are always finitely covered by a surface S of genus greater than one, so $\pi_1(S)$ is a finite index subgroup of $\pi_1(\mathcal{O})$. Given a representation $\rho : \pi_1(\mathcal{O}) \to G$ we will denote the restriction of ρ to $\pi_1(S)$ by ρ^S.

THEOREM 19. *Suppose $\pi_1(\mathcal{O}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R)$ with $[\gamma]$ an infinite order element. Let $\rho : \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ be an orbifold Fuchsian representation such that the restrictions $\rho|_{\pi_1(\mathcal{O}_L)}$ and $\rho|_{\pi_1(\mathcal{O}_R)}$ are irreducible. If S is a surface finitely covering $\mathcal{O}$ then there exists a path of representations $\rho_t^S : \pi_1(S) \to SL(n,\mathbb{R})$ such that $\rho_0^S = \rho^S$ and ρ_t^S is a Zariski dense surface Hitchin representation for each $t > 0$.*

Proof. Since $\rho : \pi_1(\mathcal{O}) \to SL(n,\mathbb{R})$ is an orbifold Hitchin representation with odd $n = 2k+1$ and $[\gamma]$ has infinite order, then $\rho([\gamma])$ has n positive distinct real eigenvalues. Moreover, since ρ is Fuchsian its image is contained in a conjugate of $SO(k, k+1)$. Using

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Proposition 18 we obtain a path of representations $\rho_t : \pi_1(\mathcal{O}) \to SL(n, \mathbb{R})$ such that $\rho_0 = \rho$ and for each $t > 0$ there is no real quadratic form J such that $\rho_t(\pi_1(\mathcal{O})) \subset SO(J)$. By Proposition 14 each $\rho_t(\pi_1(S))$ is Zariski dense in $SL(n, \mathbb{R})$.

Now consider the continuous path $[\rho_t] \in \mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ for $t \geq 0$. Its image is connected so all $PGL(n, \mathbb{R})$-conjugacy classes $[\rho_t]$ are contained in the same connected component of $\mathrm{Rep}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$. Because the representation $\rho_0 = \rho$ is Fuchsian, $[\rho_0]$ is in the Hitchin component $\mathrm{Hit}(\pi_1(\mathcal{O}), PGL(n, \mathbb{R}))$ and so is every $[\rho_t]$. Thus, by Theorem 9, each ρ_t is discrete, faithful and strongly irreducible. Since $\pi_1(S)$ has finite index in $\pi_1(\mathcal{O})$, each restriction $\rho_t^S : \pi_1(S) \to SL(n, \mathbb{R})$ is irreducible. In particular ρ_0^S is a surface Fuchsian representation. Then $[\rho_t^S]$ is a continuous path in $\mathrm{Rep}^+(\pi_1(S), SL(n, \mathbb{R}))$ with $[\rho_0^S] \in \mathrm{Hit}(\pi_1(S), SL(n, \mathbb{R}))$. Since the Hitchin component is path connected $[\rho_t^S] \in \mathrm{Hit}(\pi_1(S), SL(n, \mathbb{R}))$ for all $t \geq 0$.

To finish this section notice that the construction of the path of Zariski dense representations in the previous theorem is based on Proposition 16, so we may add the assumption of $\rho(\pi_1(\mathcal{O})) \subset SL(n, \mathbb{Q})$ to obtain that the image of every ρ_t is in $SL(n, \mathbb{Q})$ for every $t \in \mathbb{Q}$.

COROLLARY 20. *Let $\rho : \pi_1(\mathcal{O}) \to PSL(n, \mathbb{Q})$ be a representation satisfying the assumptions of Theorem 19. If S is a surface finitely covering $\mathcal{O}$ then there exists a path $\rho_t^S : \pi_1(S) \to SL(n, \mathbb{R})$ of Hitchin representations such that $\rho_0^S = \rho^S$, ρ_t^S is Zariski dense for each $t > 0$ and ρ_t^S has image in $SL(n, \mathbb{Q})$ for every $t \in \mathbb{Q}$.*

5. *Representations of* $\pi_1(\mathcal{O}_{3,3,3,3})$

In this section we look at the orbifold $\mathcal{O}_{3,3,3,3}$ and find a Fuchsian representation $\rho : \pi_1(\mathcal{O}_{3,3,3,3}) \to SL(n, \mathbb{Z})$ satisfying the assumptions of Corollary 20.

5·1. *The orbifold* $\mathcal{O}_{3,3,3,3}$

In what follows we focus on the triangle group $\Delta(3, 4, 4) \subset PSL(2, \mathbb{R})$. If we let T be the hyperbolic triangle with angles $\{\pi/3, \pi/4, \pi/4\}$, then the generators of $\Delta(3, 4, 4)$ are the rotations x and y by $2\pi/3$ and $\pi/2$ around the corresponding vertices of T. This group has presentation

$$\Delta(3, 4, 4) = \langle x, y \mid x^3 = y^4 = (xy)^4 = 1\rangle. \tag{3}$$

The fundamental domain for the action of $\Delta(3, 4, 4)$ on $\mathbb{H}^2$ is a quadrilateral with angles $\{\pi/2, \pi/3, \pi/2, \pi/3\}$. The quotient $\mathbb{H}^2/\Delta(3, 4, 4)$ is homeomorphic to the orbifold $S^2(3, 4, 4)$ whose underlying topological space is S^2 and has three cone points of orders 3, 4 and 4 (Figure 1). This defines, up to conjugation, an isomorphism $\pi_1(S^2(3, 4, 4)) \to \Delta(3, 4, 4) \subset PSL(2, \mathbb{R})$.

Let $\theta_1 = x$ and $\theta_i = y\theta_{i-1}y^{-1}$ for $i = 2, 3, 4$, then $\langle\theta_1, \ldots, \theta_4\rangle$ the quotient of $\mathbb{H}^2$ by the action of $\langle\theta_1, \ldots, \theta_4\rangle$ is homeomorphic to the orbifold $\mathcal{O}_{3,3,3,3}$ with underlying topological space S^2 and 4 cone points of order 3. By construction, we obtain that $\mathcal{O}_{3,3,3,3}$ is an index four orbifold covering of $S^2(3, 4, 4)$. If $\gamma_1, \ldots, \gamma_4$ are loops around the cone points of $\mathcal{O}_{3,3,3,3}$, then the orbifold fundamental group has the presentation

$$\pi_1(\mathcal{O}_{3,3,3,3}) = \langle\gamma_1, \ldots, \gamma_4 \mid \gamma_1^3 = \ldots = \gamma_4^3 = \gamma_1\gamma_2\gamma_3\gamma_4 = 1\rangle.$$

Identifying each γ_i with the rotation θ_i gives an isomorphism $\pi_1(\mathcal{O}_{3,3,3,3}) \cong \langle\theta_1, \ldots, \theta_4\rangle$ which defines (up to conjugation) a discrete and faithful representation

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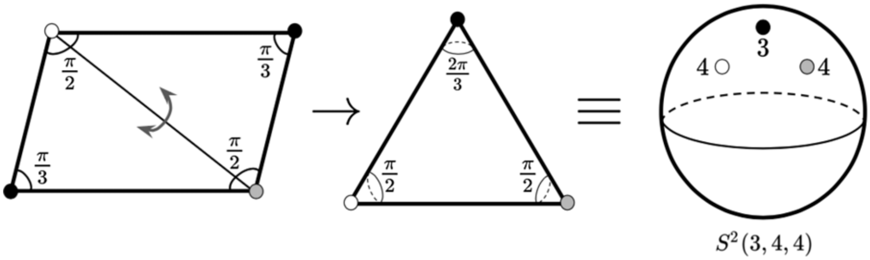

Fig. 1. Orbifold $S^2(3,4,4)$.

$$\sigma : \pi_1(\mathcal{O}_{3,3,3,3}) \longrightarrow \Delta(3,4,4) < PSL(2,\mathbb{R}). \tag{4}$$

It follows from the Borel density theorem that σ is a Zariski dense representation.

5·2. *Rational representations of $\pi_1(\mathcal{O}_{3,3,3,3})$*

We will now focus on the case $n = 2k+1$ and the representation $\omega_n \circ \sigma : \pi_1(\mathcal{O}_{3,3,3,3}) \to SL(n,\mathbb{R})$, where σ is the representation defined in (4) and $\omega_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R}) = SL(n,\mathbb{R})$ the irreducible representation introduced in Section 3·1. Since $\omega_n \circ \sigma$ is an orbifold Fuchsian representation, it is irreducible. The following result implies that we can conjugate $\omega_n \circ \sigma$ to obtain an integral representation

$$\rho : \pi_1(\mathcal{O}_{3,3,3,3}) \longrightarrow SL(n,\mathbb{Z}) < SL(n,\mathbb{R}). \tag{5}$$

PROPOSITION 21 ([**LT20**, theorem 2·1]). *Let $\omega_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ be the unique irreducible representation between these groups. Then for every odd n the restriction $\phi_n = \omega_n|_{\Delta(3,4,4)}$ is conjugate to a representation $\rho_n : \Delta(3,4,4) \to PSL(n,\mathbb{Z})$.*

Now let $\gamma \subset \mathcal{O}_{3,3,3,3}$ be a simple closed loop dividing $\mathcal{O}_{3,3,3,3}$ into two orbifolds $\mathcal{O}_L$ and $\mathcal{O}_R$ which share γ as their common boundary and have two cone points of order 3 each. Then $[\gamma] \in \pi_1(\mathcal{O}_{3,3,3,3})$ is an infinite order element and $\pi_1(\mathcal{O}_{3,3,3,3}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R)$.

PROPOSITION 22. *Let $\rho : \pi_1(\mathcal{O}_{3,3,3,3}) \simeq \pi_1(\mathcal{O}_L) *_{\langle[\gamma]\rangle} \pi_1(\mathcal{O}_R) \to PSL(n,\mathbb{Z})$ be the representation defined in (5). Then the restrictions of ρ to $\pi_1(\mathcal{O}_L)$ and $\pi_1(\mathcal{O}_R)$ are irreducible.*

Proof. To see that $\rho|_{\pi_1(\mathcal{O}_L)}$ is irreducible it suffices to see that the restriction of $\omega_n \circ \sigma$ to $\pi_1(\mathcal{O}_L)$ is irreducible. We have that $\sigma(\pi_1(\mathcal{O}_L))$ is Zariski dense in $PSL(2,\mathbb{R})$. To see that the representation $\omega_n : \sigma(\pi_1(\mathcal{O}_L)) \to PSL(n,\mathbb{R})$ is irreducible, it is enough to check that the Zariski closure of its image is irreducible. This holds since $\omega_n : PSL(2,\mathbb{R}) \to PSL(n,\mathbb{R})$ is an irreducible representation and a morphism of algebraic groups, so $\omega_n(PSL(2,\mathbb{R})) = \omega_n(\overline{\sigma(\pi_1(\mathcal{O}_L))}) \subseteq \overline{\omega_n \circ \sigma(\pi_1(\mathcal{O}_L))}$. That $\rho|_{\pi_1(\mathcal{O}_R)}$ follows from $\sigma(\pi_1(\mathcal{O}_L))$ being Zariski dense in $PSL(2,\mathbb{R})$.

Knowing that ρ is an integral orbifold Fuchsian representation, the previous proposition shows ρ satisfies the assumptions of Theorem 19. Thus we obtain the following application of Corollary 20.

THEOREM 23. *For every surface S finitely covering the orbifold $O_{3,3,3,3}$ and every odd $n > 1$ there exists a path of Hitchin representations $\rho_t : \pi_1(S) \to SL(n,\mathbb{R})$, so that:*

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(i) $\rho_0(\pi_1(S)) \subset SL(n, \mathbb{Z})$;

(ii) ρ_t *is Zariski dense for every* $t > 0$; *and*

(iii) $\rho_t(\pi_1(S)) \subset SL(n, \mathbb{Q})$ *for every* $t \in \mathbb{Q}$.

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REFERENCES

[ALS19] D. Alessandrini, G Lee and F Schaffhauser. Hitchin components for orbifolds. ArXiv:1811.05366 (2019).

[Bas80] H. Bass. Groups of integral representation type. *Pacific J. Math.* **86**(1) (1980), 15–51.

[BCL20] M. Bridgeman, R. Canary and F. Labourie. Simple length rigidity for Hitchin representations. *Adv. Math.* **360** 106901 (2020).

[BCLS15] Martin Bridgeman, Richard Canary, François Labourie and Andres Sambarino. The pressure metric for anosov representations. *Geom. Funct. Anal.* **25**(4) (2015), 1089–1179.

[BGPGW07] S. Bradlow, O. Garca-Prad, W. Goldman and A. Wienhard. Representations of surface groups: Background material for AIM workshop (2007), available online: https://www.mathi.uni-heidelberg.de/~wienhard/aim_intro_final.pdf.

[BGS10] J. Bourgain, A. Gamburd and P. Sarnak. Affine linear sieve, expanders, and sum-product. *Invent. Math.* **179**(3) (2010), 559–644.

[BL18] S. Ballas and D.D. Long. Constructing thin subgroups of SL($n+1, \mathbb{R}$) via bending. *Algebr. Geom. Topol.* **20** (4) (2020), 2071–2093.

[Cul66] W. Culver. On the existence and uniqueness of the real logarithm of a matrix. *Proc. Amer. Math. Soc.* **17**(5) (1966), 1146–1151.

[FR17] E. Fuchs and I. Rivin. Generic thinness in finitely generated subgroups of sl. *Internat. Math. Res. Notices* **2017**(17) (2017), 5385–5414.

[Fuc14] E. Fuchs. The ubiquity of thin groups. *Thin Groups and Superstrong Approximation* **61** (2014), 73–92.

[Gol87] W. Goldman. Geometric structures on manifolds and varieties of representations, the geometry of group representations. *Proc. Amer. Math. Soc. (Summer Conference 1987, Boulder, Colorado)* **74** (1987), 169–198.

[Gol88] W. Goldman. Topological components of spaces of representations. *Invent. Math.* **93**(3) (1988), 557–607.

[Gui] O. Guichard. Zariski closure of positive and maximal representations. *In preparation*.

[GW12] O. Guichard and A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.* **190**(2) (2012), 357–438.

[Hit92] N. Hitchin. Lie groups and teichmüller space. *Topology* **31**(3) (1992), 449–473.

[Lab06] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.* **165**(1) (2006), 51–114.

[LRT11] D.D. Long, A. Reid and M. Thistlethwaite. Zariski dense surface subgroups in SL($3, \mathbb{Z}$). *Geom. Topol.* **15**(1) (2011), 1–9.

[LT18] D. Long and M.B. Thistlethwaite. Zariski dense surface subgroups in SL($4, \mathbb{Z}$). *Experiment. Math.* **27**(1) (2018), 82–92.

[LT20] D.D. Long and M. Thistlethwaite. Zariski dense surface groups in SL($2k+1, \mathbb{Z}$). To appear in *Geom. Topol.* (2022).

[Sam20] A. Sambarino. Infinitesimal zariski closures of positive representations. ArXiv:2012.10276 (2020).

[Sar14] P. Sarnak. Notes on thin matrix groups. *Thin Groups and Superstrong Approximation* **61** (2014), 343–362.

[Thu78] W. Thurston. The Collected Works of William P. Thurston with Commentary: (the set) IV, The Geometry and Topology of Three-Manifolds: with a preface by Steven P. Kerckhoff (American Mathematical Association, Providence, Rhode Island, 2022).