THE DENSEST PACKING OF 9 CIRCLES IN A SQUARE

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Packing problems of this kind are obviously equivalent to the problems of placing $k$ (here 9) points in a unit square such that the minimum distance between any two of them be as large as possible. The solutions of these problems are known for $2 \leq k \leq 9$. The largest possible minimum distances $m_k$ are given in table 1, and the corresponding "best" configurations shown in figure 1.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{6} - \sqrt{2}$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>$\sqrt{2} / 2$</td>
</tr>
<tr>
<td>6</td>
<td>$\sqrt{13} / 6$</td>
</tr>
<tr>
<td>7</td>
<td>$2(2 - \sqrt{3})$</td>
</tr>
<tr>
<td>8</td>
<td>$(\sqrt{6} - \sqrt{2}) / 2$</td>
</tr>
<tr>
<td>9</td>
<td>1/2</td>
</tr>
</tbody>
</table>

Table 1
The cases $k = 2, 3, 4,$ and $5$ are solved easily. For $k = 6$ R.L. Graham obtained the solution recently. The case $k = 8$ is treated in a separate paper [1]. A proof for $k = 7$ has also been found by the author; although using essentially the same methods it is much more complicated. The case is interesting because the best configuration is not unique, one point being free to be placed anywhere in the shaded area.

For $k = 9$, which we shall solve here, the best configuration is easily guessed, and the conjecture indeed not difficult to prove. We have to show that, for any nine points $P_i (1 \leq i \leq 9)$ of a closed unit square,

$$\min_{1 \leq i < j \leq 9} d(P_i, P_j) \leq \frac{1}{2} = m_9$$

and that equality holds only for the conjectured configuration. ($d(P_i, P_j)$ denotes the distance between $P_i$ and $P_j$.)

Let $S$ be any set of nine points $P_i (1 \leq i \leq 9)$ of a closed unit square with
We shall show that there is just one such set; namely the conjectured one, for which in (1) obviously equality holds.

(1) The unit square may be covered by 9 closed squares $Q_{0i}$ ($1 \leq i \leq 9$) of side $s_0 = \frac{1}{3}$. Their diameter is $\frac{\sqrt{2}}{3} < \frac{1}{2}$, so by (1) in each of them there can be at most one point of $S$. Since there are as many points $P_i$ as squares $Q_{0i}$, in each of the squares there must lie exactly one point of $S$: $P_i \in Q_i$ ($1 \leq i \leq 9$).

(2) We shall now indicate a procedure by which the location of the points $P_i$ may be restricted to squares $Q_{1i} \subset Q_{0i}$ with side $s_1 < s_0$. Iterating the process, every $P_i$ is successively confined to squares $Q_{ni}$ of sides $s_n$ ($n = 0, 1, 2, \ldots$) ($s_0 > s_1 > s_2 > \ldots$). In every stage the square $Q_{ni}$ is in perspective to the unit square with respect to the conjectured position of $P_i$.

In every step the same method is applied. Every square $Q_{ni}$ can be reduced by its closest neighbour squares in the following way (see figure 2). Consider a rectangle, of side $s_n$ and diagonal $\frac{1}{2}$, which contains the neighbour square $Q_{nj}$ and

Figure 2
as much as possible of the square \( Q_{ni} \) to be reduced. Excluding the side which lies in \( Q_{ni} \), by (1) the rectangle can contain at most one point of \( S \). Since it contains \( P_i \in Q_{nj} \), the region in the rectangle (shaded) can be excluded from \( Q_{ni} \) as possible location of \( P_i \).

(3) By this method the four "corner" squares (i.e. the squares \( Q_{ni} \) which contain the vertices of the unit square) are reduced least, because they have only two closest neighbours. For sake of simplicity we shall also reduce the five other squares only as much as the corner squares, in order to obtain again nine squares \( Q_{n+1,i} \) of equal size. Thus it is sufficient to investigate the effect of the reducing process on a corner square. \( s_{n+1} \) is found by (see figure 3)

\[
\frac{s_n^2}{n} + \left( \frac{1}{2} + \frac{1}{2} \frac{s_n}{n} \right)^2 = \left( \frac{1}{2} \right)^2
\]

and therefore

\[
s_{n+1} = \frac{1}{2} \left( 1 + \frac{s_n}{n} - \sqrt{1 - 4\frac{s_n^2}{n}} \right).
\]

![Figure 3](https://www.cambridge.org/core)
(4) We shall now show that \( s_n \to 0 \) as \( n \to \infty \). As a consequence the points \( P_i \) must lie at the conjectured positions. In fact

\[
\frac{s_{n+1}}{s_n} = \frac{1}{2} \frac{(1 + s_n^2)^2 - (1 - 4s_n^2)}{1 + s_n + \sqrt{1 - 4s_n^2}}
\]

and

\[
\frac{s_{n+1}}{s_n} = \frac{2 + 5 s_n}{2 + 2s_n + 2\sqrt{1 - 4s_n^2}}.
\]

Since \( s_n < s_0 = \frac{1}{3} \), we have

\[
\frac{s_{n+1}}{s_n} < \frac{2 + 2s_n + 1}{2 + 2s_n + 2\sqrt{1 - 4/9}} = \frac{9 + 6s_n}{6 + 6s_n + 2\sqrt{5}}
\]

\[
< \frac{9 + 6s_n}{10 + 6s_n} = 1 - \frac{1}{10 + 6s_n} \leq 1 - \frac{1}{12}
\]

and the proof is complete.

REFERENCE


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