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FREE GROUPS, SYMMETRIC AND REDUCED PRODUCTS

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Abstract

We show that, for any Tychonoff space X with base point θ, the infinite symmetric product $SP^\infty X$ of X is a subspace of an abelian group $A(X)$ generated by X. (This clarifies the continuity of the multiplication in $SP^\infty X$.) Furthermore, $SP^\infty X$ is a retract of $A(X)$. Analogous results hold for reduced product spaces, with respect to non-abelian groups.

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1. Introduction

Dold and Thom (1958) introduced infinite symmetric products $SP^\infty X$ for all spaces X and defined a multiplication on $SP^\infty X$ which, they claim, is generally not continuous (Spanier (1959) has a simpler description of $SP^\infty X$, starting on p. 158; see footnote on p. 159). It turns out to be continuous for all Tychonoff spaces X.

James (1955) introduced reduced product spaces X_∞, which really are the non-abelian version of infinite symmetric products. For Tychonoff spaces X, all our results for $SP^\infty X$ have analogues for X_∞ with respect to non-abelian groups.

For the sake of clarity, we will now describe infinite symmetric products and reduced products in a manner which is more convenient to our work. To avoid extensive repetition, we will assume the terminology of Borges (1977).

Let X be a space with base point θ. Let $SP^0 X = \{\theta\}$. For $n \geqslant 1$, let UX^n be the quotient space of X^n which consists of unordered n-tuples $\langle x_1, \ldots, x_n \rangle$. Then let $SP^n X$ be the quotient space of UX^n which results from identifying each $\langle x_1, \ldots, x_n \rangle$ with the tuple obtained from $\langle x_1, \ldots, x_n \rangle$ by removing all θ's, except that $\langle \theta, \ldots, \theta \rangle = \langle \theta \rangle$. Let $\lambda_n \colon X^n \to SP^n X$ be the resultant quotient map. Finally, let $SP^\infty X = \sum_n SP^n X$. Define a multiplication on $SP^\infty X$ by

$$\langle x_1, \dots, x_n\rangle \circ \langle y_1, \dots, y_n\rangle = \langle x_1, \dots, x_n, y_1, \dots, y_n\rangle.$$

Clearly $\langle \theta \rangle$ is the unit element of this multiplication.

The description of the reduced product X_∞ is exactly the same as that of $SP^\infty X$, except that all tuples remain *ordered*. We let $X_n \subset X_\infty$ correspond to $SP^n X \subset SP^\infty X$.

2. Symmetric products

For any Tychonoff space X with base point θ, let βX be the Stone–Čech compactification of X with base point θ. We will make extensive use of free abelian topological groups, which were not discussed in Borges (1977). To avoid extensive repetition, we adopt the notation of Borges (1977) for free abelian topological groups, except that we replace x^{-1} by $-x$, $F_n(X)$ by $A_n(X)$, $(F(X), \mathscr{G})$ by $(A(X), \mathscr{G})$ and $(F(X), \mathscr{G}')$ by $(A(X), \mathscr{G}')$ (see p. 362 of Borges (1977)). We also let $\mathscr{G}_n = \mathscr{G} \mid A_n(X)$ and $\mathscr{G}'_n = \mathscr{G}' \mid A_n(X)$. Recall that $(A(X), \mathscr{G}')$ is a subspace of $(A(\beta X), \mathscr{G})$.

LEMMA 2.1. *Let X be a compact Hausdorff space with base point θ. Then each $SP^n X$ is a subspace of $(A_n(X), \mathscr{G}_n)$.*

PROOF. Let us consider the commutative diagram

$$\begin{array}{ccl} X^n & \xrightarrow{p'_n} & P_n = p_n(X^n) \subset A_n(X) \\ {\scriptstyle \lambda_n}\downarrow & \nearrow {\scriptstyle j_n} & \\ SP^n X & & \end{array}$$

with $p'_n = p_n \mid X^n$ and $j_n(\langle x_1, \dots, x_n\rangle) = x_1 + \dots + x_n$. Since λ_n and p'_n (recall that $p_n \colon (A_1(X))^n \to A_n(X)$) are closed continuous maps, we get that λ_n and p'_n are quotient maps. Since j_n is one-to-one, we then get that j_n is a homeomorphism.

THEOREM 2.2. *Let X be a Tychonoff space with base point θ. Then each $SP^n X$ is a subspace of $(A_n(X), \mathscr{G}'_n)$.*

PROOF. Let us consider the commutative diagrams

$$\begin{array}{ccl} (\beta X)^n & \xrightarrow{p'_n} & P_n \\ {\scriptstyle \lambda_n}\downarrow & \nearrow {\scriptstyle j_n} & \\ SP^n(\beta X) & & \end{array} \quad \text{and} \quad \begin{array}{ccl} X^n & \xrightarrow{p''_n} & P'_n = p''_n(X^n) \subset A_n(X) \\ {\scriptstyle \lambda_n}\downarrow & \nearrow {\scriptstyle j'_n} & \\ SP^n X & & \end{array}$$

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Note that $X^n = (p'_n)^{-1}(P'_n)$ (even though $X^n \neq p_n^{-1}(P'_n)$). Since p'_n is closed and continuous, P'_n is a subspace of P_n and X^n is a subspace of $(\beta X)^n$, it follows that p''_n is a quotient map. Of course, the second λ_n is a quotient map, by definition. Therefore, since j'_n is one-to-one, we get that j'_n is a homeomorphism.

LEMMA 2.3. *Let X be a compact Hausdorff space with base point θ. Then $SP^\infty X$ is a subspace of $(A(X), \mathscr{G})$.*

PROOF. Let us consider the commutative diagram

$$\begin{array}{ccc}
\bigvee_n SP^n X & \xrightarrow{\sigma} & \sum_n SP^n X = SP^\infty X \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \mu} \\
P & \xrightarrow{q'} & Q \\
\cap & & \cap \\
\bigvee_n A_n(\beta X) & \xrightarrow[q]{} & \sum_n A_n(X) = A(X),
\end{array}$$

where j, μ, σ, q' and q are onto maps, σ is the natural quotient map, $j | SP^n X = j_n$ (of Theorem 2.1), q is the natural quotient map (see, for example, Propositon 2.1(a) of Borges (1977)), and $\mu | SP^n X = j_n$.

Note that Q is closed in $A(X)$, because each $Q \cap A_n(X) = \mu(SP^n X)$, and $q^{-1}(Q) = P$. Therefore q' is also a quotient map. The map j is clearly closed, continuous and one-to-one; therefore j is a homeomorphism. Consequently μ is a homeomorphism.

THEOREM 2.4. *Let X be a Tychonoff space with base point θ. Then $SP^\infty X$ is a subspace of $(A(X), \mathscr{G}')$.*

PROOF. Let us consider the commutative diagrams

$$\begin{array}{ccc}
\bigvee_n SP^n \beta X & \xrightarrow{\sigma} & \sum_n SP^n \beta X \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \mu} \\
P & \xrightarrow{q'} & Q
\end{array}
\qquad
\begin{array}{ccc}
\bigvee_n SP^n X & \xrightarrow{\sigma} & \sum_n SP^n X \\
{\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle \mu'} \\
P' & \xrightarrow{q''} & Q' \\
\cap & & \cap \\
\bigvee_n A_n(X) & \longrightarrow & \sum_n A_n(X) = (A(X), \mathscr{G}')
\end{array}$$

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From the proof of Lemma 2.3 we get that, in the first diagram, q' is a quotient map. In the second diagram, σ is a quotient map (by definition) and j' is a homeomorphism because $j'|SP^n X = j'_n$ (in the proof of Theorem 2.2) is a homeomorphism, for each n.

We also have that Q' is closed in $(A(X), \mathscr{G}')$, because $Q' \cap A_n(X) = \mu(SP^n X)$ and $\mu(SP^n X) = \mu(SP^n \beta X) \cap A(X)$. (Clearly Theorem 2.3 of Borges (1977) remains valid for $(A(X), \mathscr{G}')$; in particular $(A(X), \mathscr{G}') = \sum_n A_n(X))$; furthermore, $(q')^{-1}(Q') = P'$. Therefore q'' is a quotient map. Consequently μ' is a homeomorphism.

Clearly the embedding of $SP^\infty X$ in $(A(X), \mathscr{G}')$ is such that the multiplication in $SP^\infty X$ is the restriction of the multiplication in $(A(X), \mathscr{G}')$. Therefore, we get the following.

COROLLARY 2.5. *If X is a Tychonoff space with base point θ, then the multiplication in $SP^\infty X$ is continuous.*

We complete this section by showing that, for the class of k_ω-spaces, $SP^\infty X$ is a retract of $(A(X), \mathscr{G})$ and, for the class of Tychonoff spaces, $SP^\infty X$ is a retract of $(A(X), \mathscr{G}')$.

THEOREM 2.6. *Let X be a Hausdorff k_ω-space with base point θ. Then $SP^\infty X$ is a retract of $(A(X), \mathscr{G})$.*

PROOF. For $a \in X$, let $\hat{a} = a$; for $a = -b$, with $b \in X$, let $\hat{a} = b$. Define $r\colon A(X) \to SP^\infty X$ by $r(k_1 a_1 + \ldots + k_j a_j) = k_1 \hat{a}_1 + \ldots + k_j \hat{a}_j$.

Clearly r is a retraction. To show that r is continuous, let us consider the following commutative diagram

$$\begin{array}{ccc} \bigvee_n (A_1(X))^n & \xrightarrow{\ p\ } & \sum_n A_n(X) = (A(X), \mathscr{G}) \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ \bigvee_n X^n & \xrightarrow[\lambda]{} & SP^\infty X \end{array}$$

where $\gamma(a_1, \ldots, a_n) = (\hat{a}_1, \ldots, \hat{a}_n)$ and $\lambda | X^n = \lambda_n$ (in the proof of Lemma 2.1). Clearly γ and λ are continuous, and p is a quotient map, by Proposition 1.2(a) of Borges (1977). Consequently, r is continuous.

THEOREM 2.7. *Let X be a Tychonoff space with base point θ. Then $SP^\infty X$ is a retract of $(A(X), \mathscr{G}')$.*

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PROOF. First note that $SP^n X$ is a subspace of $SP^n \beta X$. (Note that the map $\lambda_n: (\beta X)^n \to SP^n \beta X$ is closed and continuous, which implies that

$$\lambda_n | X^n: X^n \to (SP^n X, \text{subspace topology from } SP^n \beta X)$$

is a quotient map, because $X^n = \lambda_n^{-1}(SP^n X)$. Since $\lambda_n: X^n \to SP^n X$ is also a quotient map, by definition, we then get that $SP^n X = (SP^n X$, subspace topology from $SP^n \beta X$).)

For any Y, let $r_n = r | A_n(Y)$. From Theorem 2.6 we get that

$$r_n: A_n(\beta X) \to SP^n \beta X$$

is continuous. Consequently, $r_n: A_n(X) \to SP^n X$ is also continuous. Since $(A(X), \mathscr{G}') = \sum_n A_n(X)$, we get that $r: (A(X), \mathscr{G}') \to SP^\infty X$ is continuous.

3. Reduced products

It is easy to see that the preceding proofs remain valid for the non-abelian analogues. Therefore, we state the main results for reduced products, without proof or further comment.

THEOREM 3.1. *Let X be a Tychonoff space with base point θ. Then X_∞ is a subspace of $(F(X), \mathscr{G}')$.*

THEOREM 3.2. *Let X be a Tychonoff space with base point θ. Then X_∞ is a retract of $(F(X), \mathscr{G}')$.*

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