SOME SELECTION THEOREMS FOR MEASURABLE FUNCTIONS

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1. Introduction. Let $F: X \to Y$ be a multifunction from X to Y. Then, given measure-theoretic or topological structures on X and Y, it is possible in various ways to define the measurability of F. The selection problem is to determine which structures on X and Y and which definitions of measurability of F ensure that F will have a measurable selector. This problem has been studied recently in papers by Castaing (**2**) and Kuratowski and Ryll-Nardzewski (**6**). In the latter paper, the problem is studied for its own interest. The former uses solutions of the problem to obtain general Filippov-type theorems. (See, for example, the corollaries following Theorems 2 and 3 of Castaing's paper.) For other material on Filippov's results see, among others, (**3**; **4**; **5**; **7**; **9**).

In a slightly different direction, McShane and Warfield (**8**) investigated (and gave applications of) lifting theorems for measurable functions; i.e., given X, Y, Z, and functions $g: X \to Z$, $\phi: Y \to Z$ satisfying $g(X) \subset \phi(Y)$, they determined structures for X, Y, Z, and properties of g and ϕ which imply the existence of a measurable function $f: X \to Y$ satisfying $\phi \circ f = g$.

The theorem of Kuratowski and Ryll-Nardzewski (**6**) yields one of Castaing's theorems as a special case. In this note, we apply their results to obtain additional selection theorems. These will contain both Castaing's selection theorems and McShane-Warfield's lifting theorems (in the latter case by translating the lifting problem to a selection problem).

2. Terminology and the theorems of Kuratowski and Ryll-Nardzewski. A multifunction $F: X \to Y$ is a function whose value $F(x)$ for each $x \in X$ is a non-empty subset of Y. Equivalently, F is a subset of $X \times Y$ whose set of first elements is X. Hence, multifunctions compose as relations. If $B \subset Y$, then $F^{-1}(B)$ is defined as usual for relations, so that

$$\begin{aligned} F^{-1}(B) &= \{x \in X \mid (y, x) \in F^{-1} \text{ for some } y \in B\} \\ &= \{x \in X \mid F(x) \cap B \neq \emptyset\}. \end{aligned}$$

A function $f: X \to Y$ is a selector for the multifunction $F: X \to Y$ if and only if $f(x) \in F(x)$ for all $x \in X$. If $\mathscr{S}$ is either a σ-ring or a σ-algebra on X, if $\mathscr{E}$ is a given family of subsets of a topological space Y, and if $F: X \to Y$

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is a multifunction (or function, as the case may be) such that $F^{-1}(E) \in \mathcal{S}$ for all $E \in \mathcal{E}$, then

(i) F is $(\mathcal{A}, \mathcal{O})$-, $(\mathcal{A}, \mathcal{F})$-, or $(\mathcal{A}, \mathcal{B})$-measurable if and only if $\mathcal{S}$ is a σ-algebra and $\mathcal{E}$ is the family of open, closed, or Borel subsets of Y (the last family being the smallest σ-algebra containing all the open subsets of Y), respectively.

(ii) F is $(\mathcal{R}, \mathcal{C})$-measurable if and only if $\mathcal{S}$ is a σ-ring and $\mathcal{E}$ is the family of compact subsets of Y.

Obviously, many other definitions of measurability can be added to the above, but the above four are very common, and, in any case, are the only ones which concern us in this paper. We remark that the three definitions of measurability in (i) are equivalent to one another when applied to functions. For multifunctions, it is easy to verify that $(\mathcal{A}, \mathcal{B})$-measurability implies both $(\mathcal{A}, \mathcal{O})$- and $(\mathcal{A}, \mathcal{F})$-measurability, and that, when Y is perfectly normal, $(\mathcal{A}, \mathcal{F})$-measurability implies $(\mathcal{A}, \mathcal{O})$-measurability.

A multifunction $F: X \rightarrow Y$ is point-closed, point-compact, or point-complete if and only if each value of F is closed, compact, or complete, respectively.

In this terminology, the main results of Kuratowski and Ryll-Nardzewski particularize (with some loss of generality) to the following theorem.

THEOREM 1 (**6**, pp. 398, 400). *Let X be a set with a σ-algebra $\mathcal{S}$, Y a complete separable metric space, and $F: X \rightarrow Y$ a point-closed $(\mathcal{A}, \mathcal{F})$-measurable (alternately, $(\mathcal{A}, \mathcal{O})$-measurable) multifunction. Then F has an $(\mathcal{A}, \mathcal{F})$-measurable (equivalently, $(\mathcal{A}, \mathcal{O})$-measurable) selector.*

Remark. Castaing (**2**, Theorem 3) proved this theorem for X a compact space and $\mathcal{S}$ the σ-algebra of Borel subsets of X.

3. Selection theorems. We begin with two theorems whose proofs are inspired by, and are similar to, a proof of McShane and Warfield (**8**, Theorem 4).

THEOREM 2. *Let X be a set with σ-ring $\mathcal{S}$; let Y be a Hausdorff space which is the union of a family of at most Ω (= first uncountable ordinal) compact metrizable subspaces in such a way that any compact subset of Y lies in the union of an at most countable subfamily; and let $F: X \rightarrow Y$ be a point-closed $(\mathcal{R}, \mathcal{C})$-measurable multifunction. Then F has an $(\mathcal{R}, \mathcal{C})$-measurable selector.*

Proof. Let the family of compact sets described in the hypotheses above be $\{Y_\alpha | \alpha < \omega\}$, where ω is an ordinal less than or equal to Ω. For each $\alpha < \omega$, define

$$X_\alpha = F^{-1}(Y_\alpha) - \bigcup \{F^{-1}(Y_\beta) | \beta < \alpha\}.$$

Clearly, the X_α's are pairwise disjoint, and their union is X. Assign to each X_α the σ-ring $\mathcal{S}_\alpha$ obtained by restricting $\mathcal{S}$ to X_α. Each $\mathcal{S}_\alpha$ is, in fact, a σ-algebra on X_α (i.e., $X_\alpha \in \mathcal{S}_\alpha$), since X_α is the intersection of the countable family $\{F^{-1}(Y_\alpha) - F^{-1}(Y_\beta) | \beta < \alpha\}$, and each of the sets $F^{-1}(Y_\beta)$, $\beta \leqq \alpha$, is a member of $\mathcal{S}$. Furthermore, since $X_\alpha \in \mathcal{S}$, we have that $\mathcal{S}_\alpha \subset \mathcal{S}$ for all α.

Note that, by the definition of X_α, we have that $F(x) \cap Y_\alpha \neq \emptyset$ for all $\alpha < \omega$, $x \in X_\alpha$. Thus, for all $\alpha < \omega$, define a point-closed multifunction $F_\alpha: X_\alpha \to Y_\alpha$ by

$$F_\alpha(x) = F(x) \cap Y_\alpha \quad \text{if } x \in X_\alpha.$$

Then each F_α is $(\mathscr{A}, \mathscr{F})$-measurable, since $F_\alpha^{-1}(C) = X_\alpha \cap F^{-1}(C)$ for each closed (and therefore compact) subset C of Y_α, and since each $F^{-1}(C)$, with C compact, is in $\mathscr{S}$ by hypothesis. Y_α is of course complete with any compatible metric. Thus, by Theorem 1, F_α has an $(\mathscr{A}, \mathscr{F})$-measurable selector $f_\alpha: X_\alpha \to Y_\alpha$.

Now, define $f: X \to Y$ by

$$f(x) = f_\alpha(x) \quad \text{if } x \in X_\alpha, \alpha < \omega.$$

Clearly, f is a selector for F. Furthermore, f is $(\mathscr{R}, \mathscr{C})$-measurable. For, let C be a compact subset of Y, and let α be an at most countable ordinal such that $C \subset \bigcup \{Y_\beta | \beta \leqq \alpha\}$. Then $\beta > \alpha$ implies that

$$f_\beta^{-1}(C \cap Y_\beta) \subset F_\beta^{-1}(C) \subset \bigcup \{F_\beta^{-1}(Y_\gamma) | \gamma \leqq \alpha\} = \emptyset,$$

since $\gamma \leqq \alpha < \beta$ implies that

$$F_\beta^{-1}(Y_\gamma) = \{x \in X_\beta | F_\beta(x) \cap Y_\gamma \neq \emptyset\}$$
$$\subset \{x \in X_\beta | F(x) \cap Y_\gamma \neq \emptyset\} = X_\beta \cap F^{-1}(Y_\gamma) = \emptyset.$$

It follows that

$$f^{-1}(C) = \bigcup \{f_\beta^{-1}(C \cap Y_\beta) | \beta \leqq \alpha\}.$$

Moreover, each f_β is $(\mathscr{A}, \mathscr{F})$-measurable; therefore,

$$f_\beta^{-1}(C \cap Y_\beta) \in \mathscr{S}_\beta \subset \mathscr{S} \quad \text{for all } \beta < \omega.$$

Hence, $f^{-1}(C)$ is the union of a countable subfamily of $\mathscr{S}$.

Assuming the continuum hypothesis, the next theorem follows as a corollary to Theorem 2.

THEOREM 3. *Let X be a set with σ-ring $\mathscr{S}$, Y a separable metric space; and $F: X \to Y$ a point-closed $(\mathscr{R}, \mathscr{C})$-measurable multifunction. Then F has an $(\mathscr{R}, \mathscr{C})$-measurable selector.*

Proof. By the argument in (**8**, proof of Theorem 4), there is an ordinal ω less than or equal to the first uncountable ordinal such that the family of all compact subsets of Y can be indexed by $\{\alpha | \alpha < \omega\}$.

In the next theorem, recall that a Lusin space is a separable metrizable space which is the image of a complete separable metric space under a continuous one-to-one function.

THEOREM 4. *Let X be a set with σ-algebra $\mathscr{S}$, Y a Lusin space, and $F: X \to Y$ a point-closed $(\mathscr{A}, \mathscr{B})$-measurable multifunction. Then F has an $(\mathscr{A}, \mathscr{B})$-measurable selector.*

Proof. Let $\phi: P \to Y$ be a one-to-one continuous function from a complete separable metric space P onto Y. Define a multifunction $F_*: X \to P$ by $F_* = \phi^{-1} \circ F$. F_* is point-closed since ϕ is continuous. Moreover, F_* is $(\mathscr{A}, \mathscr{O})$-measurable. For, let O be an open subspace of P. Then O has a compatible metric which makes it a complete metric space; see (**1**, Chapter IX, § 6.1, Proposition 2). Hence, the subspace $\phi(O)$ of Y is a Lusin space. From (**1**, Chapter IX, § 6.7, Theorem 3) we conclude that $\phi(O)$ is a Borel subset of Y. It follows that

$$F_*^{-1}(O) = F^{-1}(\phi(O)) \in \mathscr{S}.$$

Applying Theorem 1, we obtain an $(\mathscr{A}, \mathscr{O})$-measurable selector $f_*: X \to P$ for F_*. Define $f = \phi \circ f_*: X \to Y$. Then f is a selector for F since

$$f(x) = \phi(f_*(x)) \in \phi(F_*(x)) = \phi \circ \phi^{-1}(F(x)) = F(x) \quad \text{if } x \in X.$$

To prove that f is $(\mathscr{A}, \mathscr{B})$-measurable, it is sufficient to prove that it is $(\mathscr{A}, \mathscr{O})$-measurable. Thus, suppose that O is open in Y. Then $\phi^{-1}(O)$ is open by the continuity of ϕ, and $f^{-1}(O) = f_*^{-1}(\phi^{-1}(O)) \in \mathscr{S}$ by the $(\mathscr{A}, \mathscr{O})$-measurability of f_*.

THEOREM 5. *Let X be a set with σ-algebra $\mathscr{S}$, Y a separable metric space, and $F: X \to Y$ a point-complete $(\mathscr{A}, \mathscr{F})$-measurable multifunction. Then F has an $(\mathscr{A}, \mathscr{F})$-measurable selector.*

Remark. This theorem generalizes Theorem 2 of Castaing (**2**). In his result, X is a compact space with $\mathscr{S}$ the corresponding family of Borel sets, and F is assumed to be point-compact rather than point-complete.

Proof. Let Z be the completion of Y, and let $i: Y \subset Z$ be the inclusion map. Then $F_* = i \circ F: X \to Z$ is a point-closed multifunction (which is equal, pointwise, to F). F_* is $(\mathscr{A}, \mathscr{F})$-measurable, since B closed in $Z \Rightarrow B \cap Y$ closed in $Y \Rightarrow F_*^{-1}(B) = F^{-1}(i^{-1}(B)) = F^{-1}(B \cap Y) \in \mathscr{S}$. Thus, by Theorem 1, F_* has an $(\mathscr{A}, \mathscr{F})$-measurable selector $f_*: X \to Z$. Since it is a selector for F_*, f_* takes all of its values in Y. Thus, $f = i^{-1} \circ f_*: X \to Y$ is a function defined on all of X. Clearly, f is a selector for F. Furthermore, f is $(\mathscr{A}, \mathscr{F})$-measurable; for, letting B be a relatively closed subset of Y, and taking closures relative to Z, we have that $f^{-1}(B) = f_*^{-1} \circ i(\bar{B} \cap Y) = f_*^{-1}(\bar{B} \cap Y) = f_*^{-1}(\bar{B}) \in \mathscr{S}$.

The remaining two theorems of this section give some criteria for deducing new selection theorems from known ones.

THEOREM 6. *Let X be a set with σ-algebra $\mathscr{S}$, Y a topological space, and $F: X \to Y$ a point-closed $(\mathscr{A}, \mathscr{F})$-measurable multifunction. If there exists a topological space Z and a closed continuous function ϕ from Z onto Y such that every point-closed $(\mathscr{A}, \mathscr{F})$-measurable multifunction from X to Z has an $(\mathscr{A}, \mathscr{F})$-measurable selector, then F has an $(\mathscr{A}, \mathscr{F})$-measurable selector.*

Remark. There are at least two valid variations of this theorem. On the one hand, we may retain the assumption of $(\mathscr{A}, \mathscr{F})$-measurability of F, but assume that Z is perfectly normal and that every point-closed $(\mathscr{A}, \mathscr{O})$-measurable multifunction from X to Z has an $(\mathscr{A}, \mathscr{O})$-measurable $(= (\mathscr{A}, \mathscr{F})$-measurable) selector. On the other hand, we may replace $(\mathscr{A}, \mathscr{F})$ by $(\mathscr{A}, \mathscr{O})$ everywhere if ϕ is open rather than closed.

Proof. Apply an argument similar to (but simpler than) that in the proof of Theorem 4 to the multifunction $F_* = \phi^{-1} \circ F: X \to Z$.

The first modification mentioned in the remark is valid since, if Z is perfectly normal, ϕ is closed, and F is $(\mathscr{A}, \mathscr{F})$-measurable, then $\phi^{-1} \circ F$ is $(\mathscr{A}, \mathscr{O})$-measurable. The second modification is obvious.

THEOREM 7. *Let X be a set with σ-ring $\mathscr{S}$, Y a topological space, and $F: X \to Y$ a point-closed $(\mathscr{R}, \mathscr{C})$-measurable multifunction. If there exists a topological space Z and a continuous function ϕ from Z onto Y such that*

(i) *every point-closed $(\mathscr{R}, \mathscr{C})$-measurable multifunction $G: X \to Z$ has an $(\mathscr{R}, \mathscr{C})$-measurable selector, and*

(ii) *$\phi^{-1}(C)$ is the at most countable union of compact subsets of Z for each compact subset C of Y,*

then F has an $(\mathscr{R}, \mathscr{C})$-measurable selector.

Proof. Given Z and ϕ as above, define a multifunction $F_*: X \to Z$ by $F_* = \phi^{-1} \circ F$. F_* is point-closed since ϕ is continuous. Moreover, F_* is $(\mathscr{R}, \mathscr{C})$-measurable. For, let C be a compact subset of Z. Then $\phi(C)$ is compact, and therefore $F_*^{-1}(C) = F^{-1}(\phi(C)) \in \mathscr{S}$. By (i), F_* has an $(\mathscr{R}, \mathscr{C})$-measurable selector $f_*: X \to Z$. Now define $f = \phi \circ f_*: X \to Y$. Clearly, f is a selector for F. Also, f is $(\mathscr{R}, \mathscr{C})$-measurable; for, if C is a compact subset of Y, it easily follows from (ii) and the $(\mathscr{R}, \mathscr{C})$-measurability of f_* that $f^{-1}(C) = f_*^{-1}(\phi^{-1}(C)) \in \mathscr{S}$.

4. Liftings are selectors. In this section we show that the lifting problem solved by McShane and Warfield in (**8**) is a special case of the selection problem solved in the preceding section.

THEOREM 8. *Let X be a set with σ-ring $\mathscr{S}$, and let Y be any space such that every point-closed $(\mathscr{R}, \mathscr{C})$-measurable multifunction $F: X \to Y$ has an $(\mathscr{R}, \mathscr{C})$-measurable selector. Then for any T_1-space Z and any continuous function $k: Y \to Z$, every $(\mathscr{R}, \mathscr{C})$-measurable function $h: X \to Z$ such that $h(X) \subset k(Y)$ can be lifted to an $(\mathscr{R}, \mathscr{C})$-measurable function $f: X \to Y$ (i.e., there exists an $(\mathscr{R}, \mathscr{C})$-measurable function $f: X \to Y$ satisfying $k \circ f = h$).*

Proof. Let h and k be as above. Then $F = k^{-1} \circ h: X \to Y$ is a point-closed multifunction since each $\{h(x)\}$ is closed, k is continuous, and $h(X) \subset k(Y)$. Moreover, F is $(\mathscr{R}, \mathscr{C})$-measurable. For, let C be a compact subset of Y. Then $k(C)$ is compact, and consequently $F^{-1}(C) = h^{-1}(k(C)) \in \mathscr{S}$. Thus,

F has an $(\mathscr{R}, \mathscr{C})$-measurable selector. It easily follows that any such selector is the desired lifting.

COROLLARY (McShane and Warfield (**8**, Theorem 4)). *Let X be a set with σ-ring $\mathscr{S}$, Z a T_1-space, and Y a Hausdorff space which is the union of a family of at most Ω compact metrizable spaces in such a way that any compact subset of Y lies in the union of an at most countable subfamily (e.g., Y separable metric if the continuum hypothesis is assumed). Let k: $Y \to Z$ be a continuous function, and h: $X \to Z$ an $(\mathscr{R}, \mathscr{C})$-measurable function such that $h(X) \subset k(Y)$. Then h can be lifted to an $(\mathscr{R}, \mathscr{C})$-measurable function f: $X \to Y$.*

Proof. Apply Theorems 2 and 8.

Added in proof. The following corollary to Theorems 2 and 3 follows easily from the continuum hypothesis and properties (A) and (L) in (E. Michael, $\aleph_0$-*spaces*, J. Math. Mech. *15* (1966), 983-1002).

COROLLARY. *Let X be a set with σ-ring $\mathscr{S}$, let Y be an $\aleph_0$-space (or, in particular, let Y be the regular quotient of a separable metrizable space), and let F: $X \to Y$ be a point-closed $(\mathscr{R}, \mathscr{C})$-measurable multifunction. Then F has an $(\mathscr{R}, \mathscr{C})$-measurable selector.*

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