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TWO BOOLEAN ALGEBRAS WITH EXTREME CELLULAR AND COMPACTNESS PROPERTIES

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1. Introduction. In this paper, we construct two kinds of Boolean algebras with extreme cellular properties and nice embedding properties. The extreme cellular properties are $\sigma - j$-linked but not $\sigma - j + 1$-linked and ccc but not $\sigma - 2$-linked. The nice embedding properties are that they are ZF-definable subalgebras of both P/F and R (see Preliminaries for notation). It is the author's opinion that R contains much of the "ZF-strength" of P/F.

In Section 3, we define a subalgebra H of R that will contain all of our examples and which is embedded in P/F.

In Section 4 the Boolean algebras yield spaces which solve a problem of E. van Douwen [**3**] in compactness theory.

Boolean algebras that are ccc but not $\sigma - 2$-linked of size continuum had previously been constructed by A. Hajnal and F. Galvin and A. Hajnal [**4**]; however they were not ZFC-demonstrably subalgebras of P/F, our example is. The author owes much to an in-depth analysis of their examples and of R.

In our conclusion, we discuss the Boolean algebras P/F versus R.

2. Preliminaries. Our set-theoretic notation is standard. We only mention that if A is a set, then $\mathscr{P}(A) = \{S:S \subseteq A\}$ and that if f is a function, then Dom f and Rng f denote the domain and range of f respectively.

Our use of Boolean algebraic concepts is elementary. The *Stone space* of a Boolean algebra B is denoted by st B and is the space of all ultrafilters on B topologized with $\{\bar{b}:b \in B\}$ as a base where $\bar{b} = \{p \in \text{st } B:b \in p\}$. Two elements b and b' of B are *disjoint* if $b \wedge b' = 0$. A subset A of B is ccc if there does not exist an uncountable pairwise disjoint subset of A. A subset A of B is *j-linked* (where $j < \omega$) if for every j-element subset F of A, $\wedge F \neq 0$. A subset A of B is $\sigma - j$*-linked* if

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$$A - \{0\} = \bigcup_{n<\omega} A_n$$

where for each $n < \omega$, A_n is j-linked.

Let X be a topological space and let

$$\tau^*(X) = \{U:U \text{ is a non-empty open subset of } X\}.$$

Consider $\tau^*(X)$ as a subset of the power set algebra $\mathscr{P}(X)$. Then, X is said to be ccc or $\sigma - j$-linked if $\tau^*(X)$ is ccc or $\sigma - j$-linked respectively. It is trivial to check that if B is a Boolean algebra, then B is ccc or $\sigma - j$-linked if and only if st B is ccc or $\sigma - j$-linked respectively.

If X is a compact space, then the *compactness number of* X, cmpn X = the least $n < \omega$ (if one exists) such that there exists an open subbase $\mathscr{S}$ of X for which every cover of X from $\mathscr{S}$ has a $\leqq n$ subcover. If no such $n < \omega$ exists, then we say that cmpn $X = \infty$. If cmpn $X = 2$, then X is said to be *supercompact* ([5]). Cmpn $X = n$ is defined in [2].

P/F denotes a Boolean algebra that is the power set algebra of a countably infinite set modulo the ideal of its finite subsets. N denotes the Baire space ω^ω with the Tychonov topology. R denotes the subalgebra of the power set algebra $\mathscr{P}(N)$ that is generated by the rectangles $\prod_{i<\omega} A_i$ of N.

3. The Boolean algebra H. For each $M \subseteq N$, set

$$\hat{M} = \{f \upharpoonright n:n < \omega \text{ and } f \in M\}.$$

Put

$$\mathscr{A} = \left\{\prod_{i<\omega} A_i\text{: for every } i < \omega,\ A_i - A_{i+1} \text{ is finite and } A_i \subseteq \omega\right\}$$

and

$$H = [\mathscr{A}] = \text{the subalgebra of } R \text{ generated by } \mathscr{A}.$$

THEOREM 3.1. *H is embeddable in P/F.*

Proof. Consider P/F as $\mathscr{P}(\hat{N})$ modulo the ideal of finite sets. Referring to [7], page 37, it suffices to define a one to one function $\varphi:\mathscr{A} \to \mathscr{P}(\hat{N})$ satisfying

$$\bigcap_{j<r} A^j - \bigcup_{j<s} B^j \neq \emptyset$$

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if and only if $\cap_{j<r} \varphi(A^j) - \cup_{j<s} \varphi(B^j)$ is infinite whenever

$$A^j = \prod_{i<\omega} A_i^j \in \mathscr{A} \quad \text{and} \quad B^j = \prod_{i<\omega} B_i^j \in \mathscr{A}.$$

Define $\varphi:\mathscr{A} \to P(\hat{N})$ by $\varphi(A) = \hat{A}$.

If $f \in \cap_{j<r} A^j - \cup_{j<s} B^j$, then

$$\{f \upharpoonright i : i < \omega\} \subseteq \bigcap_{j<r} \varphi(A^j).$$

If an infinite subset R of $\{f \upharpoonright i : i < \omega\}$ was contained in $\cup_{j<s} \varphi(B^j)$, then there would exist $j < s$ such that $R \cap \varphi(B^j)$ would be infinite. Since B^j is a closed subset of N, we would conclude that $f \in B^j$. This is a contradiction. Hence, $\cap_{j<r} \varphi(A^j) - \cup_{j<s} \varphi(B^j)$ contains a cofinite subset of $\{f \upharpoonright i : i < \omega\}$ and thus is infinite.

Conversely, if $\{s_n : n < \omega\}$ is an infinite subset of $\cap_{j<r} \varphi(A^j) - \cup_{j<s} \varphi(B^j)$, we consider two cases:

Case 1. For every $i < \omega$, $\{s_n(i) : n < \omega \text{ and } i \in \operatorname{Dom} s_n\}$ is finite. In this case, for every $i < \omega$ there exists $n_i < \omega$ such that $i \in \operatorname{Dom} s_{n_i}$. Therefore, for every $i \in \omega$,

$$s_{n_i}(i) \in \bigcap_{j<r} A_i^j.$$

Define $f \in N$ such that $s_0 \subseteq f$ and for all $i \geqq \operatorname{Dom} s_0$,

$$f(i) \in \bigcap_{j<r} A_i^j.$$

Then,

$$f \in \bigcap_{j<r} A^j - \bigcup_{j<s} B^j.$$

Case 2. There exists $i < \omega$ such that $\{s_n(i) : n < \omega \text{ and } i \in \operatorname{Dom} s_n\}$ is infinite. In this case, we choose one such $i < \omega$. Then, $\{s_n(i) : n < \omega$ and $i \in \operatorname{Dom} s_n\}$ is an infinite subset of $\cap_{j<r} A_i^j$. Since, for each $j < r$, $A_i^j - A_k^j$ is finite for every $k \geqq i$, we see that for every $k \geqq i$,

$$\{s_n(i) : n < \omega \text{ and } i \in \operatorname{Dom} s_n\} \cap \bigcap_{j<r} A_k^j$$

is infinite. Choose an $n < \omega$ such that $i \in \operatorname{Dom} s_n$. Define $f \in N$ such that $s_n \subseteq f$ and for all $k \geqq \operatorname{Dom} s_n$,

$$f(k) \in \bigcap_{j<r} A_k^j.$$

Then,

$$f \in \bigcap_{j<r} A^j - \bigcup_{j<s} B^j.$$

Remark. For each $m < \omega$ set

$$\mathscr{A}_m = \left\{ \prod_{i<\omega} A_i\text{: for each } i \geqq m, A_i - A_{i+1} \text{ is finite} \right\}$$

and set $H_m = [\mathscr{A}_m]$. Define $\varphi_m:\mathscr{A}_m \to \mathscr{P}(\hat{N})$ by

$$\varphi_m(A) = \{f \restriction n{:}n > m \text{ and } f \in A\}.$$

Just as in the theorem, φ_m extends to an embedding of H_m into P/F. $H_0 \subseteq H_1 \subseteq H_2 \ldots$. I have been unable to prove that $\cup_{m<\omega} H_m$ embeds in P/F.

4. Boolean subalgebras of H that are $\sigma - j$-linked but not $\sigma - j + 1$-linked. Fix $j \geqq 2$. Set

$$T_j = \{\pi \in N{:}\pi(0) \in \{1, \ldots, j + 1\} \text{ and for every } n < \omega, \pi(n + 1) \in \{j\pi(n) + 1, \ldots, j\pi(n) + j + 1\}\}.$$

For every $\pi \in T_j$ set

$$C_\pi = \prod_{n<\omega} (\{jn + 1, \ldots, jn + j + 1\} - \text{Rng } \pi).$$

Each C_π is a compact nowhere dense element of H. Set

$$B_j = [\{C_\pi{:}\pi \in T_j\}].$$

This is the subalgebra of H generated by $\{C_\pi{:}\pi \in T_j\}$. B_j is our ZF-definable example.

A. *If F and G are disjoint finite subsets of T_j and $\cap_{\pi\in F} C_\pi \neq \emptyset$, then there exist a finite function s and for every $k \geqq$ Dom s a subset F_k of size $\geqq j$ of $\{jk + 1, \ldots, jk + j + 1\}$ with*

$$s \times \prod_{k\geqq \text{Dom } s} F_k \subseteq \bigcap_{\pi\in F} C_\pi - \bigcup_{\pi\in G} C_\pi.$$

Proof. Choose $f \in \cap_{\pi\in F} C_\pi$. Choose $q < \omega$ such that

$$\{\{\pi(n){:}n \geqq q\}{:}\pi \in F \cup G\}$$

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is a disjoint family. Let

$$m_1 = \min\{\pi(q):\pi \in F \cup G\} \text{ and } m_2 = \max\{\pi(q):\pi \in F \cup G\}.$$

We define s as follows:

$$\begin{aligned} s(m) &= f(m) && \text{if } m < m_1 \\ &= \pi(q+1) && \text{if } m_1 \leqq m = \pi(q) \leqq m_2 \text{ for some } \pi \in G \\ &\neq \pi(q+1) && \text{if } m_1 \leqq m = \pi(q) \leqq m_2 \text{ for some } \pi \in F \\ &= jm + 1 && \text{if } m_1 < m < m_2 \text{ and } m \notin \{\pi(q):\pi \in F \cup G\}. \end{aligned}$$

For every $k \geqq \mathrm{Dom}\, s = m_2 + 1$, there is at most one $\pi \in F$ and one $r < \omega$ such that

$$\pi(r) \in \{jk + 1, \dots, jk + j + 1\}.$$

Set

$$F_k = \{jk + 1, \dots, jk + j + 1\} - \bigcup_{\pi \in F} \mathrm{Rng}\, \pi.$$

Then F_k has size $\geqq j$ and

$$s \times \prod_{k \geqq \mathrm{Dom}\, s} F_k \subseteq \bigcap_{\pi \in F} C_\pi - \bigcup_{\pi \in G} C_\pi.$$

B. *B_j is $\sigma - j$-linked.*

Proof. For every $m < \omega$ and for every $s \in \prod_{n<m} \{jn + 1, \dots, jn + j + 1\}$ set $B_s = \{b \in B_j$: for every $k \geqq \mathrm{Dom}\, s$ there exists a subset F_k of size $\geqq j$ of $\{jk + 1, \dots, jk + j + 1\}$ with $s \times \prod_{k \geqq \mathrm{Dom}\, s} F_k \subseteq b\}$. Each B_s is j-linked. Furthermore,

$$B_j - \{\emptyset\} = \bigcup_{\text{all } s} B_s.$$

Since, if $b \in B_j - \{\emptyset\}$, then there exist disjoint finite subsets F and G of T_j and an $f \in N$ with

$$f \in \bigcap_{\pi \in F} C\pi - \bigcup_{\pi \in G} C_\pi \subseteq b.$$

If $s \times \prod_{k \geqq \mathrm{Dom}\, s} F_k$ is as in the conclusion of A, then $b \in B_s$.

C. *B_j is not $\sigma - j + 1$-linked.*

Proof. Consider T_j as a subspace of N. T_j is compact. For every finite function s from ω to ω, set

$$[s] = \{\pi \in T_j : s \subseteq \pi\}.$$

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Then $\{[\pi \restriction n]:n < \omega \text{ and } \pi \in T_j\}$ is a clopen basis for T_j.

Assume

$$\{C_\pi:\pi \in T_j\} = \bigcup_{n<\omega} L_n,$$

i.e., $T_j = \cup_{n<\omega} A_n$ where

$$A_n = \{\pi \in T_j:C_\pi \in L_n\}.$$

By the Baire category theorem, there exists $n < \omega$ such that A_n is not nowhere dense. In other words, for some $\pi \in T_j$ and some $m < \omega$,

$$[\pi \restriction m + 1] \subseteq \text{cl } A_n.$$

So, we can find $\{\pi_i:1 \leqq i \leqq j + 1\} \subseteq A_n$ such that for every $1 \leqq i \leqq j + 1$,

$$\pi_i \in [\pi \restriction m + 1] \quad \text{and}$$

$$\pi_i(m + 1) = j\pi_i(m) + i = j\pi(m) + i.$$

If

$$f \in \bigcap_{i=1}^{j+1} C_{\pi_i},$$

then there exists $1 \leqq i \leqq j + 1$ such that

$$f(\pi(m)) = j\pi(m) + i.$$

So,

$$f(\pi_i(m)) = f(\pi(m)) = \pi_i(m + 1) \in \text{Rng } \pi_i$$

and hence $f \notin C_{\pi_i}$. This is a contradiction. Hence

$$\bigcap_{i=1}^{j+1} C_{\pi_i} = \emptyset$$

and L_n is not $j + 1$-linked.

D. Cmpn (st B_j) $= j + 1$.

Proof. Set

$$\mathscr{S}_j = \{\overline{N - C_\pi}:\pi \in T_j\} \cup \{\bar{C}_\pi:\pi \in T_j\}.$$

Then $\mathscr{S}_j$ is an open (and also closed) subbase for st B_j. We will show that any cover of st B_j from $\mathscr{S}_j$ has a $\leqq j + 1$ subcover. By compactness, any such cover has a finite subcover, so let

$$\text{st } B_j = \bigcup_{\pi \in F} \overline{N - C_\pi} \cup \bigcup_{\pi \in G} \bar{C}_\pi$$

where F and G are finite subsets of T_j. Then as a fixed ultrafilter will testify,

$$N = \bigcup_{\pi \in F} N - C_\pi \cup \bigcup_{\pi \in G} C_\pi.$$

If $F \cap G \neq \emptyset$, then we get a two subcover. Therefore, we assume that $F \cap G = \emptyset$. If for every $n < \omega$, there exists $1 \leqq \varphi(n) \leqq j + 1$ such that for all $\pi \in F, jn + \varphi(n) \notin \text{Rng } \pi$, then if we define $f(n) = jn + \varphi(n)$, we see that

$$f \in \bigcap_{\pi \in F} C_\pi.$$

Invoking A, we have that

$$\bigcap_{\pi \in F} C_\pi - \bigcup_{\pi \in G} C_\pi \neq \emptyset$$

which is a contradiction. Hence, there exists $n < \omega$ such that for every $1 \leqq k \leqq j + 1$ there exists $\pi_k \in F$ with $jn + k \in \text{Rng } \pi_k$. Then

$$N = \bigcup_{k=1}^{j+1} N - C_{\pi_k}$$

and thus $\{\overline{N - C_{\pi_k}}:1 \leqq k \leqq j + 1\}$ is our $\leqq j + 1$ subcover.

It remains to prove that cmpn (st B_j) $\nleqq j$. From B and C we see that st B_j is $\sigma - j$-linked but not $\sigma - j + 1$-linked; in particular st B_j is not separable. Now invoke a theorem of E. van Douwen [**3**] which states that if cmpn $X \leqq j$ and X is $\sigma - j$-linked, then X is separable.

Remark 1. Question 1 of [**3**] asks if there exists compact T_2 spaces that are $\sigma - j$-linked, not $\sigma - j + 1$-linked and of compactness number $j + 1$. The spaces st B_j are such examples.

Remark 2. If we apply the same technique when $j = 1$ to yield B_1, then st B_1 is the one point compactification of a discrete space of size continuum. Hence, st B_1 has no restrictive cellular properties.

Remark 3. In [**1**], the author has shown that there is a subalgebra B_∞ of H such that B_∞ is $\sigma - j$-linked for all $j < \omega$ but B_∞ is not σ-centered, i.e., whenever

$$B_\infty - \{\emptyset\} = \bigcup_{n<\omega} B_n$$

there exists a finite subset F of B_n for some $n < \omega$ such that $\wedge F = 0$. It follows that

$$\text{cmpn}(\text{st } B_\infty) = \infty.$$

5. A Boolean subalgebra of H that is ccc but not $\sigma - 2$-linked. For a set X, X^n denotes the set of all n-sequences composed of members of X. Set

$$T = \bigcup_{n<\omega} [2^n]^n,$$

i.e., T is the set of all n-sequences whose terms are n-sequences of 0's and 1's. T is a countable set and we will identify N with T^ω.

Let $<$ be the lexicographic order on 2^ω with greatest element 1. Set

$$C^0 = \{f \in 2^\omega : f(0) = 0\} \quad \text{and}$$

$$C^1 = \{f \in 2^\omega : f(0) = 1].$$

Set $\mathscr{L} = \{L : L$ is a $<$ increasing convergent sequence in C^1 with sup $L < 1\}$. Choose $\varphi : \mathscr{L} \to C^0$ any ZF-injection. Set

$$\mathscr{K} = \{ \{\varphi(L)\} \cup L : L \in \mathscr{L}\}.$$

$\mathscr{K}$ satisfies the following two properties: (a) if $K \neq K'$, then min $K \neq$ min K' and (b) if $S \in \mathscr{L}$, then there exists $K \in \mathscr{K}$ such that $S \subseteq K$.

Definition. If $K \in \mathscr{K}$ and $s \in T$ with Dom $s = n$, then s splits K if there exists $i < n$ such that for every $j < n$, $j \neq i$ and for every $g \in K$,

$$s(i) = (\sup K) \upharpoonright n \quad \text{and} \quad s(j) \neq g \upharpoonright n.$$

For every $K \in \mathscr{K}$ set

$$A_K = \prod_{n<\omega} \{s \in T : \text{Dom } s \geqq n \text{ and } s \text{ splits } K\}.$$

Since each $K \in \mathscr{K}$ is a nowhere dense subset of 2^ω, each $A_K \neq \emptyset$. Set

$$B_0 = [\{A_K : K \in \mathscr{K}\}].$$

B_0 is the subalgebra of H generated by $\{A_K : K \in \mathscr{K}\}$. B_0 is our ZF-definable example.

A. *Let $\mathscr{F}$ and $\mathscr{G}$ be disjoint finite subsets of $\mathscr{K}$.*

$$\bigcap_{K\in\mathscr{F}} A_K - \bigcup_{K\in\mathscr{G}} A_K \neq \emptyset$$

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if and only if

$$\{\sup K : K \in \mathscr{F}\} \cap \bigcup_{L \in \mathscr{F}} L = \emptyset.$$

Proof. (only if) Indirect proof. If $\sup K \in L$, where $K, L \in \mathscr{F}$, then choose $k < \omega$ such that

$$\sup K \restriction k \neq \sup L \restriction k.$$

If $\text{Dom } s \geqq k$, then s cannot split both K and L, hence $A_K \cap A_L = \emptyset$. (if) Direct proof. Assume $\mathscr{F} \cap \mathscr{G} = \emptyset$ and

$$\{\sup K : K \in \mathscr{F}\} \cap \bigcup_{L \in \mathscr{F}} L = \emptyset.$$

It suffices to find, for each $n < \omega$, an $s \in T$ with $\text{Dom } s \geqq n$ and such that for every $K \in \mathscr{F}$ and for every $K' \in \mathscr{G}$, s splits K but s does not split K'. To this end, fix $n < \omega$ and choose $k \geqq n$ such that

1. $|\mathscr{F} \cup \mathscr{G}| \leqq k$
2. there exists $t \in 2^k$ such that

$$t \notin \{g \restriction k : g \in \bigcup_{L \in \mathscr{F}} L\}$$

3. if $K, L \in \mathscr{F}$ and $\sup K \neq \sup L$, then

$$\sup K \restriction k \notin \{g \restriction k : g \in L\}$$

4. if $K' \in \mathscr{G}$, then

$$\min K' \restriction k \notin \{g \restriction k : g \in \bigcup_{L \in \mathscr{F}} L\}.$$

Let $\mathscr{F}' \subseteq \mathscr{F}$ be maximal with respect to the property that if $K, L \in \mathscr{F}'$, $K \neq L$, then $\sup K \neq \sup L$. It is now easy to define an $s \in T$ with $\text{Dom } s = k$ so that

$$\{\sup K \restriction k : K \in \mathscr{F}'\} \cup \{\min K' \restriction k : K' \in \mathscr{G}\} \subseteq \text{Rng } s$$

$$\subseteq \{\sup K \restriction k : K \in \mathscr{F}'\} \cup \{\min K' \restriction k : K' \in \mathscr{G}\} \cup \{t\}.$$

This s splits all $K \in \mathscr{F}$ and no $K' \in \mathscr{G}$.

In order to prove that B_0 is ccc, we first prove a lemma about 2^ω.

LEMMA. *If* $1 \leqq s < \omega$ *and if* $\{(x_0^\alpha, \dots, x_{s-1}^\alpha) : \alpha < \omega_1\} \subseteq (2^\omega)^s$ *satisfies: for each* $i < s$ *and for each* $\alpha < \beta < \omega_1$, $x_i^\alpha \neq x_i^\beta$, *then there exists a countable* $E \subseteq \omega_1$ *such that for every*

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$$f:E \to s \quad \{x^{\alpha}_{f(\alpha)}:\alpha \in E\}$$

has uncountable closure in 2^{ω}.

Proof. Since $(2^{\omega})^s$ is hereditarily separable, choose $E \subseteq \omega_1$ such that

$$\{(x^{\alpha}_0, \dots, x^{\alpha}_{s-1}):\alpha \in E\}$$

is dense in

$$\{(x^{\alpha}_0, \dots, x^{\alpha}_{s-1}):\alpha < \omega_1\}.$$

Let $f:E \to s$. Since

$$\{(x^{\alpha}_0, \dots, x^{\alpha}_{s-1}):\alpha \in E\} = \bigcup_{i<s} \{(x^{\alpha}_0, \dots, x^{\alpha}_{s-1}):f(\alpha) = i\},$$

there exists an $i < s$ such that $\{(x^{\alpha}_0, \dots, x^{\alpha}_{s-1}):f(\alpha) = i\}$ has uncountable closure in $\{(x^{\alpha}_0, \dots, x^{\alpha}_{s-1}):\alpha < \omega_1\}$. Since $\alpha < \beta < \omega_1$ implies $x^{\alpha}_i \neq x^{\beta}_i$, it must be that $\{x^{\alpha}_i:f(\alpha) = i\}$ has uncountable closure in 2^{ω}.

B. B_0 *is* ccc.

Proof. Indirect proof. Assume that

$$\left\{ \bigcap_{K \in \mathscr{F}_\alpha} A_K - \bigcup_{K \in \mathscr{G}_\alpha} A_K:\alpha < \omega_I \right\}$$

is an uncountable collection of pairwise disjoint non-$\emptyset$ elements of B_0. Therefore, for each $\alpha < \omega_1$,

$$\mathscr{F}_\alpha \cap \mathscr{G}_\alpha = \emptyset.$$

By a delta-system argument, we may assume that if $\alpha \neq \beta$, then $\mathscr{F}_\alpha \cap \mathscr{G}_\beta = \emptyset$. Hence, if $\alpha \neq \beta$, then

$$(\mathscr{F}_\alpha \cup \mathscr{F}_\beta) \cap (\mathscr{G}_\alpha \cup \mathscr{G}_\beta) = \emptyset.$$

We further assume that $\{\{\sup K:K \in \mathscr{F}_\alpha\}:\alpha < \omega_1\}$ is a delta-system with root Q and that there exists $s < \omega$ such that for every $\alpha < \omega_1$,

$$\mathscr{F}'_\alpha = \{K \in \mathscr{F}_\alpha: \sup K \notin Q\}$$

has exactly s elements. For every $\alpha < \omega_1$, put

$$\mathscr{F}'_\alpha = \{K^{\alpha}_i:i < s\}.$$

Thus, invoking A, we see that for every $\alpha < \beta$ there exist $K \in \mathscr{F}'_\alpha$ and $L \in \mathscr{F}'_\beta$ such that either $\sup K \in L$ or $\sup L \in K$. Since $\{\{\sup K:K \in \mathscr{F}'_\alpha\}:\alpha < \omega_1\}$ is an uncountable disjoint collection and each $K \in \mathscr{K}$ has only countably many elements, by restricting to an uncountable subset of ω_1, we may as well assume that if $\alpha < \beta < \omega_1$, then there exist $K \in \mathcal{F}'_\alpha$ and $L \in \mathcal{F}'_\beta$ such that sup $K \in L$.

By applying the lemma to

$$\{ (\sup K_0^\alpha, \dots, \sup K_{s-1}^\alpha) : \alpha < \omega_1 \} \subseteq (2^\omega)^s,$$

we get a countable $E \subseteq \omega_1$ such that for every

$$f:E \to s \quad \{\sup K_{f(\alpha)}^\alpha : \alpha \in E\}$$

has uncountable closure in 2^ω. Choose $\gamma < \omega_1$ such that sup $E < \gamma$. For every $\alpha \in E$ there exists $i < s$ such that

$$\sup K_i^\alpha \in \bigcup_{j<s} K_j^\gamma.$$

Define $f:E \to s$ by $f(a) =$ one such i. Then

$$\{\sup K_{f(\alpha)}^\alpha : \alpha \in E\} \subseteq \bigcup_{j<s} K_j^\gamma.$$

But $\cup_{j<s} K_j^\gamma$ has countable closure in 2^ω. This is a contradiction.

C. *B_0 is not $\sigma - 2$-linked.*

Proof. We will show that whenever $\mathcal{K} = \cup_{n<\omega} \mathcal{K}_n$, then there exists $n < \omega$ and K, L in $\mathcal{K}_n$ such that sup $K \in L$. Together with A, this implies that $\{A_K : K \in \mathcal{K}\}$ is not $\sigma - 2$-linked.

Assume

$$\mathcal{K} = \bigcup_{n<\omega} \mathcal{K}_n.$$

By induction on $n < \omega$, define two sequences $(a_n)_{n<\omega}$ and $(b_n)_{n<\omega}$ such that

1. $a_0 \in C^1 - \{1\}$ and $b_0 = 1$
2. for every $n < \omega$, if there exists $K \in \mathcal{K}_n$ such that $a_n <$ sup $K < b_n$, then $a_{n+1} =$ one such sup K and b_{n+1} is such that $a_{n+1} < b_{n+1} < b_n$; if there does not exist $K \in \mathcal{K}_n$ such that $a_n <$ sup $K < b_n$, then a_{n+1} and b_{n+1} are such that $a_n < a_{n+1} < b_{n+1} < b_n$.

Now, set $S = \{a_n : n < \omega\}$. Note that $S \in \mathcal{L}$ and for all $n < \omega$,

$$a_n < \sup S < b_n.$$

Since $\mathcal{K}$ satisfies the property (b), there exists $L \in \mathcal{K}$ such that $S \subseteq L$. Note that sup $L =$ sup S. Since $L \in \mathcal{K}$, there exists $n < \omega$ such that $L \in \mathcal{K}_n$. Since L satisfies

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$$a_n < \sup L < b_n,$$

by 2, we have that $a_{n+1} = \sup K$ for some $K \in \mathscr{K}_n$. But then $\sup K \in L$.

D. Cmpn (st B_0) = 2, *i.e.*, st B_0 *is supercompact.*

Proof. Set

$$\mathscr{S} = \{\bar{A}_K : K \in \mathscr{K}\} \cup \{\overline{N - A_K} : K \in \mathscr{K}\}.$$

Then $\mathscr{S}$ is a closed (and also open) subbase for st B_0. We will show that any 2-linked subcollection of $\mathscr{S}$ has a non-empty intersection. By compactness, it suffices to show that any finite 2-linked subcollection of $\mathscr{S}$ has a non-empty intersection; so let $\{\bar{A}_K : K \in \mathscr{F}\} \cup \{\overline{N - A_K} : K \in \mathscr{G}\}$ have the property that every pair of sets has a non-empty intersection. This means that if $K, L \in \mathscr{F}$, then $A_K \cap A_L \neq \emptyset$ and that if $K \in \mathscr{F}$ and $L \in \mathscr{G}$, then $A_K - A_L \neq \emptyset$. Hence, invoking A, we conclude that

$$\bigcap_{K \in \mathscr{F}} A_K - \bigcup_{K \in \mathscr{G}} A_K \neq \emptyset.$$

If $p \in$ st B_0 and

$$\bigcap_{K \in \mathscr{F}} A_K - \bigcup_{K \in \mathscr{G}} A_K \in p,$$

then

$$p \in \bigcap_{K \in \mathscr{F}} \bar{A}_K \cap \bigcap_{K \in \mathscr{G}} \overline{N - A_K}.$$

Remark 1. A. Hajnal had constructed a ccc poset of size continuum which was not $\sigma - 2$-linked. F. Galvin and A. Hajnal [**4**] have other examples with further properties. By standard techniques, these yield Boolean algebras, which under extra set-theoretic assumptions, are embedded in P/F. It was the desire to find examples that embed in P/F in ZFC alone that occasioned the effort. The author would like to thank Fred Galvin for his generous correspondence.

Remark 2. The role that the function $\varphi : \mathscr{L} \rightarrow C^0$ played was solely to guarantee that st B_0 would be supercompact. This had an unexpected benefit of simplifying some proofs. In fact, if one sets

$$\mathscr{K} = \{K : K \text{ is a} < \text{increasing convergent sequence in } 2^{\omega} \text{ with } \sup K < 1\}$$

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and sets

$$B_0' = [\ \{A_K : K \in \mathscr{K}\}\ ],$$

then B_0' is ccc and not $\sigma - 2$-linked; however A is no longer true and st B_0' is not supercompact by the standard subbase $\mathscr{S}$.

6. Conclusion. This conclusion is only a discussion. Proofs are not supplied.

We now discuss the mutual strengths of the rectangle algebra R and the quotient algebra P/F. How much of R is embeddable in P/F and how much of P/F is embeddable in R? It is convenient to make some definitions. A boolean algebra B is *combinatorially embedded* in a boolean algebra C if there exists a one to one mapping $\varphi : B \to C$ such that

$$\bigwedge_{i<n} b_n \neq 0 \text{ if and only if } \bigwedge_{i<n} \varphi(b_i) \neq 0.$$

A combinatorial embedding preserves the disjointness properties. Note that if φ is onto a subalgebra of C, then φ is a boolean algebraic embedding as well. A subalgebra B of P/F is *representable* if B, considered as a set of equivalence classes, has a choice function, i.e., an $h : B \to \mathscr{P}(\omega)$ such that for all $b \in B$, $h(b) \in b$. Representable subalgebras are of interest when we work in ZF.

We have seen that H is embeddable in P/F and that H contains several interesting subalgebras. H also contains the power set algebra $\mathscr{P}(\omega)$ as the subalgebra $\{A \times \omega^\omega : A \subseteq \omega\}$. Another interesting subalgebra of H is

$$E = \left[\left\{\prod_{i<\omega} A - i : A \subseteq \omega\right\}\right].$$

St E is homeomorphic to Exp $\beta\omega - [\omega]^{<\omega}$, the filter analogue of $\beta\omega - \omega$. We remind the reader that $\beta\omega$ is the Stone space of $\mathscr{P}(\omega)$, $\beta\omega - \omega$ is the Stone space of P/F and Exp $\beta\omega$ is the hyperspace of closed subsets of $\beta\omega$ with the Vietoris topology. It is well known that any boolean algebra of size ω_1 is embeddable in P/F in ZFC, so under CH, R itself is embeddable in P/F. In ZFC alone, it is unclear whether R can even be combinatorially embedded in P/F.

Problem 1. In ZFC, can R be embedded in P/F? A particularly simple subalgebra of R that the author is unable to even combinatorially embed in P/F is

$$\left[\left\{\prod_{i<\omega} A_i : \text{ for each } i < \omega,\ A_i \text{ is a singleton or is } \omega\right\}\right].$$

On the other hand, ZFC easily implies that P/F cannot be embedded in R. R contains no increasing ω_1-sequences (in fact, the simultaneously F_σ and G_δ subsets of ω^ω have this property, (cf. [6] p. 196) whereas, ZFC implies that P/F contains an increasing ω_1-sequence. We mention that it is consistent with ZF that P/F does not contain an increasing ω_1-sequence. K. Kunen has proven that ZF alone implies that P/F cannot be embedded in R. However, ZFC does imply that P/F can be combinatorially embedded in R. Using choice, let $h:P/F \to \mathscr{P}(\omega)$ be such that $h(b) \in b$. The mapping $\psi:P/F \to H$ defined by

$$\psi(b) = \prod_{i<\omega} [h(b) - i]$$

is combinatorial embedding.

P/F has a certain vague nature due to the fact that one cannot prove in ZF that it is representable. As an example of this, consider the following two statements:

1. If $\{A_n:n < \omega\}$ is a set of infinite subsets of ω such that for every $n < \omega$, $A_{n+1} - A_n$ is finite, then there exists an infinite $A \subseteq \omega$ such that for every $n < \omega$, $A - A_n$ is finite.

2. If $\{b_n:n < \omega\}$ is a set of non-0 elements of P/F such that for every $n < \omega$, $b_{n+1} \leqq b_n$, then there exists a non-0 $b \in P/F$ such that for every $n < \omega$, $b \leqq b_n$.

Statement 1 is a ZF-theorem while Statement 2 seems (the author has no proof) of necessity to require a choice principle to prove. Statement 1 is clearly the more fundamental statement about $\mathscr{P}(\omega)$. Upon closer inspection, one sees that the subalgebra H, as embedded in P/F in Theorem 3.1 is representable. This has led me to

Problem 2. In ZF, is there a representable subalgebra of P/F that cannot be embedded in R?

The point of view taken in this paper is that a successful investigation of the set algebra R will shed light on the ZF-strength of the quotient algebra P/F.

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