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DIFFERENTIAL FORMS ON STRATIFIED SPACES II

SERAP GÜRER and PATRICK IGLESIAS-ZEMMOUR✉

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Abstract

We prove that, for conelike stratified diffeological spaces, a zero-perverse form is the restriction of a global differential form if and only if its index is equal to one for every stratum.

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1. Introduction

In the previous paper 'Differential forms on stratified spaces' [1], for each zero-perverse form defined on the regular part of a diffeological stratified space, we introduced an index for every stratum, counting the number of different forms generated by the zero-perverse form around the stratum. We showed that under a natural condition on the stratification—if two points could be connected by a path cutting the singular subset in a finite number of points—the zero-perverse form is the restriction of a differential form, defined globally on the space, if and only if the index is equal to one for every stratum.

In this paper, we prove this statement for locally conelike diffeological stratified spaces, without additional conditions.

2. Locally conelike diffeological stratified spaces

We recommend Kloeckner's survey [3] for the classical topological approach to stratified spaces. On diffeology, we refer to the textbook [2], and for conelike stratified spaces, to the paper [5] by Siebenmann. The topology of a diffeological space is its natural D-topology, defined in [2, Section 2.8].

Let us begin by specifying what we understand by *stratified diffeological spaces*.

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2.1. Stratified diffeological spaces. A *stratification* of a diffeological space X by manifolds is a finite family of subspaces $\mathcal{S} = \{\mathrm{S}_i\}_{i \in \mathcal{I}}$, called *strata*, with the following properties.

(1) The strata form a partition of X.
(2) Each stratum is a manifold for the subset diffeology (see [2, Section 4.1] for *manifolds as diffeologies*).
(3) The strata satisfy the frontier condition

$$\mathrm{S}_i \cap \overline{\mathrm{S}_j} \neq \varnothing \quad \Rightarrow \quad \mathrm{S}_i \subset \overline{\mathrm{S}_j}.$$

(4) Let $m = \max\{\dim(\mathrm{S}_i)\}_{i \in \mathcal{I}}$. Then

$$\mathrm{X}_{\mathrm{reg}} = \bigcup_{i \in \mathcal{I} \mid \dim \mathrm{S}_i = m} \mathrm{S}_i$$

is an open dense subset called the *regular part* of the stratification.
(5) Let $\mathcal{I}'$ be the subset of indices $i \in \mathcal{I}$ such that $\dim(\mathrm{S}_i) < m$. If $\mathcal{I}'$ is not empty, then let $\mathcal{S}' = \{\mathrm{S}_i\}_{i \in \mathcal{I}'}$ and $\mathrm{X}' = \bigcup_{i \in \mathcal{I}'} \mathrm{S}_i$, equipped with the subset diffeology. Then $\mathcal{S}'$ is a stratification of X'.

For topological stratified spaces, the definition can be found in [4]. We assume throughout that the space X is Hausdorff, metrisable and connected for the D-topology. The subset X' which is equal to $\mathrm{X} - \mathrm{X}_{\mathrm{reg}}$ is called the *singular part* of the stratification and is sometimes denoted by $\mathrm{X}_{\mathrm{sing}}$. The elements of $\mathcal{S}'$ are called the singular strata. The first four axioms can be called the *structure axioms* and the fifth is the *recursion axiom*. By this recursion, we get a filtration of diffeological subspaces

$$\mathrm{X}_0 \subset \mathrm{X}_1 \subset \cdots \subset \mathrm{X}_\ell \subset \cdots \subset \mathrm{X}_{k-1} \subset \mathrm{X}_k,$$

where $\mathrm{X}_k = \mathrm{X}$, $\mathrm{X}_{k-1} = \mathrm{X}'$ and so on. Each term X_ℓ of the filtration is a stratified space with the corresponding subset $\mathcal{S}_\ell$ of strata, and the main strata has dimension n_ℓ. Each term is the singular part of the term before.

The next step consists of specialising the structure near the singular strata. The stratification is *locally fibred* [1], or the space is a *locally fibred stratified space*, if there exists a *tube system* $\{\pi_{\mathrm{S}} \colon \mathrm{TS} \to \mathrm{S})\}_{S \in \mathcal{S}}$ such that:

(A) TS is an open neighbourhood of S, called a *tube* over, or around, S;
(B) the map $\pi_{\mathrm{S}} \colon \mathrm{TS} \to \mathrm{S}$ is a smooth retraction which is a diffeological fibration; and
(C) for all $x \in \mathrm{TS} \cap \mathrm{TS}' \cap \pi_{\mathrm{S}'}^{-1}(\mathrm{TS})$, one has $\pi_{\mathrm{S}}(\pi_{\mathrm{S}'}(x)) = \pi_{\mathrm{S}}(x)$.

The fibre $\mathrm{F}_x = \pi_{\mathrm{S}}^{-1}(x)$, with $x \in \mathrm{S}$, is a diffeological space itself with its stratification inherited from the ambient stratification. Its intersection with the stratum is reduced to one point: $\mathrm{F}_x \cap \mathrm{S} = \{x\}$. We call it the *apex* of the fibre.

REMARK 2.1. To be locally fibred is not *a priori* recursive. Do we need a natural recursion on the local fibration? This is a legitimate question, but we do not need this property for the case treated in this paper.

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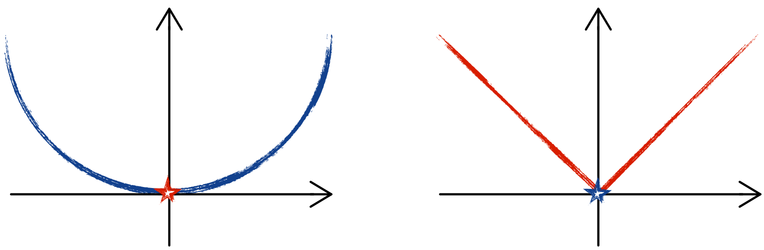

FIGURE 1. A formal cone (left) and a geometric cone (right).

2.2. Diffeological cones. Let Σ be a diffeological space. We call a *cone over* Σ the quotient space $\mathrm{Cone}(\Sigma) = \Sigma \times [0, \infty[\ /\ \Sigma \times \{0\}$, equipped with the quotient diffeology. We recall that the quotient diffeology is the finest diffeology on the quotient that makes the projection smooth [2, Section 1.50]. Consider now a diffeological space F with a smooth bijection $\phi\colon\ \mathrm{Cone}(\Sigma) \to \mathrm{F}$ such that $\phi \upharpoonright [\mathrm{Cone}(\Sigma) - \star]$ is a diffeomorphism onto its image, where $\star = \mathrm{class}(\Sigma \times \{0\})$ is the *apex* of the cone. Thus, without their apex, these spaces are equivalent to $\Sigma \times]0, \infty[$. Then, F will be called a *diffeological cone*, or simply a *cone*, and Σ is called the *base* of the cone. The cone F differs from the cone over Σ only by the germ of the diffeology at the apex.

$$\begin{array}{ccc} \Sigma \times [0, \infty[& & \\ {\scriptstyle \mathrm{class}}\downarrow & \searrow^{\Phi} & \\ \mathrm{Cone}(\Sigma) & \xrightarrow[\phi]{} & \mathrm{F} \end{array}$$

This consideration is necessary because diffeology discriminates between cones, as shown, for example, by the positive cone in $\mathbf{R}^3$ defined by $x^2 + y^2 = z^2$ and $z \geq 0$, equipped with the subset diffeology, compared with the cone over S^1, given by $\mathrm{Cone}(\mathrm{S}^1) = \mathrm{S}^1 \times [0, \infty[/\mathrm{S}^1 \times \{0\}$ (see [1, Section 3]).

An equivocal situation arises with this definition, related to the different types of stratified spaces [1, Introduction]. *Geometric stratified spaces* are the stratified spaces for which the strata are the (connected components of the) orbits of the pseudogroup of local diffeomorphisms. The *formal stratified spaces* are the other ones. The same situation happens with diffeological cones when we compare the two examples sketched in Figure 1 above: the hemisphere (left cone) defined by $x^2 + y^2 + (z - 1)^2 = 1$ with $0 \leq z < 1$ and the embedded cone (right cone) defined by $x^2 + y^2 - z^2 = 0$, both equipped with the subset diffeology of $\mathbf{R}^3$. What differentiates these two cones is the behaviour of the apex $\star$. On the left cone it is interchangeable with any other point, that is, the pseudogroup of local diffeomorphisms does not distinguish the apex from the other points. On the other hand, the apex of the right cone is alone in its orbit. This observation leads us to distinguish at least two classes of cones.

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DEFINITION 2.2. We shall say that a cone is a *geometric cone* if its apex is alone in its orbit under the pseudogroup of local diffeomorphisms. Otherwise, we shall say that it is a *formal cone*.

This remark has a real content, related to the idea about what constitutes a cone. Intuitively, the apex of a cone should be distinguished. In diffeology, it is the role of local diffeomorphisms to discriminate between points.

2.3. Differential forms on conelike stratified diffeological spaces. We have introduced the general notion of locally fibred stratified spaces in [1]. They are diffeological spaces, together with a stratification, such that the space looks like a diffeological fibre bundle [2, Sections 8.8, 8.9] on the neighbourhood of each stratum. This is *a priori* a two step generalisation of the standard situation in the classical theory of stratified spaces, where the fibre is assumed to be conelike over a stratified base (see [5]). Hence, we have a diffeological intermediary to consider, which is the analogue of the usual conelike stratified spaces.

DEFINITION 2.3. We call a *locally conelike diffeological stratified space* any locally fibred stratified diffeological space with fibres that are diffeological cones.

First, note that, in a locally conelike diffeological stratified space, the bases of the cones are stratified spaces as well. Second, the apex of the fibres is their intersection with the stratum.

Let $\alpha \in \Omega^k_{\bar{0}}[\mathrm{X}]$ be a zero-perverse form defined on the regular part $\mathrm{X}_{\mathrm{reg}}$ of a diffeological stratified space X and let ν be the index function on zero-perverse forms, defined in [1, Section 5].

PROPOSITION 2.4. *If* X *is a locally conelike diffeological stratified space, then there exists a (unique) differential form* $\underline{\alpha} \in \Omega^k(\mathrm{X})$ *such that* $\alpha = \underline{\alpha} \restriction \mathrm{X}_{\mathrm{reg}}$ *if and only if* $\nu_{\mathrm{S}}(\alpha) = 1$ *for all strata* S.

REMARK 2.5. Actually, the proposition applies to every stratum that is in the closure of the regular part. But since, by definition, $\mathrm{X}_{\mathrm{reg}}$ is dense in X, the proposition applies for all strata.

The proof of the proposition could be easily extended to the case of locally fibred diffeological stratified spaces with contractible fibres, maybe with some condition on the retraction with respect to the connected components of the regular part. But that would be a minor generalisation.

PROOF OF PROPOSITION 2.4. If $\nu_{\mathrm{S}}(\alpha) = 1$ for all $\mathrm{S} \in \mathcal{S}$, then there exists a (unique) differential form $\underline{\alpha} \in \Omega^k(\mathrm{X})$ such that $\alpha = \underline{\alpha} \restriction \mathrm{X}_{\mathrm{reg}}$. That has been proved in a general context in [1].

Let $\underline{\alpha} \in \Omega^k(\mathrm{X})$. For all strata S, let $\pi_{\mathrm{S}} : \mathrm{TS} \to \mathrm{S}$ be the retraction of the tube around a singular stratum S, which is, by hypothesis, a local fibration with fibre a cone F with base Σ. Recall that $\mathrm{S} \subset \mathrm{TS}$ and, for all $x \in \mathrm{S}$, $\pi_{\mathrm{S}}(x) = x$.

Let $\alpha = \underline{\alpha} \restriction \mathrm{X}_{\mathrm{reg}}$ and assume that α is zero-perverse with respect to these local structures. To follow the construction of the index ν, we consider the pullback of

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the tube TS by the projection of the universal covering $\mathrm{pr}\colon \tilde{\mathrm{S}} \to \mathrm{S}$, even if it is not completely necessary, since our considerations will be local. Let us denote by $\mathrm{T}\hat{\mathrm{S}}$ the pullback $\mathrm{pr}^*(\mathrm{TS})$, that is,

$$\mathrm{T}\hat{\mathrm{S}} = \{(\tilde{x}, y) \in \tilde{\mathrm{S}} \times \mathrm{TS} \mid \mathrm{pr}(\tilde{x}) = \pi_{\mathrm{S}}(y)\}.$$

This pullback is actually a tube around the embedding $\hat{\mathrm{S}}$ of the universal covering $\tilde{\mathrm{S}}$ by the smooth section $\sigma\colon \tilde{x} \to (\tilde{x}, \mathrm{pr}(\tilde{x}))$, that is,

$$\hat{\mathrm{S}} = \{(\tilde{x}, \mathrm{pr}(\tilde{x})) \mid \tilde{x} \in \tilde{\mathrm{S}}\} \subset \mathrm{T}\hat{\mathrm{S}},$$

which justifies the notation. Indeed, since $\mathrm{pr}(\tilde{x}) \in \mathrm{S}$, we have $\mathrm{pr}(\tilde{x}) = \pi_{\mathrm{S}}(\mathrm{pr}(\tilde{x}))$, which is the condition for $(\tilde{x}, \mathrm{pr}(\tilde{x}))$ to belong to $\mathrm{T}\hat{\mathrm{S}}$, and obviously $\mathrm{pr}_1 \circ \sigma = \mathbf{1}_{\tilde{\mathrm{S}}}$. Now

$$\pi_{\hat{\mathrm{S}}}\colon (\tilde{x}, y) \to (\tilde{x}, \pi_{\mathrm{S}}(y))$$

is a retraction from $\mathrm{T}\hat{\mathrm{S}}$ to $\hat{\mathrm{S}}$. The tube $\mathrm{T}\hat{\mathrm{S}}$ is also a fibration with the same fibre F; it is the deployment of the tube TS along the embedded universal covering $\hat{\mathrm{S}}$.

Hence $\mathrm{T}\hat{\mathrm{S}}$ is everywhere locally diffeomorphic to a product $\mathcal{O} \times \mathrm{F}$, where $\mathcal{O} \subset \hat{\mathrm{S}}$ is some open subset, and the projection $\pi_{\hat{\mathrm{S}}}\colon (\tilde{x}, y) \to (\tilde{x}, \pi_{\mathrm{S}}(y))$ is locally equivalent to the first projection $\mathrm{pr}_1 : \mathcal{O} \times \mathrm{F} \to \mathcal{O}$. Thus there is a local diffeomorphism ψ, defined on $\mathcal{O} \times \mathrm{F}$, into $\mathrm{T}\hat{\mathrm{S}}$ such that $\pi_{\hat{\mathrm{S}}} \circ \psi = \mathrm{pr}_1$. We always choose $\mathcal{O}$ to be connected and simply connected. The regular part of $\pi_{\hat{\mathrm{S}}}^{-1}(\mathcal{O})$ is the image by ψ of the regular part of the product $\mathcal{O} \times \mathrm{F}$, that is, $\mathcal{O} \times \mathrm{F}_{\mathrm{reg}}$. But $\mathrm{F}_{\mathrm{reg}}$ is naturally identified with $]0, \infty[\times \Sigma_{\mathrm{reg}}$ by construction. Therefore we have the following chain of square commutative diagrams.

$$\begin{array}{ccccccccc}
\mathcal{O} \times \Sigma_{\mathrm{reg}} \times]0,\infty[& = & \mathcal{O} \times \mathrm{F}_{\mathrm{reg}} & \hookrightarrow & \mathcal{O} \times \mathrm{F} & \xrightarrow{\psi} & \mathrm{T}\hat{\mathrm{S}} & \xrightarrow{\mathrm{pr}_2} & \mathrm{TS} \\
 & & & & \downarrow \pi_\star & & \downarrow \pi_{\hat{\mathrm{S}}} & & \downarrow \pi_{\mathrm{S}} \\
\downarrow \mathrm{pr}_1 & & \downarrow \mathrm{pr}_1 & & \mathcal{O} \times \{\star\} & \xrightarrow{\psi|_{\mathcal{O}\times\{\star\}}} & \hat{\mathrm{S}} & \xrightarrow{\mathrm{pr}_2} & \mathrm{S} \\
 & & & & \downarrow \mathrm{pr}_1 & & \downarrow \mathrm{pr}_1 & & \| \\
\mathcal{O} & \xrightarrow[\mathbf{1}_{\mathcal{O}}]{} & \mathcal{O} & \xrightarrow[\mathbf{1}_{\mathcal{O}}]{} & \mathcal{O} & \hookrightarrow & \tilde{\mathrm{S}} & \xrightarrow[\mathrm{pr}]{} & \mathrm{S}
\end{array}$$

We need to make a remark here about the arrow $\psi|_{\mathcal{O}\times\{\star\}}\colon \mathcal{O} \times \{\star\} \to \hat{\mathrm{S}}$ that associates $(\tilde{x}, \star)$ with $(\tilde{x}, x = \mathrm{pr}(\tilde{x}))$. For geometric cones, this arrow exists by construction since the apex is alone in its orbit under the pseudogroup of local diffeomorphisms. For formal cones only, the local triviality should be understood in the category of stratified spaces.

Let us now introduce the notation

$$\hat{\alpha} = \mathrm{pr}_2^*(\underline{\alpha}) \in \Omega^k(\mathrm{T}\hat{\mathrm{S}}) \quad \text{and} \quad \tilde{\alpha} = \hat{\alpha} \restriction \mathrm{T}\hat{\mathrm{S}}_{\mathrm{reg}} = \mathrm{pr}_2^*(\alpha) \in \Omega^k(\mathrm{T}\hat{\mathrm{S}}_{\mathrm{reg}}).$$

Decompose $\mathrm{F}_{\mathrm{reg}}$ in terms of connected components, giving

$$\mathcal{O} \times \mathrm{F}_{\mathrm{reg}} = \mathcal{O} \times \coprod_a \mathrm{F}^a_{\mathrm{reg}} = \mathcal{O} \times \coprod_a \Sigma^a_{\mathrm{reg}} \times]0, \infty[\quad \text{with } a \in \pi_0(\Sigma_{\mathrm{reg}}).$$

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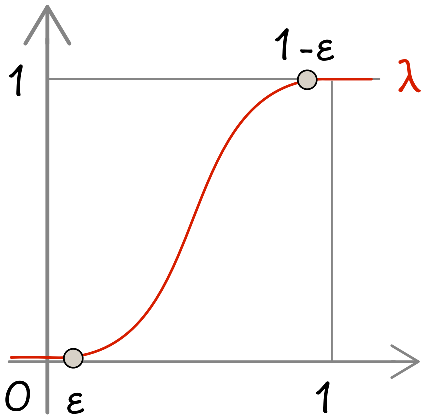

FIGURE 2. The smashing function λ.

We get a two-square diagram summarising the situation.

$$\begin{array}{ccccc}
\mathcal{O} \times \coprod_a \Sigma^a_{\mathrm{reg}} \times\,]0,\infty[& \hookrightarrow & \mathcal{O} \times \mathrm{F} & \xrightarrow{\psi} & \mathrm{T}\hat{\mathrm{S}} \\
\mathrm{pr}_1 \downarrow & & \downarrow \mathrm{pr}_1 & & \downarrow \mathrm{pr}_1 \\
\mathcal{O} & \xrightarrow[\mathbf{1}_{\mathcal{O}}]{} & \mathcal{O} & \hookrightarrow & \tilde{\mathrm{S}}
\end{array}$$

Now consider an n-plot P in $\tilde{\mathrm{S}}$ defined on some open set U and $r_0 \in \mathrm{U}$. There exists an open neighbourhood $\mathrm{V} \subset \mathrm{U}$ of r_0 such that $\mathrm{Q} = \mathrm{P} \restriction \mathrm{V}$ takes its values in some trivialisation of the open set $\mathcal{O}$, which we assume to be connected and simply connected. Let $y \in \mathrm{F}_{\mathrm{reg}} = \coprod_a \Sigma^a_{\mathrm{reg}} \times\,]0,\infty[$, so that $y = \Phi(z,t)$ for some $z \in \Sigma^a_{\mathrm{reg}}$ and $t \in\,]0,\infty[$, where Φ has been introduced in Section 2.2. Consider the (stationary) smooth path

$$\gamma\colon\ s \mapsto \Phi(z, \lambda(s)t),$$

where λ is the smashing function described by Figure 2. Now:

(1) $\gamma(s) = \star$ is the apex of F for all $s \in\,]-\infty, \varepsilon]$;

(2) $\gamma(s) \in \mathrm{F}^a_{\mathrm{reg}}$ for all $s \in\,]\varepsilon, \infty[$; and

(3) in particular, $\gamma(0) = \star$ and $\gamma(1) = y$.

Next, let $\mathrm{Q} \times \gamma$ be the plot $(r,s) \mapsto (\mathrm{Q}(r), \gamma(s))$, into $\mathcal{O} \times \mathrm{F}$, when (r,s) runs over $\mathrm{V} \times \mathbf{R}$. Fix $(u_i)_{i=1}^k$ with $u_i \in \mathbf{R}^n$ and $(\tau_i)_{i=1}^k$ with $\tau_i \in \mathbf{R}$. Consider the real smooth function defined on $\mathrm{V} \times \mathbf{R}$,

$$f(r,s) = \hat{\alpha}(\psi \circ (\mathrm{Q} \times \gamma))_{(r,s)}(u_i, \tau_i)_{i=1}^k.$$

Indeed, $\psi \circ (\mathrm{Q} \times \gamma)$ is a plot in $\mathcal{O} \times \mathrm{Y}$, ψ is a local diffeomorphism into $\mathrm{T}\hat{\mathrm{S}}$, $\hat{\alpha}$ is a k-form on $\mathrm{T}\hat{\mathrm{S}}$ and the $(u_i, \tau_i)_{i=1}^k$ have been fixed.

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(1) First, consider $s \in \,]-\infty, \varepsilon[$, so that $\gamma(s) = \star$. Set $\tilde{x}_r = \mathrm{Q}(r)$, so that we have $\psi(\mathrm{Q}(r), \gamma(s)) = \psi(\tilde{x}_r, \star) = (\tilde{x}_r, x_r)$, with $x_r = \mathrm{pr}(\tilde{x}_r)$. Then

$$\begin{aligned} f(r,s) &= \hat{\alpha}((r,s) \mapsto \psi(\tilde{x}_r, \star))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \hat{\alpha}((r,s) \mapsto (\tilde{x}_r, x_r))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \hat{\alpha}((r,s) \mapsto r \mapsto (\tilde{x}_r, x_r))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \mathrm{pr}_1^*(\hat{\alpha}(r \mapsto (\tilde{x}_r, x_r)))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \hat{\alpha}(r \mapsto (\tilde{x}_r, x_r))_r(u_i)_{i=1}^k \\ &= \underline{\alpha}(r \mapsto x_r)_r(u_i)_{i=1}^k \quad \text{because } \hat{\alpha} = \mathrm{pr}_2^*(\underline{\alpha}) \\ &= \underline{\alpha}(\mathrm{pr} \circ \mathrm{Q})_r(u_i)_{i=1}^k. \end{aligned}$$

Therefore

$$f \restriction \mathrm{V} \times \,]-\infty, \varepsilon[\, = \mathrm{pr}^*(\underline{\alpha})(\mathrm{Q})_r(u_i)_{i=1}^k.$$

(2) Next, consider $s \in \,]\varepsilon, +\infty[$. Then $\gamma(s) \in \mathrm{F}^a_{\mathrm{reg}}$, $\mathrm{Q} \times \gamma$ is a plot in $\mathcal{O} \times \mathrm{F}^a_{\mathrm{reg}}$ and $\psi \circ (\mathrm{Q} \times \gamma)$ is a plot of $\{\mathrm{T}\hat{\mathrm{S}}\}_a$, the connected component of $\mathrm{T}\hat{\mathrm{S}} = \mathrm{pr}^*(\mathrm{TS})$ associated with the component $\mathrm{F}^a_{\mathrm{reg}} = \Sigma^a_{\mathrm{reg}} \times \,]0, \infty[$ (see [1, Section 5, Step 1]). Hence, according to [1, (♠)], $\hat{\alpha} \restriction \{\mathrm{T}\hat{\mathrm{S}}\}_a = \mathrm{pr}_1^*(\bar{\alpha}_a)$, where $\bar{\alpha}_a \in \Omega^k(\tilde{\mathrm{S}})$. Thus

$$\begin{aligned} f(r,s) &= \hat{\alpha}((r,s) \mapsto \psi(\tilde{x}_r, \gamma(s)))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \mathrm{pr}_1^*(\bar{\alpha}_a)((r,s) \mapsto \psi(\tilde{x}_r, \gamma(s)))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \bar{\alpha}_a(\mathrm{pr}_1 \circ [(r,s) \mapsto \psi(\tilde{x}_r, \gamma(s))])_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \bar{\alpha}_a((r,s) \mapsto \mathrm{pr}_1 \circ \psi(\tilde{x}_r, \gamma(s)))_{(r,s)}(u_i, \tau_i)_{i=1}^k \\ &= \bar{\alpha}_a((r,s) \mapsto \tilde{x}_r)_{(r,s)}(u_i, \tau_i)_{i=1}^k \quad \text{since } \mathrm{pr}_1 \circ \psi = \mathrm{pr}_1 \\ &= \bar{\alpha}_a(r \mapsto \tilde{x}_r)_r(u_i)_{i=1}^k. \end{aligned}$$

Therefore

$$f \restriction \mathrm{V} \times \,]\varepsilon, \infty[\, = \bar{\alpha}_a(\mathrm{Q})_r(u_i)_{i=1}^k.$$

(3) In conclusion, $f_r : s \mapsto f(r,s)$ is a smooth function defined on $\mathbf{R}$ for all $r \in \mathcal{O}$. On the interval $]-\infty, \varepsilon[$, f_r is constant and equal to $\mathrm{pr}^*(\underline{\alpha})(\mathrm{Q})_r(u_i)_{i=1}^k$, and on $]\varepsilon, -\infty[$, f_r is constant and equal to $\bar{\alpha}_a(\mathrm{Q})_r(u_i)_{i=1}^k$. Since f_r is smooth, then continuous, these two constants must be equal. Since that is true locally for any r and any vectors u_i in $\mathbf{R}^n$,

$$\bar{\alpha}_a = \mathrm{pr}^*(\underline{\alpha} \restriction \mathrm{S})$$

for all indices a. That is, $\nu(\mathrm{S}) = 1$. This applies to every stratum that is in the closure of $\mathrm{X}_{\mathrm{reg}}$. Since, by definition, $\mathrm{X}_{\mathrm{reg}}$ is dense in X, it follows that $\nu(\mathrm{S}) = 1$ for all strata. □

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SERAP GÜRER, Galatasaray University, Ortaköy,
Çirağan Cd. No: 36, 34349 Beşiktaş, Istanbul, Turkey
e-mail: sgurer@gsu.edu.tr

PATRICK IGLESIAS-ZEMMOUR, Aix Marseille Univ.,
CNRS, Centrale Marseille, I2M, Marseille, France
and
Einstein Institute of Mathematics Edmond J. Safra Campus,
The Hebrew University of Jerusalem Givat Ram. Jerusalem, 9190401, Israel
e-mail: piz@math.huji.ac.il