\[
\sum_{i=1}^{5} i^2 F_i = 25F_7 - 9F_8 + 2F_9 - 8 = 25 \times 13 - 9 \times 21 + 2 \times 34 - 8 = 196.
\]

Using identity (4), a formula for \( \sum_{i=1}^{n} (2i - 1)^2 F_{2i - 1} \) can be developed:

\[
\sum_{i=1}^{n} (2i - 1)^2 F_{2i - 1} = (4n^2 - 4n + 9)F_{2n} - 8nF_{2n-1}. \tag{10}
\]

Using the identity \( \sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1 \) [2, 3], it is a good exercise to verify that

\[
\sum_{i=1}^{n} i^2 F_{2i} = n^2 F_{2n+1} - (2n - 1)F_{2n} + 2F_{2n-1} - 2. \tag{11}
\]

Formulas 9-11 have analogous results to Lucas numbers:

\[
\sum_{i=1}^{n} i^2 L_i = n^2 L_{n+2} - (2n - 1)L_{n+3} + 2L_{n+4} - 18 \tag{12}
\]

\[
\sum_{i=1}^{n} (2i - 1)^2 L_{2i - 1} = (2n - 1)^2 L_{2n} - 8(n - 1)L_{2n-1} + 8L_{2n-2} - 18 \tag{13}
\]

\[
\sum_{i=1}^{n} i^2 L_{2i} = n^2 L_{2n+1} - (2n - 1)L_{2n} + 2L_{2n-1} \tag{14}
\]

References
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85.10 The Secret Santa problem revisited

Introduction
In [1] the authors ask and answer the question ‘Among those permutations of \( n \) different objects having no 1-cycles (no invariant elements), what fraction have no 2-cycles?’ Here is a different, more general way to come to their conclusion.

If \( T_n \) is the number of permutations having no 1 or 2-cycles, then \( T_0 = 1 \) and \( T_1 = T_2 = 0 \) (by inspection), and, for \( n > 2 \),
\( T_n = (n - 1)T_{n-1} + (n - 1)(n - 2)T_{n-3} \).

To see this, note that, if the \( n \) distinct objects are the integers 1 through \( n \), then any admissible permutation can be found in exactly one of two distinct ways:

1. Take any of the admissible permutations of the integers \( \{1,2,\ldots,n-1\} \) and, in its representation as a product of cycles, insert \( n \) immediately after any of the \( n - 1 \) integers.
2. Take any admissible permutation of the \( n - 3 \) integers obtained by removing two numbers from the set \( \{1,2,\ldots,n-1\} \) (giving \( \binom{n-1}{2} \) \( T_{n-3} \) possibilities) and attach a 3-cycle consisting of the last two deleted numbers and \( n \) (which can be chosen in two ways).

If \( A_n = T_n/n! \), then \( nA_n = (n - 1)A_{n-1} + A_{n-3} \) as the authors in [1] remark. Letting \( A(x) = \sum_{n\geq 3} A_n x^n \) you find \((1 - x)A'(x) = x^2(1 + A(x))\). Solving the differential equation,

\[
A(x) = -1 + \exp\left[-(x + \frac{1}{2}x^2)\right].
\]

Now \( A(z) \), conceived as a function of a complex variable \( z \), has a pole at \( z = 1 \) with a residue of \( -e^{-3/2} \). Thus \( A(z) + e^{-3/2}/(z - 1) \) has no poles in the finite plane (is an entire function) and hence the coefficients in its power series about the origin (which are \( A_n - e^{-3/2} \)) tend to zero as \( n \) tends to infinity (for the series converges in \( |z| < R \) for arbitrary \( R \)). Therefore \( A_n \rightarrow e^{-3/2} \) as \( n \rightarrow \infty \). But \( D_n \), the number of permutations with no 1-cycles, is asymptotic to \( e^{-1}n! \). From which it follows that \( T_n/D_n \rightarrow e^{-1/2} \) as noted in [1].

References

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85.11 An old limit revisited

Every calculus student knows the limit \( e = \lim_{h \to 0} (1 + h)^{1/h} \) or that, \( e = (1 + h)^{1/h} \) ‘when \( h \) is small’. A natural question is then ‘how small does \( h \) have to be?’ If \( h \) has to be very small then that is a problem: a calculator (even a computer) finds it hard to distinguish between \( h \) and 0.

We start by finding a series expansion of \((1 + h)^{1/h}\). Taking the natural logarithm of \((1 + h)^{1/h}\), we have

\[
\log(1 + h)^{1/h} = \frac{1}{h} \log(1 + h) = 1 - \frac{h}{2} + \frac{h^2}{3} - \ldots \quad (1)
\]