NATURAL PARTIAL ORDERS
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1. Introduction. Let $n$ be an ordinal. A partial ordering $P$ of the ordinals $T = T(n) = \{w: w < n\}$ is called natural if $x P y$ implies $x < y$.

A natural partial ordering, hereafter abbreviated NPO, of $T(n)$ is thus a coarsening of the natural total ordering of the ordinals. Every partial ordering of a finite set $S$ is isomorphic to a natural partial ordering. This is a consequence of the theorem of Szpilrajn (5) which states that every partial ordering of a set may be refined to a total ordering. In this paper we consider only natural partial orderings. In the first section we obtain theorems about the lattice of all NPO's of $T(n)$. In the succeeding sections, for $n \leq \omega$ we associate a subgroup of the upper triangular group $TL(n)$ of $n \times n$ matrices. We obtain necessary and sufficient conditions for an NPO to be associated with a normal subgroup of $TL(n)$; we show that these "normal" NPO's form a distributive sublattice and for finite $n$ we count the order of this lattice. We show that this number,

$$\binom{2n}{n} / (n + 1),$$

is exactly the number of non-isomorphic partial orderings $P$ of $\{0, \ldots, n - 1\}$ which do not have a sub-partially ordered set like either of those shown in Figure 1.

Since a relation $P$ on $T$ is a subset of $T \times T$ we shall use whichever of the notations $(x, y) \in P$, $x P y$ or $(x, y) \in P$, $x \sim P y$ seems most convenient. The collection of NPO's of $T$ will be denoted by $\mathfrak{N}$ or by $\mathfrak{N}(n)$ if it is important.

Received June 17, 1966. This research was partially sponsored by National Science Foundation Grant GP-3909.
to emphasize the ordinal \( n \). It is natural to order \( R \) itself by set inclusion: we write \( P \subseteq Q \) if \( x P y \) implies \( x Q y \) for all \((x, y)\) in \( T \times T\). For any ordinal \( n \) we have the minimal ordering \( \phi = \{(x, x) : x < n\} \) and the maximal ordering \( \Omega = \{(x, y) : x < y < n\} \). Clearly \( \phi \leq P \leq \Omega \) for every \( P \). If \( x < y \) we define the NPO \( \Gamma(x, y) = \{(r, s) : r = s \text{ or } (r, s) = (x, y)\} \), which is easily seen to be a minimal element over \( \phi \). In fact the following theorem is easily proved.

**Theorem 1.1.** For any ordinal \( n \), the set \( R(n) \) of all natural partial orderings of \( T(n) = \{w : w < n\} \) is a complete, compactly generated* point lattice in which the set of compact elements is the set of points \( \{\Gamma(x, y) : 0 \leq x < y < n\} \).

We shall omit the proof of this theorem and the following standard useful result.

**Lemma 1.1.** If \( R \) is a subset of \( R \), then \((x, y) \in \bigcup R \) if and only if there exists an integer \( r \) and a sequence \( x = u_1 < \ldots < u_r = y \) such that for each \( i, 1 \leq i < r \), there exists \( R_i \in R \) such that \( u_i R_i u_{i+1} \).

Figure 2 shows \( R(3) \).

The next sequence of lemmas characterizes coverings in \( R(n) \) and shows that the lattice is lower semi-modular.

**Lemma 1.2.** Let \( P \in R \). Let \( x < y \) and \((x, y) \notin P \). If \( Q = P \cup \Gamma(x, y) \) and \((u, v) \neq (x, y)\), then \( u Q v \) if, and only if, \( u P v \) or \( u P x \) and \( y P v \).

*For a definition of “compactly generated” see (2).
Proof. Clearly either of these two conditions is sufficient for \( u \leq v \). Let us suppose that \( u \leq v \) by means of the sequence \( u = u_1 R_1 u_2 \ldots R_{i-1} u_i = v \) where \( R_i = \Gamma(x, y) \). First we may assume without loss of generality that \( u_i \neq u_{i+1} \). Now it follows that \( u_i < u_{i+1} \) and so for at most one index \( i \) we can have \( R_i = \Gamma(x, y) \). Indeed if \( u_i \Gamma(x, y) u_{i+1} \) and \( u_i \neq u_{i+1} \), then \( u_i = x \); the condition \( u_i < u_{i+1} \) guarantees that at most one term is \( x \). If no \( R_i = \Gamma(x, y) \), then \( u P v \). If \( R_i = \Gamma(x, y) \), then
\[
  u = u_1 P u_i, \quad u_i = x \Gamma(x, y) y = u_{i+1} P u_r = v
\]
and the necessity is proved.

Clearly \( P = \bigcup \{ \Gamma(x, y) : x P y \} \) but more generally if \( P \subseteq Q \), then \( Q = P \cup \{ \Gamma(x, y) : (x, y) \in P - Q \} \).

Lemma 1.3. If \( P \) and \( Q \) belong to \( \mathfrak{R} \), then \( P \) is covered by \( Q \) if, and only if, \( P - Q = \{(x, y)\} \).

Proof. The sufficiency is clear since \( Q \) differs from \( P \) by exactly one ordered pair. Conversely, suppose that \( Q \) covers \( P \) and that \( (x, y) \) and \( (u, v) \in Q - P \). Clearly \( P \subseteq P \cup \Gamma(x, y) \subseteq Q \). Thus \( P \cup \Gamma(x, y) = Q \) and so
\[
  (u, v) \in P \cup \Gamma(x, y).
\]
From Lemma 1.2, \( u P v \) or \( u P x \) and \( y P v \). By assumption, the former cannot hold; hence the latter condition does and \( u \leq x \) and \( y \leq v \). Similarly we have \( x \leq u \) and \( v \leq x \); thus \( (u, y) = (x, y) \).

To establish lower semi-modularity we need a better characterization of a covering.

Lemma 1.4. If \( Q = P \cup \Gamma(x, y) \) covers \( P \) in \( \mathfrak{R} \), then \( y \) is a \( P \)-maximal element among \( A(x) = \{ z : (x, z) \in P \} \); dually, \( x \) is a \( P \)-minimal element among \( B(x) = \{ z : (z, y) \notin P \} \).

Proof. Suppose there were a \( z \) such that \( x \sim P z \) and yet \( y P z \). Since we have \( (x, z) \in P \cup \Gamma(x, y) \), then \( P \subseteq P \cup \Gamma(x, y) \subseteq P \cup \Gamma(x, y) \). But since \( P \cup \Gamma(x, y) \) covers \( P \), it must be that \( P \cup \Gamma(x, z) = P \cup \Gamma(x, y) \), or that \( (x, y) \in P \cup \Gamma(x, z) \). But from Lemma 1.2 this entails \( x P x \) and \( z P y \); thus \( z = y \).

Lemma 1.4 gives us a way to find all the covers, if any exists, for an arbitrary \( P \in \mathfrak{R} \). We may proceed as follows. Let \( (u, v) \notin P \) and \( u \leq v \). Choose a \( P \)-maximal element \( y \) from \( A(u, v) = \{ z : u \sim P z \} \). Having chosen \( y \), choose a minimal element \( x \) from \( B(u, y) = \{ z : z P u \} \). Having chosen \( y \), choose a minimal element \( x \) from \( B(u, y) = \{ z : z P u \} \). We can now show that \( P \cup \Gamma(x, y) \) covers \( P \). We must show that \( y \) is a maximal element in \( A(x) \). In the contrary case, as in Figure 3, suppose that \( y P z \)
and \( x \sim P z \). It follows that \( u \sim P z \) for otherwise \( x P u \) implies \( x P z \), a contradiction. But \( u \sim P z \) and \( v P z \) implies that \( z \in A(u, v) \), while \( y P z \) contradicts the choice of \( y \). Similarly it follows that \( x \) is a \( P \)-minimal element in \( B(y) \).

For infinite ordinals, \( P \) need have no covering element. Indeed, for \( T(\omega + 1) = \{0, 1, 2, \ldots, \omega\} \), if \( P \) is as shown in Figure 4, then \( Q \supseteq P \) implies that \( P - Q \) has the form \( \{(0, m) : m > m_0\} \).

**Theorem 1.2.** The lattice \( \mathbb{R} \) of natural partial orderings is lower semi-modular in the sense that if \( Q \) is covered by \( P \cup Q \), then \( P \) covers \( P \cap Q \).

**Proof.** Let \( P \cup Q = Q \cup \Gamma(u, v) \). We shall show that \( P - P \cap Q = \{(u, v)\} \). We may suppose that \( P \not\subseteq Q \) and so \( P - Q \cap P \) is not empty. Now suppose that \( (r, s) \in P - (P \cap Q) \). Thus \( (r, s) \in Q \); however, \( (r, s) \in P \subset Q \cup \Gamma(u, v) \). Thus \( r Q u \) and \( v Q s \). Now \( u \sim Q s \) (otherwise \( r Q s \)) and dually \( r \sim Q v \). Thus \( s \in A(u) \) and \( r \in B(v) \), but by Lemma 1.4, \( v \) is a \( Q \)-maximal element in \( A(u) \) and \( u \) is a \( Q \)-minimal element in \( B(v) \). From \( v Q s \) and \( r Q u \) we conclude that \( (r, s) = (u, v) \). Thus \( P - P \cap Q = \{(u, v)\} \).
The lattice \( R(n) \) is not complemented and the next result is a necessary condition for an element of \( R(n) \) to possess a complement. If \( n \leq \omega \), this condition is also sufficient. Necessary and sufficient conditions in the general case are not known.

**Lemma 1.5.** A necessary condition for an NPO, \( P \), to have a complement in \( R(n) \) is

\[ (*) \quad \text{For all } x, y \text{ such that } x P y, \text{ either } y = x + 1 \text{ or there exists } (u, v) \neq (x, y) \text{ with } x \leq u < v \leq y \text{ and } u P v. \]

**Proof.** Suppose that \( Q \) is a complement for \( P \) and that \( x P y \) and \( y \neq x + 1 \). If there is a \( w \) such that \( x < w < y \) and \( x P w \) or \( w P y \), then \( (*) \) holds. In the contrary case, suppose that for all \( w \) such that \( x < w < y \), then \( x \sim P w \) and \( w \sim P y \). However, since \( x (P \cup Q) w \), it follows that either \( x Q w \) or there exists \( u \) and \( v \) such that \( x < u P v \) and \( (*) \) holds. Similarly from \( w (P \cup Q) y \), \( w P y \) or the condition \( (*) \) is satisfied. However, if \( x Q w \) and \( w Q y \), then \( x Q y \) and hence \( (x, y) \in P \cap Q \neq \phi \), a contradiction. Thus \( (*) \) is necessary.

**Theorem 1.3.** If \( n \leq \omega \), there exists a complement for \( P \) in \( R(n) \) if and only if \( (*) \) holds.

**Proof of the sufficiency of \( (*) \).** Let \( \hat{P} = \{(x, y) : x = y \text{ or } (x < y \text{ and for all } w, x \leq w < y \text{ implies } w \sim P (w + 1)\}\}. \) We claim that \( \hat{P} \) is an NPO. Clearly it is reflexive and antisymmetric. To prove transitivity we suppose that \( x \hat{P} y \) and \( y \hat{P} z \). Suppose that \( x < y \) and that there exists \( w \) such that \( x < w < z \) and \( w P (w + 1) \). By assumption \( y \hat{P} z \), and thus \( w < y \), a contradiction of \( x \hat{P} y \). Thus \( \hat{P} \) is a partial ordering.

To prove that \( P \cap \hat{P} = \phi \) we note that since \( n \leq \omega \), if \( x P y \), then \( y \) is finite and a finite number of applications of \( (*) \) implies that there exists \( w \) such that \( x \leq w < y \) and \( w P (w + 1) \). Thus \( x \sim \hat{P} y \). We prove that \( P \cup \hat{P} = \Omega \) by showing by induction on \( y - x \) that if \( x \leq y \), then \( x \sim \hat{P} y \). If \( y - x = 1 \), then either \( x P y \) or \( x \hat{P} y \) by definition. In general, if \( x \sim P y \), either \( w \sim P (w + 1) \) for all \( w \) such that \( x < w < y \), in which case \( x \hat{P} y \), or there exist \( w \) such that \( w P (w + 1) \). But then, by induction, \( x \sim (P \cup \hat{P}) w \) and \( (w + 1) \sim (P \cup \hat{P}) y \); hence \( x \sim (P \cup \hat{P}) y \).

We remark that \( \hat{P} = \bigcup \{ \Gamma(u, u + 1) : u \sim P (u + 1) \} \) and thus if \( Q \) is any complement for \( P \), it follows that \( Q \supseteq \hat{P} \). Condition \( (*) \) is not sufficient if \( \omega < n \) as the example of Figure 5 shows.

2. The upper triangular group. For this and subsequent sections we require that \( n \leq \omega \). It will be convenient to let \( T \subseteq \{1, 2, \ldots \} \) and \( T(n) = \{1, 2, \ldots, n\} \). With each NPO, \( P \), of \( T \) we associate a group \( G_P \) of non-singular \( n \times n \), upper triangular matrices with entries in GF(2). Actually the field plays no role in our considerations; we could have used an arbitrary field, but
by choosing GF(2), all our entries are either 0 or 1 and certain of our calculations are easier. The group of all non-singular upper triangular matrices we shall denote by TL(n). In this section we show that the mapping \( P \to G_P \) is a lattice isomorphism of \( \mathfrak{M}(n) \) into TL(n).

**Definition.** Let \( P \in \mathfrak{M}(n) \). Let \( G_P = \{ M = (m_{ij}) : m_{ii} = 1, \text{ and if } i \sim P j \text{ then } m_{ij} = 0 \} \). In particular, note that \( G_P \subseteq TL(n) \).

Clearly every matrix in \( G_P \) has 1 on the main diagonal and the identity matrix \( I \in G_P \) for all \( P \). Only the entries which must be 0 have been specified by \( P \).

**Theorem 2.1.** For all natural partial orderings, \( P, G_P \), is a group.

**Proof.** Let \( A = (a_{ij}) \) and \( B = (b_{jk}) \) belong to \( G_P \). Let \( AB = C = (c_{ik}) \). Thus

\[
C_{ik} = \sum_{1 \leq j \leq k} a_{ij} b_{jk} = \sum_{1 \leq j \leq k < l} a_{ij} b_{jk}.
\]

If \( i \sim P k \), then clearly for each \( j \) either \( i \sim P j \) or \( j \sim P k \); thus \( c_{ik} = 0 \) and hence \( C \in G_P \). If \( n \) is finite, closure of course suffices to establish that \( G_P \) is a group. If \( n = \omega \), then we must also show that if \( A \in G_P \), then \( A^{-1} \in G_P \).

Let \( A = (a_{ij}) \) and \( A^{-1} = (b_{jk}) \); we do know that \( A^{-1} \in TL(n) \). We shall show, by induction on \( (k - i) \), that if \( i \sim P k \), then \( b_{ik} = 0 \). For \( (k - i) = 0 \) the result is vacuously true. Suppose then that \( i \sim P k \) while if \( s - r < (k - i) \) and \( r \sim P s \), then \( b_{rs} = 0 \). From \( I = AA^{-1} \) we have

\[
0 = \delta_{ik} = \sum_{t < j \leq k} a_{ij} b_{jk}.
\]

When \( i < j \) the term \( a_{ij} b_{jk} \) is zero for if \( a_{ij} = b_{jk} = 1 \), then \( i P j \) and by induction \( j P k \); hence \( i P k \), a contradiction. Thus \( 0 = a_{ij} b_{ik} = b_{ik} \).

It is easy to see that \( G_\Phi \) is the identity group and that if \( P = \Gamma(x, y) \), then \( G_P \), which we denote by \( G(x, y) \), is a group of order 2 whose non-identity
element is $I + E(x, y)$, where $E(x, y)$ is the $n \times n$ matrix whose only non-zero entry is in row $x$ and column $y$. For our next result we need the familiar fact that $E(x, y)E(u, v) = \delta_{yu} E(x, v)$.

**Theorem 2.2.** Let $n$ be a finite ordinal. Let $P$ be a natural partial ordering of $T(n)$. The group $G_P$ is generated by \{ $I + E(x, y)$: $x P y$, $x \neq y$ \}; indeed $G_P$ is the subgroup union of the subgroups \{ $G(x, y)$: $x P y$, $x \neq y$ \}.

**Proof.** Since it is evident that if $x \neq y$ and $x P y$, then $I + E(x, y) \in G_P$, we have only to prove that $G_P \subseteq \bigcup \{ G(x, y): x P y, x \neq y \}$. Now it is clear that if $M = (m_{ij}) \in G_P$, then $M = I + \sum E(u, v)$, the sum ranging over precisely those pairs $(u, v)$ such that $u \neq v$ and $m_{uv} \neq 0$. First of all, note that if $m_{ij} \neq 0$, then $i P j$, and so $I + E(i, j)$ belongs to the set the theorem alleges to be a generating set for $G_P$. We shall show that if $M \in G_P$, then $M \in \bigcup \{ G(x, y): x P y, x \neq y \}$ by an induction on the number $\lambda$ of non-zero entries in $M$. Note that $\lambda$ is the sum of $n$ and the number of terms in $\sum E(u, v)$. If $\lambda = n$, then $M = I$ and the result holds. In general we may write

$$M = I + \sum (u, v) = I + \sum_{(u,v) \neq (r,s)} E(u, v) + E(r, s),$$

where $(r, s)$ is chosen so that $r$ is a minimal value occurring among the first coordinates of the pairs occurring in the sum. (Equivalently, $r$ is a minimal row index occurring among all non-zero entries in $M$ lying off the main diagonal.) Now we compute

$$\left( I + \sum_{(u,v) \neq (r,s)} E(u, v) \right) (I + E(r, s)) =$$

$$I + \sum_{(u,v) \neq (r,s)} E(u, v) + E(r, s) + \sum_{(u,v) \neq (r,s)} \delta_{sr} E(u, v).$$

However, in the last sum, $\delta_{sr} = 0$ for all $v$ since $r \leq u < v$ by the choice of $r$. Thus

$$M = \left( I + \sum_{(u,v) \neq (r,s)} E(u, v) \right) (I + E(r, s)).$$

By the induction hypothesis the first factor belongs to $\bigcup \{ G(x, y): x P y, x \neq y \}$ as does the second; hence $M$ does as well. Thus the theorem holds.

**Theorem 2.3.** The mapping $P \rightarrow G_P$ is a lattice isomorphism from $\mathcal{H}(n)$ into a sublattice of the lattice of all subgroups of $TL(n)$. Indeed $G_{P \wedge Q} = G_P \cap G_Q$ and $G_{P \vee Q} = G_P \cup G_Q$.

**Proof.** First note that $P \subseteq Q$ if and only if $G_P \subseteq G_Q$. This is an immediate corollary of the preceding theorem and from this remark we learn that the mapping $P \rightarrow G_P$ is one-to-one. The equality $G_{P \wedge Q} = G_P \cap G_Q$ is easily seen to hold, primarily because both $P \cap Q$ and $G_P \cap G_Q$ are set intersections.
Because the mapping is order-preserving, to complete the proof we need only show that \( G_{P \cup Q} \subseteq G_P \cup G_Q \). From the preceding theorem it will suffice to show that \( a (P \cup Q) b \) and \( a \neq b \) implies that \( I + E(a, b) \in G_P \cup G_Q \). We begin by computing the commutator:

\[
C(x, y; u, v) = [(I + E(x, y)), (I + E(u, v))] = [(I + E(x, y))(I + E(u, v))]^2 \\
= (I + E(x, y) + E(u, v) + \delta_{yu} E(x, v))^2 \\
= I + E(x, y) + E(u, v) + \delta_{yu} E(x, v) + E(x, y) \\
+ 0 + \delta_{yz} E(u, y) + 0 + E(u, v) + \delta_{yu} E(x, v) + 0 + 0 \\
+ \delta_{yu} E(x, v) + 0 + 0 + 0
\]

\[
= \begin{cases} 
I & \text{if } y \neq u \text{ and } v \neq x, \\
I + E(u, y) & \text{if } v = x \text{ and } u \neq y, \\
I + E(x, v) & \text{if } v \neq x \text{ and } u = y.
\end{cases}
\]

Now we shall show that if \( a = u_1 < u_2 < \ldots < u_r = b \) is a sequence such that \( u_i R_i u_{i+1} \) where \( R_i = P \) or \( Q \), then \( I + E(a, b) \in G_P \cup G_Q \). We proceed by induction on \( r \). If \( r = 1 \), then \( I + E(a, b) \in G_P \) or \( G_Q \). If \( r = 2 \), then \( I + E(a, b) = C(u_1, u_2; u_2, u_3) \in G_P \cup G_Q \). Thus, in any event we have that \( I + E(u_1, u_3) \in G_P \cup G_Q \). By induction \( I + E(u_3, u_r) \in G_P \cup G_Q \) and thus \( G_P \cup G_Q \) contains \( C(u_1, u_3; u_3, u_r) = I + E(a, b) \).

**Definition.** Let \( P \) and \( Q \) be natural partial orderings of \( T(n) \). \( P \) and \( Q \) are called **isomorphic** (\( P \cong Q \)) if there is a permutation \( w \) of \( T(n) \) such that \( i P j \) if and only if \( w(i) Q w(j) \). We say that \( P \) is **unique** if \( P \cong Q \) implies that \( P = Q \). \( P \) and \( Q \) are called **dual** provided \( i P j \) if and only if \( (n + 1 - j) Q (n + 1 - i) \).

We note that \( \Gamma(x, y) \) and \( \Gamma(u, r) \) are always isomorphic while \( \phi \) and \( \Omega \) are unique. In Figure 2, NPO’s \( A \) and \( B \) are unique and mutually dual.

**Theorem 2.4.** If \( P \) and \( Q \) are isomorphic, then \( G_P \) and \( G_Q \) are isomorphic, indeed \( G_Q = \Pi^{-1}G_P \Pi \) if \( \Pi \) is the permutation matrix corresponding to the permutation \( \pi \) which takes \( P \) into \( Q \).

**Proof.** We need only verify that in general if \( \Pi \) is the permutation matrix \( \Pi = (\pi_{ij}) \), where \( \pi_{ij} = 1 \) if and only if \( \pi(i) = j \), then

\[
\Pi^{-1}E(x, y)\Pi = E(\pi(x), \pi(y)).
\]

**Theorem 2.5.** If \( P \) and \( Q \) are dual, then the groups \( G_P \) and \( G_Q \) are isomorphic.

**Proof.** It is easy to verify that the map

\[
I + \sum E(x, y) \to I + E(n + 1 - y, n + 1 - x)
\]

is an anti-isomorphism of \( G_P \) onto \( G_Q \) and so the mapping

\[
I + \sum E(x, y) \to [I + \sum E(n + 1 - y, n + 1 - x)]^{-1}
\]

is the desired group isomorphism.

https://doi.org/10.4153/CJM-1968-055-7 Published online by Cambridge University Press
3. Normal natural partial orders. In this section we determine necessary and sufficient conditions on $P$ such that $G_P$ is normal in $TL(n)$; moreover, Theorem 3.7 shows that if $P$ has no restriction to a subset yielding either of the four-element abstract partially ordered sets of Figure 1, then there exists $Q \cong P$ such that $G_Q$ is normal in $TL(n)$. The sublattice of normal NPO's is shown to be distributive.

Definition. A natural partial ordering $P$ of $T(n)$ is called normal if $G_P$ is a normal subgroup of $TL(n)$.

Theorem 3.1. A natural partial ordering $P$ is normal if and only if the following two conditions hold:

1. If $i P k$, $i \neq k$, and $r < i$, then $r P k$.
2. If $k P i$, $i \neq k$, and $s > i$, then $k P s$.

(This theorem asserts that $P$ is normal if and only if it has no configurations like those in Figure 6.)

Proof of Theorem 3.1. $G_P$ is normal in $TL(n)$ if and only if for all pairs $(i, k)$ such that $i \neq k$ and $i P k$, and for all pairs $(r, s)$ such that $r < s$, it is the case that $(I + E(r, s))(I + E(i, k))(I + E(r, k)) \in G_P$.

A direct calculation shows this product to be equal to

\[
I + E(i, k) \quad \text{if } r \neq k \text{ and } s \neq i,
I + E(i, k) + E(r, k) \quad \text{if } s = i,
I + E(i, k) + E(i, s) \quad \text{if } k = r.
\]

(The case $r = k$ and $s = i$ cannot arise.) Thus $G_P$ is normal if and only if $i P k$ and $r < i$ implies $r P k$ and $i P k$ and $k < s$ implies $i P s$. In Figure 2, the normal partial orders are $\phi$, $\Gamma(1, 3)$, $A$, $B$, and $\Omega$.

Our next theorem characterizes normal NPO's as those that commute with all other natural partial orderings in a product of relations. If $P$ and $Q$ are relations on a set $J$, their product $PQ$ is the set of pairs $\{(x, y) : \text{there exists } z \text{ such that } x P z \text{ and } z Q y\}$. It is well known (1) that if an algebra has permuting congruence relations, then the lattice of permuting congruence relations is modular.

Theorem 3.2. A natural partial ordering $P$ is normal if and only if $PQ = QP$ for all natural partial orderings $Q$. 

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Proof. Let us suppose that \( PQ = QP \) for all \( Q \in \mathcal{R}(n) \). Suppose \( i P k \) and \( r < i \). Let \( Q = \Gamma(r,i) \). Thus \( r Q i \) and \( i P k \). Since \( PQ = QP \), there is an \( x \) such that \( r P x \) and \( x Q k \). But \( x Q k \) implies \( x = k \); hence \( r P k \). Dually we verify that \( i P k \) and \( k < s \) implies \( i P s \). Consequently, suppose that \( P \) is normal. If \( i P k \) and \( k Q s \), then \( i < k < s \) and so \( i P s \) from Theorem 3.1. Hence \( i Q i \) and \( i P s \), and thus \( PQ \leq QP \). Similarly, if \( r Q i \) and \( i P k \), then \( r P k \) so that \( r P k \) and \( k Q k \); thus \( QP \leq PQ \). It is a well-known result (and easily proved) that if relations \( P \) and \( Q \) on a set \( T \) permute, then \( P \cup Q = PQ = QP \).

Theorem 3.3. If either \( P \) or \( Q \) is a normal natural partial ordering then \( x (P \cup Q) y \) if and only if \( x P y \) or \( x Q y \).

Proof. If \( P \) is normal, Theorem 3.2 yields \( P \cup Q = PQ \). Hence, if \( x (P \cup Q) y \), we may suppose that \( x P z \) and \( z Q y \) for some \( z \). If \( z = x \), then \( x Q y \) while if \( z \neq x \), then, since \( z \leq y \), Theorem 3.1 implies that \( x P y \).

Corollary. The normal natural partial orders form a distributive sublattice \( \mathfrak{D}(n) \) of \( \mathcal{R}(n) \).

Proof. From the lattice isomorphism \( P \rightarrow P \) and the fact that the join and meet of normal subgroups is normal follows the fact that the normal partial orders form a sublattice. The lattice is distributive because we have shown that, as relations, the meet and join of normal partial orders are set meet and set join.

Our next theorem establishes the fact that if \( P \) is unique, then \( P \) is a normal NPO.

Theorem 3.4. Let \( P \in \mathcal{R}(n) \). If there is a triple \((i, k; j) \) in \( P \) such that \( j < i \neq k \), \( i P k \), and \( j \sim P k \), then there is \( Q \in \mathcal{R}(n) \) such that \( P \cong Q \) and \( Q \neq P \).

Proof. For a fixed \( k \) let \( C(k) = \{ (s, r) : r P k, s < r \) and \( s \sim P k \} \). By hypothesis, \( C(k) \) is non-empty. Let \( (u, v) \) be chosen in \( C(k) \) such that \((v - u)\) is minimal. Let \( Q \) be the partial ordering obtained by interchanging \( u \) and \( v \) as indicated in Figure 7. Specifically

\[
\begin{align*}
x Q y \text{ if } & \begin{cases}
\{x, y\} \cap \{u, v\} = \emptyset \text{ and } x P y.
\end{cases} \\
x = v \text{ and } u P y \text{ or } y = v \text{ and } x P u.
\end{align*}
\]

(Note that \( u \sim P v \) so that \( u \sim Q v \) and \( v \sim P u \).) We claim that \( Q \) is a natural

\[
\begin{array}{c}
P \quad Q \\
\begin{array}{c}
\circ v \\
\circ u (u \leq v)
\end{array} & \begin{array}{c}
\circ k \\
\circ u (u \leq v)
\end{array}
\end{array}
\]

Figure 7.
partial ordering and so \( Q \cong P \) since the permutation \( \pi = (u, v) \) provides the isomorphism. First we show that \( x Q y \implies x < y \). Since \( u < v \), the questionable cases are \( x = v \) or \( y = u \). Consider first the case \( v Q y \). Thus \( u P y \) and so \( u < y \). If \( y P k \), then \( u P k \), contrary to the choice of \( y \). Thus \( y \sim P k \). If \( y < v \), then \( (y, v) \in C(k) \), but then since \( u < y \), we should have \( v - y < v - u \), contrary to the choice of \((u, v)\). Similarly, suppose that \( x Q u \), that is, \( x P v \). Thus \( x < v \) and if \( u < x \), then \( (u, x) \in C(k) \) while

\[
x - u < v - u,
\]

contrary to the choice of \((u, v)\).

Now it is easy to see that \( Q \) is a partial ordering and since \( u \sim P k \) while \( u Q k \), it is clear that \( P \neq Q \).

**Corollary.** If \( P \) is unique, then \( P \) is normal.

**Proof.** This, together with Theorem 3.1, is essentially the contrapositive statement of the theorem or its dual.

Hereafter we shall suppose that \( n \) is always finite. Our notation is

\[
T = T(n) = \{1, \ldots, n\}.
\]

**Definition.** Let \( P \) be a partial ordering of a set \( S \). The depth of \( x \in S \), denoted \( D_P(x) \) or more briefly \( D(x) \), is defined by

\[
D(x) = \max\{r: \text{there is a chain } x = x_1 P x_2 P \ldots P x_r \text{ such that } x_i \neq x_{i+1}\}.
\]

**Lemma 3.1.** If \( y P x \) and \( y \neq x \), then \( D(y) > D(x) \). If \( D(y) = r \) and \( y = y_1 P y_2 P \ldots P y_r \) is a chain such that \( y_i \neq y_{i+1} \), then \( D(y_i) = r - i + 1 \).

**Definition.** Let \( P \) be a natural partial ordering of \( T(n) \). An element \( x \) is said to be of class 1 if \( y P x \) whenever \( D(y) > D(x) \). The elements of class 1 will be denoted \( C_1 \), or \( C_P 1 \) if it is important to denote the partial ordering.

An interesting result which we shall not prove since we make no use of it is that \( P \) is unique if and only if every element is of class 1. The following result is immediate.

**Lemma 3.2.** Let \( P \) be a natural partial ordering of \( T(n) \). \( x \in C_1 \) if and only if \( y P x \) for all \( y \) such that \( D(y) = D(x) + 1 \).

Now we can easily prove a lemma which turns out to be extremely useful.

**Lemma 3.3.** Let \( P \) be a natural partial ordering of \( T(n) \). Let

\[
H(r) = \{x: D(x) = r\}.
\]

If for some integer \( r \) for which \( H(r) \neq \emptyset \) but \( H(r) \cap C_1 = \emptyset \), then \( P \) contains four elements, \( r, s, u, v \), such that \( r P s, u P v, r \sim P v \), and \( u \sim P s \).

**Proof.** If \( m \) denotes the maximum depth of an element in \( T(n) \), then every element of depth \( m \) is of class 1, since the condition of the definition is vacuous.
Let $r < m$, and for each element $x$ in $H(r + 1)$, let

$$U(x) = \{y : x P y \text{ and } D(y) = r\}.$$ 

Of course $U(x)$ is non-empty. Now select $x_0 \in H(r + 1)$ such that

$$|U(x_0)| \leq |U(x)|$$

for all $x \in H(r + 1)$.

Let $y_0 \in U(x_0)$. We shall show that $y_0 \in C_1$ or the alleged configuration appears in $P$. Suppose that $x \in H(r + 1)$ and $x \sim P y_0$. Consider $U(x)$. If there is a $y \in U(x)$ such that $y \not\in U(x_0)$, then for $x_0, y_0, x, y$ we have $x_0 P y_0, x P y, x_0 \sim P y$ and $x \sim P y_0$. If this configuration is assumed not to occur, then it must be that $U(x) \subseteq U(x_0)$; however since $|U(x_0)| \leq |U(x)|$, it follows that $U(x) = U(x_0)$ and so $x \in H(r + 1)$ must imply $x P y_0$ and so $y_0 \in C_1$ by the previous lemma.

**Lemma 3.4.** If $P$ is a normal NPO, then for all $r$, if $H(r) \neq \emptyset$, then $H(r) \cap C_1 \neq \emptyset$.

**Proof.** Let $H(r) \cap C_1 = \emptyset$, while $H(r) \neq \emptyset$. From the lemma we have that $P$ has a configuration $r P s, u P v, r P v$, and $u P s$. Since $P$ is normal, we must have $r \geq u$, and conversely $u \geq r$; hence $r = u$, a contradiction.

**Lemma 3.5.** Let $P$ be a normal NPO. If $D(x) = D(y)$, $x \in C_1$, and $y \not\in C_1$, then $x > y$. Moreover, if $D(z) > D(w)$, then $z < w$.

**Proof.** Since $y \not\in C_1$, then there must exist $z$ such that $D(z) > D(y)$ and $z \sim P y$. Since $x \in C_1$, we have $z P x$, and now since $P$ is normal, it must be that $y < x$. To prove the second assertion note first that if $w \in C_1$, then $z P w$ and so $z < w$. Thus we may assume that $w \not\in C_1$. From Lemma 3.4 there is an $x \in C_1$ such that $D(x) = D(w)$. Thus $z P x$ and if $w < z$, then from normality it follows that $w P x$; hence $D(w) > D(x)$, a contradiction.

**Theorem 3.5.** Let $P$ be a normal NPO. Let $r$ be an integer such that $H(r) \neq \emptyset$. Then there exist integers, $a \leq b < c$, such that $x \in H(r) \cap C_1$ if and only if $b \leq x < c$ while $x \in H(r) - C_1$ if and only if $a \leq x < b$.

**Proof.** From Lemma 3.4, $H(r) \cap C_1 \neq \emptyset$ and so let $c = \max\{x : x \in H(r) \cap C_1\}$, $b = \min\{x : x \in H(r) \cap C_1\}$, and $a = \min\{x : x \in H(r)\}$. Clearly $D(x) = r$ implies $a \leq r < c$ and Lemma 3.5 shows that if $x \in H(r) - C_1$, then $x < b$. Conversely, suppose first that $b \leq x < c$. If $D(x) > r$, then $x P b$ since $b \in C_1$ and thus $x < b$; hence $r \geq D(x)$. If $D(x) < r = D(b)$, then $c < x$, by Lemma 3.5. Hence $D(x) = r$ and again from Lemma 3.5, since $b \in C_1$, it follows that $x \in C_1$ also. Secondly suppose that $a \leq x < b$. If $D(x) > r = D(a)$, then $x < a$; if $D(x) < r = D(b)$, then $b < r$; hence $D(x) = r$. However, Lemma 3.5 and the choice of $b$ show that $x \not\in C_1$; thus $x \in H(r) - C_1$. 

https://doi.org/10.4153/CJM-1968-055-7 Published online by Cambridge University Press
COROLLARY. Let $P$ be a normal NPO. If $\pi$ is a permutation such that

(1) $\pi$ fixes every element not in $H(1) \cap C1$,
(2) $\pi$ fixes $H(1) \cap C1$,
then $i P j$ if and only if $\pi(i) P \pi(j)$.

Proof. Suppose that $i \neq j$ and $i P j$. Since $i \notin H(1) \cap C1$, $\pi(i) = i$. If $j \notin H(1) \cap C1$, there is nothing to prove. If $j \in H(1) \cap C1$, then $\pi(j) \in H(1) \cap C1$ and so $i P \pi(j)$. The converse follows in a similar manner.

THEOREM 3.6. If $P$ and $Q$ are normal NPO's, then $P \cong Q$ implies that $P = Q$.

Proof. Let $\pi$ be the permutation of $T(n)$ such that $i P j$ if and only if $\pi(i) Q \pi(j)$. Clearly we must have $D(i) = D(\pi(i))$ and $i \in C_P 1$ if and only if $\pi(i) \in C_Q 1$. By Theorem 3.5 the maximal elements in $C_P 1$ and in $C_Q 1$ satisfy $b \leq x \leq n$ for some integer $b$ which must be the same for $P$ and $Q$. Thus $\pi$ can only permute these elements. Now if $b = 1$, the proof is complete. If $1 < b$, then consider the restriction of $P$ and $Q$ to $T(b - 1)$. Let us denote these restrictions by $\hat{P}$ and $\hat{Q}$, respectively. It is easy to verify that $\hat{P}$ and $\hat{Q}$ are normal NPO's of $T(b - 1)$; moreover the restriction $\hat{\pi}$ of $\pi$ to $\{1, \ldots, b - 1\}$ is a permutation which shows that $\hat{P} \cong \hat{Q}$. By induction, $\hat{P} = \hat{Q}$ and hence $P = Q$ since the addition of $\{b, \ldots\}$ to $T(b - 1)$ is the addition of class one elements with respect to both $P$ and $Q$.

COROLLARY. No two normal natural partial orderings are isomorphic.

Our next theorem gives necessary and sufficient conditions for $P$ to be normal based on the sub-partially ordered sets that can be obtained from $P$ by restriction. We begin by noting that if $P$ is normal, then $P$ cannot have configurations like those of Figure 1. We shall now show that the absence of these configurations in $P$ implies that there is an isomorphic NPO, $Q$, such that $Q$ is normal.

The next lemma is an obvious "renumbering" principle which we list for easy reference.

LEMMA 3.6. Let $P$ be an NPO. Let $i, j$ be such that

\[ \{x: x P i, x \neq i\} = \{x: x P j, x \neq j\} \text{ and } \{y: i P y, i \neq y\} = \{y: j P y, j \neq y\}; \]

then $P$ is left invariant under the transposition $\pi = (i, j)$.

Note that if $i$ and $j$ are maximal elements, then the second of these conditions is vacuous.

THEOREM 3.7. Let $P$ be an NPO of $T(n)$. There is an NPO, $Q$, of $T(n)$ such that $P \cong Q$ and $Q$ is normal if and only if, for all sets of four distinct elements $r, s, t, u$

(1) $r P s P t$ implies $u P t$ or $r P u$
and

(2) $r P s$ and $t P u$ implies $r P u$ or $t P s$.  

https://doi.org/10.4153/CJM-1968-055-7 Published online by Cambridge University Press
Proof. If \( P \cong Q \) and \( P \) satisfies conditions (1) and (2), then so does \( Q \). However, if \( Q \) is normal, then, as we have remarked, the conditions are necessary.

Our proof is by induction on \( n \) and for fixed \( n \) on the number of triples \((i, k; j)\) such that \( i P k \), \( i \neq k \) and either \( j < i \) and \( j \sim P k \) or \( j > k \) and \( i \sim P j \). For brevity we call such triples "bad" \( P \)-triples. Of course, if there are no bad \( P \)-triples, then \( P \) is normal.

Our first reduction is to show that we may assume \( n \in C1 \). In any event it follows from condition (2) and Lemma 3.3 that there are depth one elements of class 1; let \( x \) be one such. If \( n \notin C1 \), let \( \pi \) be the transposition \((x, n)\) and let \( R \) be the partial order defined by \( u R v \) if and only if \( \pi^{-1}(u) P \pi^{-1}(v) \). It is easily verified that \( R \) is a partial ordering. There are two interesting cases in the argument to show that \( R \) is natural: \( u = n \) or \( n = x \). If \( n R v \), then \( x P \pi^{-1}(v) \) and as \( D(x) = 1 \), then \( v = n \). If \( u R x \), then \( \pi^{-1}(u) P n \) and as \( x \in C1 \) it follows that either \( D(\pi^{-1}(u)) = 1 \) or \( \pi^{-1}(u) P x \). In the former case we must have \( \pi^{-1}(u) = n \), or \( u = x \), while in the latter \( \pi^{-1}(u) < x \), and hence \( \pi^{-1}(u) = u < x \). It now follows that \( n \in C_{\pi} 1 \), for if \( D(y) > 1 \), then \( y P x \) and \( y = \pi(y) R n \). Hereafter we shall assume that \( n \in C_{\pi} 1 \).

Our second reduction is to assume that \( \hat{P} \), the restriction of \( P \) to \( T(n - 1) \), is normal. Since conditions (1) and (2) hold for \( P \), they hold a fortiori for \( \hat{P} \), and hence by induction there is an NPO, \( \hat{Q} \), isomorphic to \( \hat{P} \). We now wish to adjoin \( n \) to \( T(n - 1) \) and extend \( \hat{Q} \) to \( \hat{Q} \) so that \( Q \cong P \); the restriction of this \( Q \) to \( T(n - 1) \) is clearly \( \hat{Q} \). To do this, let \( \pi \) be the permutation mapping \( P \) onto \( \hat{Q} \).

We omit the verification that \( Q \) is an NPO and that if \( \pi \) is extended to a permutation of \( T(n) \) by defining \( \pi(n) = n \), then \( Q \cong P \) via \( \pi \). Finally, \( n \in C_{\pi} 1 \) since in general \( x \in C_{\pi} 1 \) if and only if \( \pi(x) \in C_{\pi} 1 \) and in this case \( \pi(n) = n \). Thus we assume hereafter that \( n \in C_{\pi} 1 \) and that the restriction \( \hat{P} \) of \( P \) to \( T(n - 1) \) is a normal NPO.

Now suppose that \( P \) has a bad triple. Since \( \hat{P} \) is normal, the bad triple must involve \( n \) and since \( n \in C1 \) it must be of the form \( i P n, i \neq n, j < i \), and \( j \sim P n \). We shall show that we can permute \( i \) and \( j \) to obtain an NPO, \( \hat{Q} \), with fewer bad triples. Our result will then follow by induction.

First, it must be that \( D(j) = 1 \); otherwise \( j P n \). Second, it must be that \( D(i) = 2 \), and even more, that \( i \) is maximal in \( \hat{P} \), for otherwise \( \hat{P} \) would have a bad triple. Third, if \( x P i \), then \( x P j \) by condition (1). Conversely, if \( x P j \), then \( x P i \), for otherwise \((x, j; i)\) is a bad triple \((i > j)\) in \( \hat{P} \). Thus \( \{x: x P i, x \neq i\} = \{x: x P j, x \neq j\} \). Now in \( \hat{P} \), since

\[
D_{\hat{P}}(i) = D_{\hat{P}}(j) = 1,
\]

https://doi.org/10.4153/CJM-1968-055-7 Published online by Cambridge University Press
the conditions of Lemma 3.6 are satisfied so that the permutation \( \pi = (i, j) \) of \( T(n - 1) \) leaves \( \hat{P} \) invariant. Now regard \( \pi \) as a permutation on \( T(n) \) and let \( Q \) be defined by \( u \ Q \ v \) if and only if \( \pi^{-1}(u) \ P \pi^{-1}(v) \). \( Q \) is an NPO, \( Q \cong P, \) \( \hat{Q} = \hat{P} \), and \( Q \) is normal, and we claim that \( Q \) has fewer bad triples than \( P \). We show that there is a 1-1 mapping of the bad triples of \( Q \) into and not onto the bad triples of \( P \). Since \( \hat{Q} = \hat{P} \), any bad triple of \( Q \) must include \( n \).

Suppose that \((x, n; y)\) is a bad \( Q \)-triple. If neither \( x \) nor \( y \) is \( i \) or \( j \), \((x, n; y)\) is a bad \( P \)-triple. Suppose \((j, n; y)\) is a bad \( Q \)-triple because \( y < j \) and \( j \sim Qn \); then \( y \neq i \). Now \((i, n; y)\) is a bad \( P \)-triple since \( y < j < i \). Suppose that \((x, n; j)\) is a bad \( Q \)-triple because \( j < x \) and \( j \sim Qn \). Now \( x \neq i \) as \( j \sim Pn \); hence \((x, n; i)\) is a bad \( P \)-triple, if \( i < x \). If \( i > x \), we argue that \((x, n; j)\) is a bad \( P \)-triple. Suppose that \((x, n; i)\) with \( i < x \) is a bad \( Q \)-triple; then \((x, n; j)\) is a bad \( P \)-triple. Note that here \( i < x \) so that the triple \((x, n; j)\) with \( i > x \) of the previous case is a different triple from this one. Thus to each bad \( Q \)-triple there corresponds a bad \( P \)-triple, different bad \( Q \)-triples corresponding to different bad \( P \)-triples. In addition the \( Q \)-triple \((j, n; i)\) corresponding to the old bad \( P \)-triple \((i, n; j)\) is not bad! Thus \( Q \) has at least one less bad triple than \( P \) and the proof is complete.

4. The order of \( T(n) \). In this section we shall show that if \( n \) is finite, the number of elements in \( T(n) \) is

\[
\binom{2n}{n} / (n + 1).
\]

In view of Theorems 3.6 and 3.7, this number is the number of abstract non-isomorphic partially ordered sets which do not contain copies of either configurations in Figure 1 as sub-partially ordered sets.

Our first count is obtained by showing a 1-1 correspondence between the NPO’s of \( T(n) \) and the set of lattice paths counted by Feller (3). Let \( \mathfrak{W}(n) \) be the set of non-negative, integer-valued step functions, \( f \), defined on \([0, n]\) such that

1. \( 0 < f(x) \leq x \),
2. \( f(x) = f([x]) \),
3. \( f(0) = 0 \) and \( f(n) = n \),
4. \( f \) is monotone non-decreasing.

The graph of \( f \) can be thought of as a polygonal path from \((0, 0)\) to \((n, n)\) with jumps at points with integer coordinates, always lying below the line \( y = x \). Through the transformation \((i, j) \rightarrow (i + j, j - i)\) the path corresponds to what is called a lattice path \( \{s_1, s_2, \ldots, s_{2n}\} \) from the origin through the points \((a, s_a)\) to the point \((2n, 0)\) such that all \( s_a \geq 0 \) and \( s_{a+1} - s_a = \pm 1 \). The number of these paths has been determined (3, p. 72) to be

\[
\binom{2n}{n} / (n + 1).
\]
We shall now show that the relation \( C \) defined by \( f \leq g \) if and only if \( f(x) \leq g(x) \) makes \( \mathcal{M}(n) \) into a lattice and that there is a lattice isomorphism from \( \mathcal{M}(n) \) onto \( \mathcal{D}(n) \).

**Definition.** Let \( f \in \mathcal{M}(n) \). Let \( P_f \) be defined as follows:

(i) \( x P_f x \) for all \( x \in T(n) \).

(ii) If \( x \neq y \), then \( x P_f y \) if and only if \( x - 1 < f(y - 1) \).

**Lemma 4.1.** For all \( f, P_f \) is a normal partial ordering of \( T(n) \).

**Proof.** First note that if \( x P_f y \) and \( x < y \), then \( x - 1 < f(y - 1) \) implies that \( x < y \) and thus \( P_f \) is a natural relation. \( P_f \) is then easily seen to be reflexive and antisymmetric. Suppose \( x P_f y \) and \( y P_f x \). We exclude trivial cases by supposing that \( x \neq y \). Then

\[
x - 1 < f(y - 1) \leq (y - 1) < f(x - 1);
\]

hence \( x P_z \). Thus \( P_f \) is a natural partial ordering.

To show that \( P_f \) is normal we verify the two conditions of Theorem 3.1. First, suppose that \( i P k \) and \( r \leq i \). Thus \( r - 1 < i - 1 < f(k - 1) \); hence \( r P k \). Second, if \( k P i \) and \( s > i \), then \( f(s - 1) > f(i - 1) > k - 1 \); hence \( k P s \).

**Lemma 4.2.** Let \( P \) be a normal natural partial ordering. Let \( f \) be the function defined by:

\[
f(x) = \begin{cases} 0 & \text{if } 0 < x < 1, \ f(n) = n, \text{ and for } 1 \leq x < n; \\ f(x) = f([x]) = \max\{y: y = 0 \text{ or } y P (x + 1) \text{ and } y \neq x + 1\}. \end{cases}
\]

Then \( f \in \mathcal{M}(n) \) and \( P_f = P \).

**Proof.** It is clear that \( f \) has integer values, that \( 0 \leq f(x) \leq x \), that \( f(0) = 0 \) and \( f(n) = n \), and that \( f([x]) = f(x) \). We shall prove that \( x_1 < x_2 \) implies that \( f(x_1) \leq f(x_2) \). We may suppose that \( f(x_1) = y_1 \neq 0 \). Thus \( y_1 P (x_1 + 1) \) and since \( x_1 + 1 < x_2 + 1 \) it follows from the normality of \( P \) that \( y_1 P x_2 + 1 \). Hence by definition \( y_1 \leq f(x_2) \).

Now suppose \( x P_f y \) and \( x \neq y \). Then \( x - 1 < f(y - 1) \). The case \( f(y - 1) = 0 \) is thus impossible since \( x \geq 1 \). Thus \( f(y - 1) = \max\{z: z P y, z \neq y\} \). Thus \( f(y - 1) P y \) and since \( x \leq f(y - 1) \) and \( P \) is normal, it follows that \( x P y \). Thus \( P_f \subseteq P \). Conversely, suppose \( x P_f y \) and \( x \neq y \). Then \( (x - 1) P y \) since \( P \) is normal and so \( x - 1 \leq f(y - 1) = \max\{z: z P y, z \neq y\} \). Hence \( x P_f y \) and so \( P_f = P \).

**Lemma 4.3.** \( f \subseteq g \) in \( \mathcal{M}(n) \) if and only if \( P_f \subseteq P_g \) in \( \mathcal{M}(n) \).

**Proof.** Suppose first that \( f \subseteq g \). If \( x P_f y \), then \( x - 1 < f(y - 1) \leq g(y - 1) \) and so \( x P_f y \). Conversely, suppose \( P_f \subseteq P_g \) and consider \( x \). If \( g(x) < f(x) \), then \( (g(x) + 1) P_f (x + 1) \) and so \( (g(x) + 1) P_g (x + 1) \), but this entails \( g(x) < g(x) \), a contradiction. Hence \( f(x) \leq g(x) \).
Corollary. \( f = g \) if and only if \( P_f = P_g \). Moreover, given \( f \in \mathcal{M}(n) \), if \( f_1 \) is the function constructed in Lemma 4.2 corresponding to the partial order \( P_f \), then \( f = f_1 \).

**Proof.** The first part of the corollary is clear. Lemma 4.3 gives \( P_P = P_f \) and thus \( f_1 = f \).

**Theorem 4.1.** \( \mathcal{M}(n) \) is a lattice under the partial ordering \( \subseteq \). The mapping \( f \rightarrow P_f \) is a lattice isomorphism of \( \mathcal{M}(n) \) onto \( \mathcal{D}(n) \).

**Proof.** It is a standard result that \( \mathcal{M}(n) \) is a complete lattice in which the greatest lower bound of a set \( \mathcal{R} \) of functions in \( \mathcal{M}(n) \) is the function \( g \) such that \( g(x) = \min\{r(x) : r \in \mathcal{R}\} \); dually the least upper bound is the function \( h \) such that \( h(x) = \max\{r(x) : r \in \mathcal{R}\} \). From Lemma 4.2 the mapping is onto and from Lemma 4.3 it is a lattice isomorphism.

The integer
\[
\binom{2n}{n} / (n + 1)
\]
is the number of ways of inserting parentheses in a string of \( n + 1 \) symbols so as to interpret the string as an element in a binary (non-associative) system. This number is also the number of lattice paths and the connection between these two problems is through the Lukasiewicz parenthesis-free notation.* Thus, for example, \(((a + b) + (c + d))\) becomes \(+ + ab + cd\). By replacing \(+ \) by \( 1 \) and \( a, b, c \) by \(-1 \) and deleting the superfluous last symbol \( d \), we obtain the sequence \( 1, 1, -1, -1, 1, -1 \). The \( i \)th partial sums of this series is the integer \( s_i \) of the associated path \((s_1, s_2, s_3, s_4, s_5, s_6)\). The formula
\[
\binom{2n}{n} / (n + 1)
\]
for the number of ways of parenthesizing \( a_1 + a_2 + \ldots + a_{n+1} \) is developed in (4).

It is interesting to note that Feller’s count of the number of paths \((s_1, \ldots, s_{2n})\) and Hall’s count of the number of ways of inserting parentheses in \( a_1, \ldots, a_{n+1} \) coincide since each shows that this number, \( f_{n+1} \), satisfies the recursion \( f_{n+1} = f_0 f_n + f_1 f_{n-1} + \ldots + f_n f_0 \). Our second development of the count of \( \mathcal{D}(n) \) proceeds along somewhat different lines and is of interest in itself.

We begin with the observation that since \( \mathcal{D}(n) \) is a finite distributive lattice, every element of \( \mathcal{D} \) has a unique expression as an irredundant meet of meet irreducibles. Moreover, in a distributive lattice a meet of meet irreducibles is irredundant if and only if no two irreducibles in the set are comparable. We shall characterize the meet irreducibles of \( \mathcal{D} \) and count the number of subsets of this set in which no two elements are comparable.

*We are indebted to Professors Marshall Hall, Jr., and Donald Knuth for these remarks.
**Lemma 4.4.** A normal NPO, $P$, is a meet irreducible in $\mathfrak{D}(n)$ if and only if $P = \Omega$ or there exists a pair $(r, s)$, $r < s$, such that $P = \bigcup \{ \Gamma(u, v) : u < r$ or $s < v \}$.

*Proof of sufficiency.* Let $(r, s)$ be given. Define $P$ by the formula of the lemma. We omit the proof that $P$ is a normal NPO. It is easy to see that $u P v$ if and only if $u = v$ or $u < r$ or $s < v$. A typical $P$ is shown in Figure 8.

Suppose that $P \subseteq R$ in $\mathfrak{D}$ and $(a, b) \in R - P$. Thus $r \leq a < b \leq s$. Since $R$ is normal, it follows (Theorem 3.1) that $r R b$ and thus, for the same reason, that even $r R s$. Now if $P$ is meet reducible, say, $P = R_1 \cap \ldots \cap R_k$ with $P \neq R_i$, then $r R_i s$ and hence $r P s$, a contradiction.

*Proof of necessity.* Let $P \neq \Omega$ be meet irreducible. Choose $r$ and $s$ such that $r \sim P s$ and $(s - r)$ is maximal. Let $\hat{P} = \bigcup \{ \Gamma(u, v) : u < r$ or $s < b \}$. We claim that $P = \hat{P}$. To show that $P \subseteq \hat{P}$ suppose that $a P b$. If $a < r$ or $s < b$, then $a \hat{P} b$. However, if $r \leq a < b \leq s$ it follows from normality that $r P b$ and this in turn implies that $r P s$; a contradiction. Hence $a P b$ implies either $a < r$ or $s < b$ and so $P \subseteq \hat{P}$.

Our proof that $P \supseteq \hat{P}$ is in three stages. First, from the choice of the pair $(r, s)$ we have that $u < r$ and $x \gg s$ imply $u P x$; similarly $s < v$ and $y \leq r$ imply $y P s$. Second, we claim that $Q = P \cup \Gamma(r, s)$ (i.e., $P \cup \{(r, s)\}$ by Theorem 3.3) is itself normal. We prove this by verifying the criteria of Theorem 3.1. Since $P$ is normal, we need only check a triple $(r, s; t)$; however, our first remark shows that $x > s$ implies $r P t$, and dually. In particular, note that $Q$ covers $P$ in $\mathfrak{R}(n)$ from Lemma 1.3; a fortiori $Q$ covers $P$ in $\mathfrak{D}$. Thus since $P$ is meet irreducible, $Q$ is the unique element of $\mathfrak{D}$ covering $P$. 

![Figure 8.]
Third, we claim that for all \( u \), if \( u < r \), then \( u \not\sim R y \) for all \( y \not\succ u \). This, and its dual which we omit, proves that \( P \supseteq \bar{P} \). If our assertion is not true, choose \( u \) minimal such that \( u < r \) and there exists a \( y \) such that \( u \not\sim R y \). For this \( u \) choose \( w = \max\{x : u \not\sim R x\} \). Now we claim that \( R = P \cup \Gamma(u, w) \) is normal. Indeed, from Theorem 3.1, arguing as before, if \( R \) is not normal, it follows that either there exists \( s \) with \( s < u \) and \( s \not\sim R w \) (a contradiction of the choice of \( u \)) or there exists \( t \) with \( w < t \) such that \( u \not\sim R t \) (a contradiction of the choice of \( w \)). But manifestly, \( Q \neq R \), a contradiction of the meet irreducibility of \( P \).

In view of this lemma the following notation is sensible.

**Notation.** If \( P \) is a meet irreducible in \( \mathfrak{D} \) and \( P = \bigcup\{ \Gamma(u, v) : u < r \) or \( s < v\} \) we write \( P = [r, s] = P_{r,s} \). The set of intervals

\[ \mathfrak{R}(n) = \mathfrak{R} = \{[a, b] : 1 \leq a < b \leq n\} \]

is then isomorphic to the set of meet irreducibles of \( \mathfrak{D}(n) \).

The ordering of \( \mathfrak{D} \) induces a partial ordering of \( \mathfrak{R} : [r, s] \leq [u, v] \) if and only if \( P_{r,s} \subseteq P_{u,v} \). Clearly \([r, s] \leq [u, v] \) if and only if \( r \leq u < v \leq s \); thus this ordering is dual to subset inclusion of the intervals on the real line.

From our previous remarks it now remains only to count the number of subsets of \( \mathfrak{R} \) in which no two intervals are comparable. To this end we define a function \( K(r, s) \) for all pairs \( (r, s) \), \( r \leq s \). If \( r < s \), let \( K(r, s) \) be the number of subsets \( E \subseteq \mathfrak{R} \) such that

1. \( E \subseteq \{[u, v] : u \leq r, u < v, \) and \( v \leq s\}, \)
2. no two intervals in \( E \) are comparable.

If \( r = s \), let \( K(s, s) = K(s - 1, s) \), and \( K(1, 1) = 0 \).

It is clear that \( K(1, s) = s - 1 \) if \( 1 < s \) and that the number of elements in \( \mathfrak{D} \) is \( 1 + K(n, n) \) since we have not yet counted the meet irreducible \( \Omega \).

**Theorem 4.2.** If \( r < s \), then \( K(r, s) \) satisfies the recursion relation

\[ K(r, s) = K(r, s - 1) + K(r - 1, s) + 1 \]

which is satisfied by

\[ K(r, s) = -1 + \frac{1 + (s - r)}{1 + (s + r)} \left( r + s + 1 \right) \cdot \]

**Proof.** We observe that

\[ K(r, s) = K(r, s - 1) + r + \sum_{i=1}^{r-1} K(i, s - 1). \]

The first term counts all sets \( E \) in which every index is at most \( n - 1 \). The second term counts all singleton sets \( E \) of the form \( E = [i, s] \). The third sum counts all sets \( E \) which are not singletons and in which an interval \([i + 1, s]\) occurs. Note that since \( E \) cannot contain two comparable intervals, only one interval of the form \([i + 1, s]\) can occur. Also if \([i + 1, s] \in E \) and
If \([h, j] \in E\), then \(j \leq s - 1\) and \(h \leq i\) and so \(E - \{[i + 1, s]\}\) is counted in \(K(i, s - 1)\). From this equation the recursion of the theorem follows by a direct calculation. Similarly, it is a simple matter to verify that

\[
-1 + \frac{1 + (s - r)}{1 + (r + s)} \left(1 + s + r\right)
\]

satisfies both the recursion and the initial conditions \(K(1, 1) = 0\) and \(K(1, s) = s - 1\).

**Corollary.**

\[
1 + K(n, n) = 1 + K(n - 1, n) = \left(\binom{2n}{n - 1}\right) / n = \left(\binom{2n}{n}\right) / (n + 1).
\]

**References**