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GENERAL PROJECTION SYSTEMS AND RELAXED COCOERCIVE NONLINEAR VARIATIONAL INEQUALITIES

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Abstract

We explore the *solvability* of a general system of nonlinear relaxed cocoercive variational inequality (SNVI) problems based on a new projection system for the direct product of two nonempty closed and convex subsets of real Hilbert spaces.

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1. Introduction

Projection-type systems are frequently used in convergence analysis for solutions of variational inequality problems arising in several fields, for instance, in complementarity theory, convex quadratic programming, optimization and control theory, and variational problems. Projection methods (sometimes referred to as Galerkin methods) are also applied in different contexts, especially to inner approximation schemes for A-proper nonlinear equations [9] where solutions are approximated as strong limits of solutions of corresponding simpler systems of finite-dimensional equations. Recently, Chang *et al.* [1] considered the application of the general two-step model [7] for projection methods to the approximation solvability of nonlinear strongly monotone inequality problems in Hilbert spaces. They generalized the iterative algorithm used in [6, 7] along with some special iterative algorithms of interest. In this paper we explore, based on a general system of projection-type methods, the approximation solvability of a system of nonlinear relaxed cocoercive variational inequalities in Hilbert spaces. The notion of relaxed cocoercivity generalizes the notion of monotonicity as well as

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of strong monotonicity. The obtained results extend and improve the results in [1], [5], [6] and [7]. For more details, we refer the reader to [1–9].

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $S : K_1 \times K_2 \to H_1$ and $T : K_1 \times K_2 \to H_2$ be any mappings on $K_1 \times K_2$, where K_1 and K_2 are nonempty closed convex subsets of H_1 and H_2, respectively. We consider a system of nonlinear variational inequality (SNVI) problems: find an element $(x^*, y^*) \in K_1 \times K_2$ such that

$$\langle \rho S(x^*, y^*), x - x^* \rangle \geq 0 \quad \forall x \in K_1 \quad \text{and} \tag{1.1}$$

$$\langle \eta T(x^*, y^*), y - y^* \rangle \geq 0 \quad \forall y \in K_2, \tag{1.2}$$

where $\rho, \eta > 0$.

The SNVI (1.1)–(1.2) problem is equivalent to the projection formulae

$$x^* = P_K [x^* - \rho S(x^*, y^*)] \qquad \text{for } \rho > 0 \quad \text{and}$$
$$y^* = Q_K [y^* - \eta T(x^*, y^*)] \qquad \text{for } \eta > 0,$$

where P_K is the projection of H_1 onto K_1 and Q_K is the projection of H_2 onto K_2.

We note that the SNVI (1.1)–(1.2) problem extends the NVI problem: determine an element $x^* \in K_1$ such that

$$\langle S(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K_1.$$

Also, we note that the SNVI (1.1)–(1.2) problem is equivalent to a system of nonlinear complementarities (SNCs): find $(x^*, y^*) \in K_1 \times K_2$ such that $S(x^*, y^*) \in K_1^*$, $T(x^*, y^*) \in K_2^*$, and

$$\langle \rho S(x^*, y^*), x^* \rangle = 0 \quad \text{for } \rho > 0, \quad \langle \eta T(x^*, y^*), y^* \rangle = 0 \quad \text{for } \eta > 0,$$

where K_1^* and K_2^*, respectively, are polar cones to K_1 and K_2 defined by

$$K_1^* = \{f \in H_1 : \langle f, x \rangle \geq 0, \quad \forall x \in K_1\} \quad \text{and}$$
$$K_2^* = \{g \in H_2 : \langle g, y \rangle \geq 0, \quad \forall g \in K_2\}.$$

Now, we recall some auxiliary results and notions crucial to the problem at hand.

LEMMA 1.1. *For an element $z \in H$, we have $x \in K$ and $\langle x - z, y - x \rangle \geq 0$, $\forall y \in K$ if and only if $x = P_K(z)$.*

LEMMA 1.2 ([2]). *Let $\{\alpha^k\}$, $\{\beta^k\}$ and $\{\gamma^k\}$ be three nonnegative sequences such that*

$$\alpha^{k+1} \leq (1 - t^k)\alpha^k + \beta^k + \gamma^k \quad \text{for } k = 0, 1, 2, \ldots,$$

where $t^k \in [0, 1]$, $\sum_{k=0}^{\infty} t^k = \infty$, $\beta^k = o(t^k)$ and $\sum_{k=0}^{\infty} \gamma^k < \infty$. Then $\alpha^k \to 0$ as $k \to \infty$.

For H a Hilbert space, a mapping $T : H \to H$ is called monotone if $\forall x, y \in H$, $\langle T(x) - T(y), x - y\rangle \geq 0$. The mapping T is *(r)-strongly* monotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y\rangle \geq r||x - y||^2 \quad \text{for a constant } r > 0.$$

This implies that $\|T(x) - T(y)\| \geq r\|x - y\|$, that is, T is (r)-expansive, and when $r = 1$, it is expansive. The mapping T is called (s)-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geq 0$ such that $\|T(x) - T(y)\| \leq s\|x - y\|$, $\forall x, y \in H$. The mapping T is called (μ)-cocoercive if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y\rangle \geq \mu||T(x) - T(y)||^2 \quad \text{for a constant } \mu > 0.$$

Clearly, every (μ)-cocoercive mapping T is $(1/\mu)$-Lipschitz continuous. We note that T is called relaxed (γ)-cocoercive if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\|T(x) - T(y)\|^2, \quad \forall x, y \in H.$$

We say that T is (r)-strongly pseudomonotone if there exists a constant $r > 0$ such that

$$\langle T(y), x - y\rangle \geq 0 \quad \Rightarrow \quad \langle T(x), x - y\rangle \geq r\,\|x - y\|^2, \quad \forall x, y \in H,$$

and that T is relaxed (γ, r)-cocoercive if there exist constants $\gamma, r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\,\|T(x) - T(y)\|^2 + r\,\|x - y\|^2.$$

This implies

$$\langle T(x) - T(y), x - y\rangle \geq (-\gamma)\,\|T(x) - T(y)\|^2,$$

that is, T is relaxed (γ)-cocoercive.

We define T to be relaxed (γ, r)-pseudococoercive if there exist positive constants γ and r such that $\forall x, y \in H$

$$\langle T(y), x - y\rangle \geq 0 \quad \Rightarrow \quad \langle T(x), x - y\rangle \geq (-\gamma)\|T(x) - T(y)\|^2 + r\|x - y\|^2.$$

2. General projection methods

In this section, we discuss the approximation-solvability of the SNVI (1.1)–(1.2) problem based on the following algorithms.

ALGORITHM 1. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = \left(1 - a^k - b^k\right)x^k + a^k P_K\left[x^k - \rho S\left(x^k, y^k\right)\right] + b^k u^k \quad \text{and}$$
$$y^{k+1} = \left(1 - \alpha^k - \beta^k\right)y^k + \alpha^k Q_K\left[y^k - \eta T\left(x^k, y^k\right)\right] + \beta^k v^k,$$

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where $\rho, \eta > 0$ are constants, and $\{u^k\}$ and $\{v^k\}$, respectively, are bounded sequences in K_1 and K_2. The sequences $\{a^k\}$, $\{b^k\}$, $\{\alpha^k\}$ and $\{\beta^k\}$ are in $[0, 1]$ with $(k \geq 0)$

$$0 \leq a^k + b^k \leq 1, \quad 0 \leq \alpha^k + \beta^k \leq 1.$$

ALGORITHM 2. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = \left(1 - a^k - b^k\right) x^k + a^k P_K \left[x^k - \rho S\left(x^k, y^k\right)\right] + b^k u^k \quad \text{and}$$
$$y^{k+1} = \left(1 - a^k - b^k\right) y^k + a^k Q_K \left[y^k - \eta T\left(x^k, y^k\right)\right] + b^k v^k,$$

where $\rho, \eta > 0$ are constants, and $\{u^k\}$ and $\{v^k\}$, respectively, are bounded sequences in K_1 and K_2. The sequences $\{a^k\}$ and $\{b^k\}$ are in $[0, 1]$ with $(k \geq 0)$

$$0 \leq a^k + b^k \leq 1.$$

ALGORITHM 3. For an arbitrarily chosen initial point $(x^0, y^0) \in K_1 \times K_2$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = \left(1 - a^k\right) x^k + a^k P_K \left[x^k - \rho S\left(x^k, y^k\right)\right] \quad \text{and}$$
$$y^{k+1} = \left(1 - a^k\right) y^k + a^k Q_K \left[y^k - \eta T\left(x^k, y^k\right)\right],$$

where $\rho, \eta > 0$ are constants. The sequence $\{a^k\} \subset [0, 1]$ for $k \geq 0$.

Next, we consider, based on Algorithm 2, the approximation solvability of the SNVI (1.1)–(1.2) problem involving strongly monotone and Lipschitz continuous mappings in Hilbert space settings.

THEOREM 2.1. *Let H_1 and H_2 be two real Hilbert spaces and K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be relaxed (γ, r)-cocoercive and (μ)-Lipschitz continuous in the first variable and let S be (ν)-Lipschitz continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be relaxed (λ, s)-cocoercive and (β)-Lipschitz continuous in the second variable and let T be (τ)-Lipschitz continuous in the first variable. Let $\|\cdot\|^*$ denote the norm on $H_1 \times H_2$ defined by*

$$\|(x, y)\|^* = (\|x\| + \|y\|) \quad \forall (x, y) \in H_1 \times H_2.$$

In addition, let

$$\theta + \eta\tau = \sqrt{1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2} + \eta\tau < 1,$$
$$\sigma + \rho\nu = \sqrt{1 - 2\eta r + 2\eta\lambda\beta^2 + \eta^2\beta^2} + \rho\nu < 1,$$

let $(x^*, y^*) \in K_1 \times K_2$ form a solution to the SNVI (1.1)–(1.2) *problem, and let sequences $\{x^k\}$ and $\{y^k\}$ be generated by* Algorithm 2.

Furthermore, let

(i) $0 \leq a^k + b^k \leq 1$,
(ii) $\sum_{k=0}^{\infty} a^k = \infty$ *and* $\sum_{k=0}^{\infty} b^k < \infty$, *and*
(iii) $0 < \rho < 2r/\mu^2$ *and* $0 < \eta < 2s/\beta^2$.

Then the sequence $\{(x^k, y^k)\}$ *converges to* (x^*, y^*).

PROOF. Since $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI (1.1)–(1.2) problem, it follows that

$$x^* = P_K[x^* - \rho S(x^*, y^*)] \quad \text{and} \quad y^* = Q_K\,[y^* - \eta T(x^*, y^*)].$$

Applying Algorithm 2, we have

$$\begin{aligned}
&\left\|x^{k+1} - x^*\right\| \\
&\quad = \left\|(1 - a^k - b^k)x^k + a^k P_K\left[x^k - \rho S(x^k, y^k)\right] + b^k u^k \right. \\
&\qquad \left. -(1 - a^k - b^k)x^* - a^k P_K\,[x^* - \rho S(x^*, y^*)] - b^k x^*\right\| \\
&\quad \leq (1 - a^k - b^k)\left\|x^k - x^*\right\| \\
&\qquad + a^k \left\|P_K\left[x^k - \rho S(x^k, y^k)\right] - P_K\,[x^* - \rho S(x^*, y^*)]\right\| + Mb^k \\
&\quad \leq (1 - a^k)\left\|x^k - x^*\right\| \\
&\qquad + a^k \left\|x^k - x^* - \rho\left[S(x^k, y^k) - S(x^*, y^k) + S(x^*, y^k) - S(x^*, y^*)\right]\right\| + Mb^k \\
&\quad \leq (1 - a^k)\left\|x^k - x^*\right\| + a^k\left\|x^k - x^* - \rho\left[S(x^k, y^k) - S(x^*, y^k)\right]\right\| \\
&\qquad + a^k\rho\left\|\left[S(x^*, y^k) - S(x^*, y^*)\right]\right\| + Mb^k,
\end{aligned}$$

where $M = \max\{\sup\|u^k - x^*\|, \sup\|v^k - y^*\|\} < \infty$.

Since S is relaxed (γ, r)-cocoercive and (μ)-Lipschitz continuous in the first variable, and S is (ν)-Lipschitz continuous in the second variable, we have in light of part (i) of Theorem 2.1 that

$$\begin{aligned}
&\left\|x^k - x^* - \rho\left[S(x^k, y^k) - S(x^*, y^k)\right]\right\|^2 \\
&\quad = \|x - x^*\|^2 - 2\rho\left\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\right\rangle \\
&\qquad + \rho^2\left\|S(x^k, y^k) - S(x^*, y^k)\right\|^2 \\
&\quad = \|x - x^*\|^2 - 2\rho\left\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\right\rangle \\
&\qquad + (\rho^2 + 2\rho\gamma)\left\|S(x^k, y^k) - S(x^*, y^k)\right\|^2 \\
&\quad \leq \left\|x^k - x^*\right\|^2 - 2\rho r\left\|x^k - x^*\right\|^2 + (\rho^2 + 2\rho\gamma)\mu^2\left\|x^k - x^*\right\|^2 \\
&\quad = \left[1 - 2\rho r + \rho^2\mu^2 + 2\rho\gamma\mu^2\right]\left\|x^k - x^*\right\|^2.
\end{aligned}$$

As a result, letting $\theta = \sqrt{1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2}$ we have

$$\left\|x^{k+1} - x^*\right\| \leq \left(1 - a^k\right)\left\|x^k - x^*\right\| + a^k\theta\left\|x^k - x^*\right\| + a^k\rho\nu\left\|y^k - y^*\right\| + Mb^k. \quad (2.1)$$

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Similarly, letting $\sigma = \sqrt{1 - 2\eta r + 2\eta\lambda\beta^2 + \eta^2\beta^2}$ we have

$$\|y^{k+1} - y^*\| \le (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\eta\tau\|x^k - x^*\| + Mb^k. \quad (2.2)$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \\
&\quad \le (1 - a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| + a^k\eta\tau\|x^k - x^*\| + Mb^k \\
&\qquad + (1 - a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\rho\nu\|y^k - y^*\| + Mb^k \\
&\quad = [1 - (1-\delta)a^k](\|x^k - x^*\| + \|y^k - y^*\|) + 2Mb^k,
\end{aligned}$$

where $\delta = \max\{\theta + \eta\tau, \sigma + \rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|\cdot\|^*$.

If we set

$$\alpha^k = \|x^k - x^*\| + \|y^k - y^*\|, \quad t^k = (1-\delta)a^k, \quad \beta^k = 2Mb^k$$

for $k = 0, 1, 2, \ldots$, in Lemma 1.2, and apply (i) and (ii), we conclude that $\|x^k - x^*\| + \|y^k - y^*\| \to 0$ as $k \to \infty$. Hence $\|x^{k+1} - x^*\| + \|y^{k+1} - y^*\| \to 0$.

Consequently, the sequence $\{(x^k, y^k)\}$ converges strongly to (x^*, y^*), a solution to the SNVI (1.1)–(1.2) problem. This completes the proof. □

Note that the proof of the following theorem follows rather directly without using Lemma 1.2.

THEOREM 2.2. *Let H_1 and H_2 be two real Hilbert spaces and let K_1 and K_2, respectively, be nonempty closed convex subsets of H_1 and H_2. Let $S : K_1 \times K_2 \to H_1$ be relaxed (γ, r)-cocoercive and (μ)-Lipschitz continuous in the first variable and let S be (ν)-Lipschitz continuous in the second variable. Let $T : K_1 \times K_2 \to H_2$ be relaxed (λ, s)-cocoercive and (β)-Lipschitz continuous in the second variable and let T be (τ)-Lipschitz continuous in the first variable. Let $\|\cdot\|^*$ denote the norm on $H_1 \times H_2$ defined by*

$$\|(x, y)\|^* = (\|x\| + \|y\|) \quad \forall (x, y) \in H_1 \times H_2.$$

In addition, let

$$\theta + \eta\tau = \sqrt{1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2} + \eta\tau < 1,$$
$$\sigma + \rho\nu = \sqrt{1 - 2\eta r + 2\eta\lambda\beta^2 + \eta^2\beta^2} + \rho\nu < 1,$$

let $(x^*, y^*) \in K_1 \times K_2$ form a solution to the SNVI (1.1)–(1.2) *problem, and let sequences $\{x^k\}$ and $\{y^k\}$ be generated by* Algorithm 3. *Furthermore, let*

(i) $0 \le a^k \le 1$,

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(ii) $\sum_{k=0}^{\infty} a^k = \infty$ *and*
(iii) $0 < \rho < 2r/\mu^2$ *and* $0 < \eta < 2s/\beta^2$.

Then the sequence $\{(x^k, y^k)\}$ *converges strongly to* (x^*, y^*).

PROOF. Since $(x^*, y^*) \in K_1 \times K_2$ forms a solution to the SNVI (1.1)–(1.2) problem, it follows that

$$x^* = P_K[x^* - \rho S(x^*, y^*)] \quad \text{and} \quad y^* = Q_K[y^* - \eta T(x^*, y^*)].$$

Applying Algorithm 3, we have

$$\begin{aligned}\|x^{k+1} - x^*\| &= \|(1-a^k)x^k + a^k P_K[x^k - \rho S(x^k, y^k)] \\ &\quad -(1-a^k)x^* - a^k P_K[x^* - \rho S(x^*, y^*)]\| \\ &\le (1-a^k)\|x^k - x^*\| \\ &\quad + a^k \|P_K[x^k - \rho S(x^k, y^k)] - P_K[x^* - \rho S(x^*, y^*)]\| \\ &\le (1-a^k)\|x^k - x^*\| \\ &\quad + a^k \|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k) + S(x^*, y^k) - S(x^*, y^*)]\| \\ &\le (1-a^k)\|x^k - x^*\| + a^k\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\| \\ &\quad + a^k\rho\|[S(x^*, y^k) - S(x^*, y^*)]\|.\end{aligned}$$

Since S is relaxed (γ, r)-cocoercive and (μ)-Lipschitz continuous in the first variable, and S is (ν)-Lipschitz continuous in the second variable, we have

$$\begin{aligned}&\|x^k - x^* - \rho[S(x^k, y^k) - S(x^*, y^k)]\|^2 \\ &\quad = \|x - x^*\|^2 - 2\rho\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\rangle \\ &\qquad + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\ &\quad = \|x - x^*\|^2 - 2\rho\langle S(x^k, y^k) - S(x^*, y^k), x^k - x^*\rangle \\ &\qquad + \rho^2\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\ &\quad \le \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 + \rho^2\mu^2\|x^k - x^*\|^2 \\ &\qquad + 2\rho\gamma\|S(x^k, y^k) - S(x^*, y^k)\|^2 \\ &\quad \le \|x^k - x^*\|^2 - 2\rho r\|x^k - x^*\|^2 + \rho^2\mu^2\|x^k - x^*\|^2 + 2\rho\gamma\mu^2\|x^k - x^*\|^2 \\ &\quad = [1 - 2\rho r + \rho^2\mu^2 + 2\rho\gamma\mu^2]\|x^k - x^*\|^2.\end{aligned}$$

Setting $\theta = \sqrt{1 - 2\rho r + 2\rho\gamma\mu^2 + \rho^2\mu^2}$ it follows that

$$\|x^{k+1} - x^*\| \le (1-a^k)\|x^k - x^*\| + a^k\theta\|x^k - x^*\| + a^k\rho\nu\|y^k - y^*\|. \tag{2.3}$$

Similarly, setting $\sigma = \sqrt{1 - 2\eta r + 2\eta\lambda\beta^2 + \eta^2\beta^2}$ we have

$$\|y^{k+1} - y^*\| \le (1-a^k)\|y^k - y^*\| + a^k\sigma\|y^k - y^*\| + a^k\eta\tau\|x^k - x^*\|. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\left\|x^{k+1}-x^*\right\| + \left\|y^{k+1}-y^*\right\| \\
&\quad \le (1-a^k)\left\|x^k-x^*\right\| + a^k\theta\left\|x^k-x^*\right\| + a^k\eta\tau\left\|x^k-x^*\right\| \\
&\qquad + (1-a^k)\left\|y^k-y^*\right\| + a^k\sigma\left\|y^k-y^*\right\| + a^k\rho\nu\left\|y^k-y^*\right\| \\
&\quad = \left[1-(1-\delta)a^k\right]\left(\left\|x^k-x^*\right\| + \left\|y^k-y^*\right\|\right) \\
&\quad \le \prod_{j=0}^{k}\left[1-(1-\delta)a^j\right]\left(\left\|x^0-x^*\right\| + \left\|y^0-y^*\right\|\right),
\end{aligned}$$

where $\delta = \max\{\theta+\eta\tau, \sigma+\rho\nu\}$ and $H_1 \times H_2$ is a Banach space under the norm $\|\cdot\|^*$.

Since $\delta < 1$ and $\sum_{k=0}^{\infty} a^k$ is divergent, it follows that

$$\lim_{k\to\infty}\prod_{j=0}^{k}\left[1-(1-\delta)a^j\right] = 0 \quad \text{as } k \to \infty.$$

Therefore

$$\left\|x^{k+1}-x^*\right\| + \left\|y^{k+1}-y^*\right\| \to 0,$$

and consequently the sequence $\{(x^k, y^k)\}$ converges strongly to (x^*, y^*), a solution to the SNVI (1.1)–(1.2) problem. This concludes the proof. □

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