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ON THE VOLUME DISTRIBUTION OF THE TYPICAL POISSON–DELAUNAY CELL

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Abstract

A method of obtaining the distribution of the volume of the typical cell of a Delaunay tessellation generated by a Poisson process in $\mathbb{R}^d$ is developed and used to derive the density when $d = 1, 2, 3$.

DELAUNAY TESSELLATION; EXACT VOLUME DISTRIBUTION

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1. Introduction

Consider a stationary Poisson point process in $\mathbb{R}^d$ with intensity ρ. Any $d + 1$ points from this process define almost surely an open ball B which contains the $d + 1$ points in its boundary. If no points from the Poisson process are contained in B, then the simplex with vertices at the $d + 1$ points is a so-called Delaunay cell. The collection of all such Delaunay cells constitutes almost surely a tessellation of the space $\mathbb{R}^d$ (cf. Rogers (1964)). It is well known that the volume V_d of the typical Delaunay cell has moments

$$E(V_d^k) = \frac{\Gamma(d^2/2)\Gamma(d+k)\Gamma\{(d^2+dk+k+1)/2\}\Gamma^{d-k+1}\{(d+1)/2\}}{\Gamma(d)\Gamma\{(d^2+1)/2\}\Gamma\{(d^2+dk)/2\}\Gamma^{d+1}\{(d+k+1)/2\}\{2^d\pi^{(d-1)/2}\rho\}^k} \times \prod_{i=2}^{d+1} \frac{\Gamma\{(k+i)/2\}}{\Gamma(i/2)} \tag{1.1}$$

for $k = 1, 2, 3, \cdots$ (cf. Miles (1972), (1974) and Møller (1989)).

The objective in this paper is to demonstrate how (1.1) can be used to derive the exact distribution of V_d. The interesting special cases for $d = 1$, $d = 2$ and $d = 3$ are obtained. The first case is an exponential distribution with parameter ρ while the second case is expressed in terms of the modified Bessel function.

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2. Distribution of the volume

First, it will be shown that the moments in (1.1) determine the distribution uniquely by verifying the absolute convergence of the series,

$$\sum_{k=1}^{\infty} E(V_d^k)t^k/k! \tag{2.1}$$

for some $t>0$ (cf. Rao (1965), p. 86 (2b.5.5)). From (1.1), we have

$$\begin{aligned}&\frac{E(V_d^{k+1})t^{k+1}/(k+1)!}{E(V_d^k)t^k/k!}\\ &=\frac{t(d+k)(\{d^2+1+(d+1)k\}/2)_{(d+1)/2}(2^d\pi^{(d-1)/2}\rho)^{-1}}{(1+k)\Gamma\{(d+1)/2\}(\{d^2+dk\}/2)_{d/2}(\{d+1+k\}/2)_{1/2}}\end{aligned} \tag{2.2}$$

where $(\alpha)_\beta$ stands for $\Gamma(\alpha+\beta)/\Gamma(\alpha)$. Applying Luke (1969), p. 33(11),

$$1/(z)_b=z^{-b}[1+(-b)(b-1)/(2z)+\cdots],\qquad |\arg z|\leqq \pi-\delta,\ \delta>0$$

to the right side of (2.2), taking the limits as $k\to\infty$ and simplifying, we have the following condition for (2.1) to be absolutely convergent:

$$t<(4d)^{d/2}\pi^{(d-1)/2}\rho\Gamma[(d+1)/2]/(d+1)^{(d+1)/2}. \tag{2.3}$$

Since the right side of (2.3) is positive, it is possible to choose $t>0$ guaranteeing the absolute convergence of (2.1). Hence the moments in (1.1) determine the distribution of V_d uniquely.

Let

$$f_\rho(v)=(2\pi iv)^{-1}\int_{L_1} v^{-k}E(V_d^k)dk \tag{2.4}$$

where L_1 is a suitably chosen Mellin–Barnes contour $(c-i\infty,\ c+i\infty)$ and $E(V_d^k)$ is defined by (1.1) for complex k (see Erdélyi et al. (1953)). Since the density function defined in (2.4) is unique and has moments (by the Mellin transform) given by (1.1) for $k=1,2,\cdots$, this is the density function corresponding to the volume V_d.

Clearly, $f_1(v)=f_\rho(v/\rho)/\rho$ so it suffices to evaluate $f(v)=f_1(v)$ in the following.

The relations (2.4) and (1.1), on using the multiplication formula for the gamma function and simplifying, yield

$$f(v)=A_d(2\pi iv)^{-1}\int_L \nabla_d(k)(B_dv^2)^{-k}dk \tag{2.5}$$

where L encloses all the poles of the integrand,

$$A_d=\frac{2^{d-1/2}(d+1)^{d^2/2}\Gamma(d^2/2)\Gamma^d\{(d+1)/2\}}{\pi d^{(d^2-1)/2}\Gamma(d)\Gamma\{(d^2+1)/2\}\prod_{i=2}^{d}\Gamma(i/2)}, \tag{2.6}$$

$$B_d=\left[\frac{2^{d-1}\pi^{(d-1)/2}d^{d/2}\Gamma\{(d+1)/2\}}{(d+1)^{(d+1)/2}}\right]^2, \tag{2.7}$$

and

$$\nabla_d(k)=\frac{\prod_{r=2}^{d}\Gamma(r/2+k)\prod_{r=0}^{d}\Gamma\left[\frac{d^2+1+2r}{2(d+1)}+k\right]}{\prod_{r=1}^{d-1}\Gamma(d/2+r/d+k)\Gamma^{d-1}\{(d+1)/2+k\}}. \tag{2.8}$$

The integral in (2.5) can be evaluated as the sum of the residues at the poles of $\nabla_d(k)$ given in (2.8). This provides $f(v)$, in the general case as an infinite series involving gamma, psi and zeta functions. The technique (see Mathai and Rathie (1971), Rathie (1989)) is well known; as an example, the result for $d=3$ is derived in the next section by using the theory of residues.

Computer programs are available from the author for calculating numerically the density and distribution functions with reasonably good precision.

3. Particular cases

In this section the density of V_d is evaluated for $d=1, 2, 3$.

(a) For $d=1$, (2.5) reduces to

$$\begin{aligned} f(v)&=2\pi^{-1/2}(2\pi iv)^{-1}\int_L\Gamma(k+1/2)\Gamma(k+1)(v^2/4)^{-k}dk \\ &=(2\pi i)^{-1}\int_M\Gamma(t)v^{-t}dt, \end{aligned} \tag{3.1}$$

on using the duplication formula for gamma functions. Here M encloses the poles of $\Gamma(t)$. Hence, $f(v)=\exp(-v)$, $v>0$, i.e. V_1 is exponentially distributed with parameter 1. This result is well known.

(b) For $d=2$, (2.5) yields

$$\begin{aligned} f(v)&=3\pi^{-1/2}(2\pi iv)^{-1}\int_L\frac{\Gamma(k+1)\Gamma(k+5/6)\Gamma(k+7/6)}{\Gamma(k+3/2)}(4\pi^2v^2/27)^{-k}dk \\ &=(8/9)\pi vK_{1/6}^2(2\pi v/3\sqrt{3}), \qquad v>0 \end{aligned} \tag{3.2}$$

using Erdélyi (1954), p. 371. Here $K_{1/6}(\cdot)$ is the modified Bessel function of order 1/6. This is an interesting compact form for the density function of V_2. The plot of $f(v)$ is given in Fig. 1. The software MATHEMATICA was used to produce this plot.

(c) For $d=3$, (2.5) takes the following form:

$$f(v)=A_3(2\pi iv)^{-1}\int_L\frac{\Gamma^2(k+3/2)\Gamma(k+5/4)\Gamma(k+7/4)}{(k+1)\Gamma(k+11/6)\Gamma(k+13/6)}(B_3v^2)^{-k}dk. \tag{3.3}$$

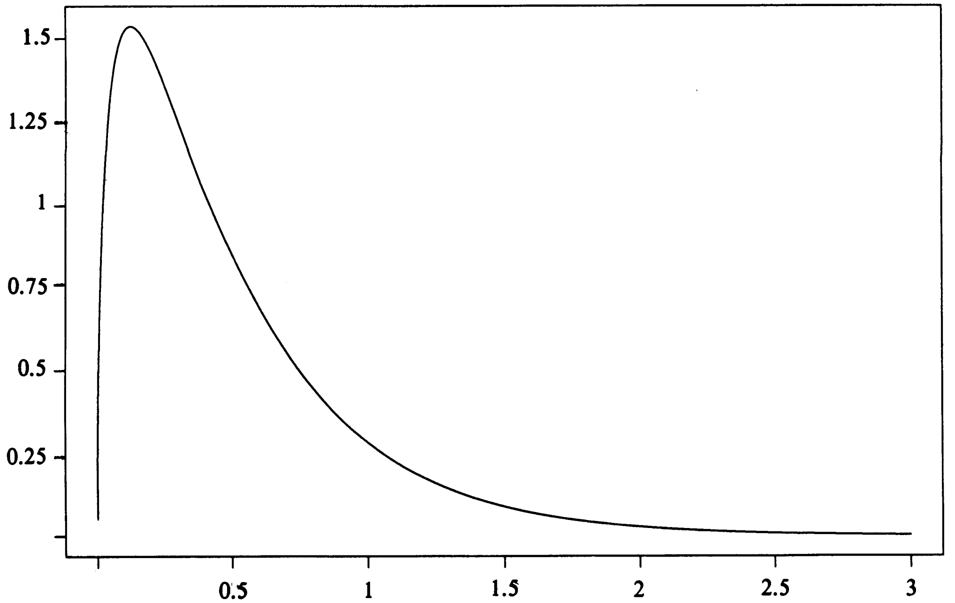

Figure 1. Plot of $f(v)$ when $d = 2$

The integrand in (3.3) has simple poles at $k = -1$, $k = -t - 5/4$, $t = 0, 1, 2, \cdots$, $k = -s - 7/4$, $s = 0, 1, 2, \cdots$ and poles of order 2 at $k = -t - 3/2$, $t = 0, 1, 2, \cdots$. Evaluating the contour integral in (3.3) as the sum of the residues at the poles, we have

$$f(v) = A_3\left[Pv - \sum_{t=0}^{\infty} Q_t v^{2t+3/2} - \sum_{t=0}^{\infty} R_t v^{2t+5/2} - \sum_{t=0}^{\infty} S_t v^{2t+2}\{-\ln(B_3 v^2) + T_t\}\right], \tag{3.4}$$

where

$$A_3 = 560\sqrt{2}/(81\pi), \qquad B_3 = 27\pi^2/16,$$

$$P = B_3\pi\Gamma(1/4)\Gamma(3/4)/[\Gamma(5/6)\Gamma(7/6)],$$

$$Q_t = (-1)^t\Gamma^2(-t+1/4)\Gamma(-t+1/2)B_3^{t+5/4}/[(t+1/4)\Gamma(-t+7/12)\Gamma(-t+11/12)t!],$$

$$R_t = (-1)^t\Gamma^2(-t-1/4)\Gamma(-t-1/2)B_3^{t+7/4}/[(t+3/4)\Gamma(-t+1/12)\Gamma(-t+5/12)t!],$$

$$S_t = \Gamma(-t-1/4)\Gamma(-t+1/4)B_3^{t+3/2}/[(t+1/2)\Gamma(-t+1/3)\Gamma(-t+2/3)(t!)^2],$$

$$T_t = 2\psi(t+1) + \psi(-t-1/4) + \psi(-t+1/4) - \psi(-t+1/3) - \psi(-t+2/3) + (t+1/2)^{-1},$$

where $\psi(\cdot)$ is the psi function (see Erdélyi et al. (1953)).

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