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SPLASH FORMATION AT THE NOSE OF A SMOOTHLY CURVED BODY IN A STREAM

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Abstract

In two-dimensional flow past a body close to a free surface, the upwardly diverted portion may separate to form a splash. We model the nose of such a body by a semi-infinite obstacle of finite draft with a smoothly curved front face. This problem leads to a nonlinear integral equation with a side condition, a separation condition and an integral constraint requiring the far-upstream free surface to be asymptotically plane. The integral equation, called Villat's equation, connects a natural parametrisation of the curved front face with the parametrisation by the velocity potential near the body. The side condition fixes the position of the separation point, whereas the separation condition, known as the Brillouin-Villat condition, imposes a continuity relation to be satisfied at separation. For the described flow we derive the Brillouin-Villat condition in integral form and give a numerical solution to the problem using a polygonal approximation to the front face.

1. Introduction

In two-dimensional flow past a body close to a free surface, the upwardly diverted portion may separate to form a splash. We model the nose of such a body by a semi-infinite obstacle of finite draft with a smoothly curved front face, assuming that the speed of the flow is high enough for gravity effects to be negligible in the neighbourhood of the body. We assume that the pressure is minimal on the free surface [1] or equivalently, in view of the Euler equations, that the curvature of the free streamline is positive towards the cavity.

For obstacles with a convex front face of finite curvature, this imposes a restriction on where the free streamline separates from the body. Namely, a Brillouin-Villat condition [1] must be fulfilled at the separation point, which can be shown to be equivalent to the requirement that the curvature of the free streamline be continuous at

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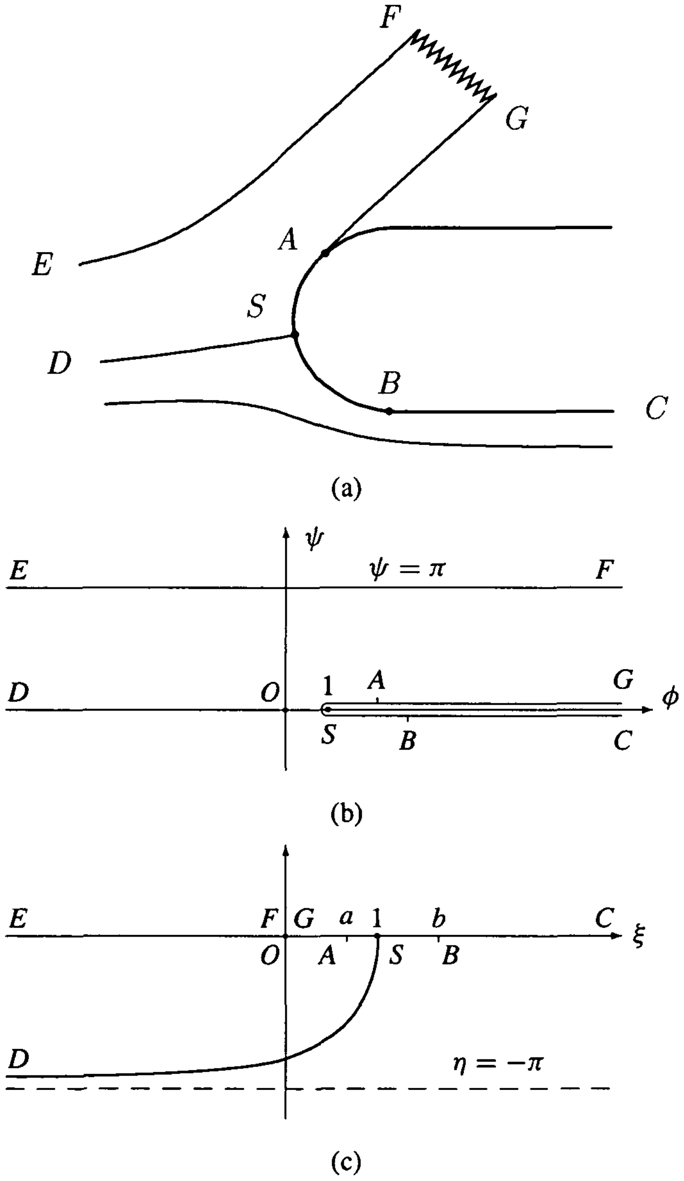

FIGURE 1. Flow in the z-, f - and ζ-planes .

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detachment. A straightforward demonstration of this fact for the geometry concerned is given in Section 2, for the general case see [1].

The flow is sketched in the complex $z = x + \mathrm{i}\, y$ plane in Figure 1(a). Our task is to find an analytic complex velocity potential $f(z) = \phi + \mathrm{i}\,\psi$. We normalise velocities so that the uniform stream is of magnitude $U = 1$, and distances so that the splash jet has thickness π. Hence the flow in the f-plane is as in Figure 1(b) with the streamline $\psi = \pi$ being the free surface, and the streamline $\psi = 0$ bifurcating from a single stagnation streamline originating upstream into two branches. This f-plane is then mapped to a lower half ζ-plane by the conformal mapping

$$f = \zeta - \log \zeta \tag{1}$$

and the flow in the $\zeta = \xi + \mathrm{i}\,\eta$ plane is as shown in Figure 1(c) (see, for example, [4]).

The origin FG in the ζ-plane is far downstream in the splash jet. Similarly, the region CDE at infinity in the lower half ζ-plane corresponds to the lower portion of the actual flow, with C far downstream on the lower surface of the body, D far upstream on the stagnation streamline and E far upstream on the free surface. The stagnation point S lies at $\zeta = 1$, and we introduce real constants a, b with $a < 1 < b$ such that $\zeta = a$ is the image of the separation point A and $\zeta = b$ is the image of B, the point where the front face meets the bottom plane surface.

The flow problem is solved in the lower-half ζ-plane, with the logarithmic hodograph

$$\Omega = \tau - \mathrm{i}\,\theta = \log f'(z) = \log(u - \mathrm{i}\, v) \tag{2}$$

as dependent variable. Note that the fluid velocity magnitude is $q = \exp(\tau)$, and θ is the angle that the velocity bears to the positive x-axis. To relate $\Omega(\zeta)$ to the physical plane z, we need to express z as a function of ζ, by integrating the equation

$$\frac{dz}{d\zeta} = (1 - \zeta^{-1}) \exp(-\Omega(\zeta)) \tag{3}$$

which follows from (1) and (2).

The free-surface boundary condition is constancy of pressure, and in the absence of gravity, this demands constancy of fluid velocity magnitude $q = 1$, or $\tau = 0$. On the other hand, the body boundary condition can be specified semi-inversely by specifying the angle θ on the body. In the ζ-plane, these boundary conditions are applied on the real axis $\zeta = \xi - \mathrm{i}\,0$. Namely, the free surface condition must be fulfilled in the interval $(-\infty, a)$ and the body boundary condition in the interval $(a, +\infty)$.

First the body is specified semi-inversely by giving $\theta = \Theta(\xi)$ for $a < \xi < b$. If the body's slope is the tangent of an angle $\Theta_0(\xi)$ which varies from $\Theta_0(a)$ to $\Theta_0(b) = \pi$

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as we move from the separation point to bottom surface, then we have

$$\Theta(\xi) = \begin{cases} \Theta_0(\xi), & \text{if } a < \xi < 1 \\ \Theta_0(\xi) - \pi, & \text{if } 1 < \xi < b. \end{cases} \tag{4}$$

Note that Θ is simply a "shifted" form of Θ_0, introducing the necessary 180° change in streamline direction at the stagnation point $\xi = 1$. Given $\Theta_0(\xi)$, the body's outline can be found through numerical integration of (3).

Direct formulation and treatment of the problem imply, however, that the outline of the obstacle is given in the physical plane z. This does not immediately provide us with the dependence $\Theta_0(\xi)$, as it is not known in advance how the complex velocity potential behaves near the body. The equation that connects the natural parametrisation of the outline with its parametrisation by the velocity potential is called Villat's equation [1]. We derive such an equation for the geometry described above. If the separation point is known, this equation requires a side condition for the solution to be unique. When it is not known where separation occurs, Villat's equation must also be supplied with the Brillouin-Villat condition.

In Section 2 the semi-inverse solution to the problem is obtained and its behaviour at infinity and near the separation point is considered. In Section 3 we give a numerical solution to the direct problem using a polygonal approximation to the front face and reduction to a system of a finite number of nonlinear equations, which are treated by Newton's method.

2. Semi-inverse solution

We suppose throughout this section that $\Theta(\xi)$ is a known function. In order to obtain the solution to the flow problem we introduce an auxiliary hodograph variable $\chi(\zeta)$ related to the logarithmic hodograph as

$$\chi(\zeta) = (\zeta - a)^{1/2}\Omega(\zeta).$$

The branch of the square root used can be defined explicitly as $w^{1/2} \equiv |w|^{1/2} \exp\{\mathrm{i} \arg(\zeta)/2\}$, where $\arg(\zeta) \in [-\pi, \pi)$. Denoting the real and imaginary parts of $\chi(\zeta)$ by $R(\xi, \eta)$ and $I(\xi, \eta)$ respectively and taking account of the relation $\tau = 0$ holding on EA we get the boundary conditions for $I(\xi, \eta)$:

$$I(\xi, 0) = \begin{cases} 0, & \text{if } \xi < a \\ -\theta(\xi)(\xi - a)^{1/2}, & \text{if } \xi \geq a. \end{cases} \tag{5}$$

Since $I(\xi, \eta)$ is an imaginary part of an analytic function, it satisfies the Laplace equation in the lower half-plane of ζ. Assuming that $I(\xi, \eta)$ is bounded as $|\zeta| \to \infty$

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and making use of Green's function for the Laplace equation in the lower half-plane subject to the Dirichlet boundary condition (for example, [3]), we obtain:

$$I(\xi,\eta)=\frac{\eta}{\pi}\int_a^b\frac{\Theta(s)(s-a)^{1/2}}{(\xi-s)^2+\eta^2}ds. \tag{6}$$

It was also taken into account here that $\theta(s)$ vanishes for $s>b$.

To determine the complex conjugate $R(\xi,\eta)$ we note that $I(\xi,\eta)$ is the imaginary part of the analytic function

$$\chi_0(\zeta)=\frac{1}{\pi}\int_a^b\frac{\Theta(s)(s-a)^{1/2}}{(s-\zeta)}ds.$$

Therefore, $\chi(\zeta)=\chi_0(\zeta)+C$, where C is a real constant to be found. On account of (3) the following equation holds on the free surface:

$$\frac{dy}{d\xi}=(1-\xi^{-1})\sin\theta(\xi).$$

Furthermore, if $\xi<a$,

$$\theta(\xi)=-R(\xi,0)|\xi-a|^{-1/2}. \tag{7}$$

Hence the constant C must equal zero, otherwise the free surface will not be asymptotically plane far upstream. Thus, $R(\xi,\eta)$ has the form:

$$R(\xi,\eta)=\frac{1}{\pi}\int_a^b\frac{\Theta(s)(s-a)^{1/2}(s-\xi)}{(\xi-s)^2+\eta^2}ds. \tag{8}$$

Note that for a particular case of a step-wise constant $\Theta(s)$ the integrals in (6) and (8) can be evaluated explicitly, which would allow the solution to be presented in a closed form, see [5] where a rectangular body in a similar flow is considered.

Clearly, $\theta(\xi)$ must be bounded, so it follows from (7) that $R(a-0,0)=0$, which gives us an integral constraint on physically acceptable functions $\Theta(\xi)$:

$$\int_a^b\Theta(s)(s-a)^{-1/2}ds=0. \tag{9}$$

As seen from the derivation, condition (9) is connected with the behaviour of streamlines far upstream and at the separation point. Let us consider in more detail how the free streamline behaves near the separation point. Putting $\eta=0$ and subtracting (9) from the integral in (8) we rewrite $R(\xi,0)$ in the form:

$$R(\xi,0)=\frac{(\xi-a)}{\pi}\int_a^b\frac{\Theta(s)}{(s-a)^{1/2}(s-\xi)}ds. \tag{10}$$

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Assuming that the front face is smooth near the separation point and substituting (10) into (7), we find after some manipulations that $\theta(a-0) = \Theta(a+0)$, that is, the streamline leaves the body smoothly. Further examination of (10) gives the formula (see Appendix)

$$\theta'(\xi) = K(a-\xi)^{-1/2} + \theta'(a+0) + O((a-\xi)^{1/2}), \tag{11}$$

where $\xi < a$ and

$$\pi K = \Theta(a)(b-a)^{-1/2} - \int_0^{(b-a)^{1/2}} (\Theta(a+y^2) - \Theta(a))y^{-2}dy.$$

Formula (11) gives the asymptotic behaviour of the free streamline's curvature $\kappa(\xi) \equiv \theta'(\xi)/(1-\xi^{-1})$ in the neighbourhood of the separation point. Suppose that the body is convex and its curvature is finite near the separation. If $K \neq 0$ the free streamline either enters the body ($K < 0$) or has a negative curvature ($K > 0$). Both cases are physically unacceptable and we are left with the only possibility $K = 0$. The condition $K = 0$, or equivalently $\theta'(a-0) = \Theta'(a+0)$, is called the Brillouin-Villat condition. Here the Brillouin-Villat condition was derived in integral form; other forms of this condition, obtained for a different geometry of the obstacle could be found, for example, in [1], [2] or [6].

Now let us turn to asymptotics of streamlines far from the front face of the body. The streamlines are represented in the ζ-plane by the curves

$$\zeta = \frac{(\psi+\beta)}{\sin\beta}e^{i\beta}, \tag{12}$$

where ψ, the value of the stream function, is fixed for each curve and β varies from $(-\pi)$ to $\beta_\infty \equiv \min(-\psi, 0)$. Let the streamline corresponding to a value ψ of the stream function be described in the physical z-plane by the function $z = z(\beta, \psi)$. We are going to examine the asymptotic behaviour of this function at small $|\zeta|$ and at large $|\operatorname{Re}\zeta|$.

It follows from (6) and (8) that, as $|\zeta| \to 0$,

$$\chi(\zeta) = -\alpha a^{1/2} + O(|\zeta|), \tag{13}$$

where

$$\alpha = -\frac{1}{a^{1/2}\pi}\int_a^b \frac{(s-a)^{1/2}}{s}\Theta(s)ds. \tag{14}$$

Formula (13) and Equation (3) yield the relation

$$z(\zeta) = C_1 - e^{i\alpha}\log\zeta + O(|\zeta|),$$

valid at small $|\zeta|$ for some constant C_1. On the free streamline originating from the separation point, we have $\zeta = \xi > 0$ and therefore

$$z(\xi) = C_1 - e^{i\alpha}\log\xi + O(|\xi|).$$

If $\zeta = \xi < 0$, that is, on the free streamline originating far upstream,

$$z(\xi) = C_1 - e^{i\alpha}\log|\xi| + i\pi e^{i\alpha} + O(|\xi|).$$

These observations show that, after separation, the splash ultimately forms a jet at the angle α to the horizontal given by (14), the thickness of the jet being π.

If $|\zeta| \to \infty$, then $\chi(\zeta) = O(|\zeta|^{-1})$ and, by virtue of (3),

$$z(\zeta) = C_2 + \zeta - \log\zeta + O(|\zeta|^{-1/2}).$$

Now letting β in $z(\beta, \psi)$ go to $-\pi + 0$ we obtain that the far upstream streamline level is $\operatorname{Im} C_2 + \psi$, where $\psi \in (-\infty; \pi]$. If $\psi < 0$, that is, for streamlines below the body, the far downstream streamline level is $\operatorname{Im} C_2 + \psi$. Therefore, the streamlines below the stagnation streamline asymptotically retain their level. We note that, due to the chosen normalisation, the magnitude of the draft and the jet thickness are fixed and equal π.

3. Direct problem and numerical procedure

Until now we considered the flow problem in a semi-inverse formulation: first the dependence $\Theta_0(\xi)$ was chosen and only after that, the shape of the body's front face was determined. The latter is done through numerical integration of (3) in the interval $a < \xi < b$. In the direct treatment of the problem, the choice of $\Theta_0(\xi)$ must fulfill the requirement that the resulting front face of the body coincide with a given profile AB. Let AB be parametrised as $\Theta_0 = \theta_0(l)$ where l is the arc-length measured from B. If we determine the connection between l and ξ, the required dependence $\Theta_0(\xi) \equiv \theta_0(l(\xi))$ will be found.

Noting that, in view of (3), $dl = -|1 - \xi^{-1}|e^{-\tau}d\xi$ we deduce that $l(\xi)$ satisfies the integral equation

$$l(\xi) = \int_\xi^b d\xi\, S(\xi)\exp\left\{-\frac{(\xi - a)^{1/2}}{\pi}\int_a^b \frac{\theta_0(l(s))}{(s - \xi)(s - a)^{1/2}}ds\right\}, \tag{15}$$

further referred to as Villat's equation [1]. In (15)

$$S(\xi) = \left[\frac{(1 - a)^{1/2} + (\xi - a)^{1/2}}{(b - a)^{1/2} + (\xi - a)^{1/2}}\right]^2 \frac{(b - \xi)}{\xi}$$

and the integral in the exponent is understood in the Cauchy principal value sense. Equation (15) must be supplied with the side condition

$$l(a) = L \tag{16}$$

and the consequence of condition (9)

$$\int_a^b \frac{\theta_0(l(s))}{\sqrt{s-a}} ds = 2\pi((b-a)^{1/2} - (1-a)^{1/2}). \tag{17}$$

The side condition fixes the position of the separation point. The meaning of condition (9) was described earlier. If the separation point is not given, the solution to (15) must also satisfy the Brillouin-Villat separation condition.

Existence and uniqueness theory for Villat's equation based on the Leray-Schauder fixed point theorem is discussed in detail in [1]. For computational purposes though, it is more convenient not to reduce relation (3) to (15) but to directly proceed from it. Our approach employs polygonal approximation of the body's contour. Let the front face of the obstacle be the curve $z = z(t)$, $t \in [0, T]$. We assume that $z(0) = 0$ corresponds to point B and $z(T)$ is the separation point A, which for the moment is considered to be given. Introducing a uniform mesh $t_1 = 0 < t_2 < \cdots < t_n = T$, we approximate the front face of the obstacle by a polygonal line that joins the points $z_k \equiv z(t_k)$, $(k = 1, \dots, n)$. The angle between the link $[z_k, z_{k+1}]$ and the positive x-axis will be denoted by Θ_k^0. For each selection $\{\xi_k\}$ $(k = 1, \dots, n)$ such that

$$\xi_1 > 1, \quad 0 < \xi_n < 1, \quad \xi_{k+1} < \xi_k, \tag{18}$$

we define a step-wise constant function $\Theta_0(\xi)$:

$$\Theta_0(\xi) = \Theta_k^0, \quad \text{if } \xi \in (\xi_{k+1}, \xi_k].$$

Now consider the initial value problem

$$\begin{aligned} \frac{dz}{d\xi} &= (1-\xi^{-1})\frac{q(1,\xi_n,\xi)}{q(\xi_1,\xi_n,\xi)} \exp\{Q(\xi,\{\xi_k\}) + \mathrm{i}\,\Theta_0(\xi)\}, \\ z|_{\xi=\xi_1} &= 0, \end{aligned} \tag{19}$$

where

$$Q(\xi,\{\xi_k\}) = \frac{1}{\pi}\sum_{r=1}^{n-1} \Theta_r^0 \log\left|\frac{q(\xi_r,\xi_n,\xi)}{q(\xi_{r+1},\xi_n,\xi)}\right|$$

and

$$q(\alpha,\beta,\gamma) \equiv \frac{(\alpha-\beta)^{1/2} + (\gamma-\beta)^{1/2}}{(\alpha-\beta)^{1/2} - (\gamma-\beta)^{1/2}}.$$

The solution of this problem on $[\xi_n, \xi_1]$ will be denoted by $z(\xi, \{\xi_k\})$. We have to find a vector $\{\xi_k\}$ so that the conditions

$$z_k = z(\xi_r, \{\xi_k\}), \quad (r = 2, \dots, n), \tag{20}$$

$$\sum_{k=1}^{n-1} \Theta_k^0 \left(\sqrt{\xi_k - \xi_n} - \sqrt{\xi_{k+1} - \xi_n}\right) = \pi \left(\sqrt{\xi_1 - \xi_n} - \sqrt{1 - \xi_n}\right) \tag{21}$$

are fulfilled. Relations (19) and (21) follow from (3) and (17) respectively. Equations (20) and (21) form a system of n equations for n unknowns $\{\xi_k\}$ and are solved numerically with the aid of the C05NBF NAG-routine. Theoretically it makes no difference whether the real or imaginary part of (20) is used, although practically it is better to choose the real part for Θ_r^0 close to 0 or π and the imaginary part for Θ_r^0 close to $\pi/2$. To guarantee that the conditions (18) are satisfied throughout the computation, variables $\{\xi_k\}$ are replaced by new variables $\{h_k\}$ as follows:

$$\xi_{n-k} = \xi_{n-k+1} + (\xi_1 - \xi_n) h_{n-k}^2 \Big/ \left(1 + \sum_{r=2}^{n-1} h_r^2\right), \quad k = 1, \dots, n-2,$$

where $\xi_1 = 1 + h_1^2$ and $\xi_n = 1/(1 + h_n^2)$. When the values of $\{\xi_k\}$ are found, $\Theta_0(\xi)$ can be used for computing streamlines. In particular, the free streamline that extends beyond the separation point A is obtained through integration of the equation

$$\frac{dz}{d\xi} = (1 - \xi^{-1}) e^{i\theta(\xi)} \tag{22}$$

from ξ_n to zero. Here

$$\theta(\xi) = \sum_{k=1}^{n-1} \frac{2\Theta_k^0}{\pi} \left\{ \arctan \sqrt{\frac{\xi_k - \xi_n}{\xi_n - \xi}} - \arctan \sqrt{\frac{\xi_{k+1} - \xi_n}{\xi_n - \xi}} \right\}$$
$$- 2 \left\{ \arctan \sqrt{\frac{\xi_1 - \xi_n}{\xi_n - \xi}} - \arctan \sqrt{\frac{1 - \xi_n}{\xi_n - \xi}} \right\}.$$

If the separation point is not given, we determine its location by moving $z(T)$ along the body's outline until the coefficient K in (11) vanishes. For the purpose of illustration let us consider the family of circular arc front faces:

$$x = -R \sin t, \quad y = R(1 - \cos t),$$

where $t \in (0, \gamma)$ and γ belongs to the interval $(0, \pi)$. Determination of $\gamma = \gamma_0$, for which the condition $K = 0$ holds, solves the problem of locating the separation point for obstacles with a round front. We recall that the draft and the jet thickness

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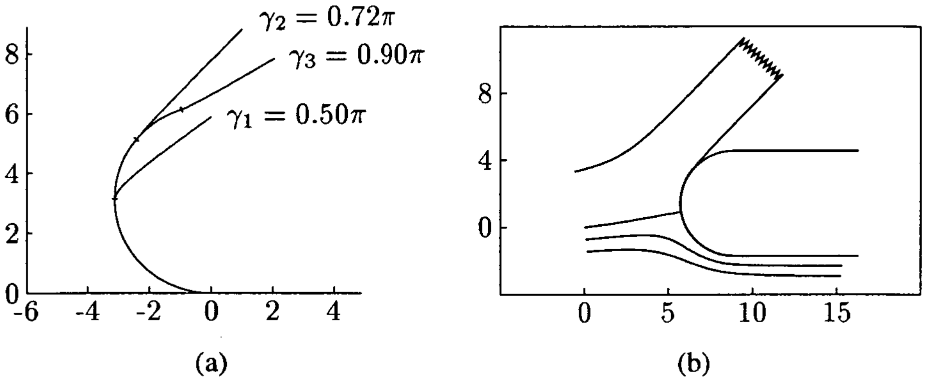

FIGURE 2. Locating the separation point and the computed flow for $R = \pi$.

are of the same value, which is π for the normalised flow. If $\gamma \neq \gamma_0$ and therefore $K \neq 0$, the free streamline either enters the obstacle ($K < 0$) or does not comply with the convexity requirement ($K > 0$). To find the location of the separation point numerically we let γ change from 0 with a small step, each time obtaining $\Theta_0(\xi)$ and thereby the free streamline extending beyond A. The angle γ after which the free streamline changes the direction of its concavity gives us an approximate value for γ_0. Figure 2(a) exhibits results of computation for $R = \pi$, $n = 89$ and three different values of γ: $\gamma_1 = 0.50\pi$, $\gamma_2 = 0.72\pi$ and $\gamma_3 = 0.90\pi$. Strokes show the position of points A for respective γ_j. The sign of $\theta'(\xi)$ on the free streamline is established by considering the difference $\Delta = \theta(a - \delta) - \theta(a)$ for small positive δ. Here Δ is negative for γ_1 and positive for γ_3. Smooth separation occurs near $\gamma_2 = 0.72\pi$. The flow itself is shown in Figure 2(b). If we increase the number of links, say double it, the position of the resulting separation point will slightly change, such changes becoming less and less significant for further increases of n. A similar procedure can be applied to more general obstacles.

The polygonal approximation used here is quite acceptable for the aim of flow visualisation because the geometry of the flow in the z-plane depends on $\Theta(\xi)$ integrally or, in other words, is found through averaging of $\Theta(\xi)$ with one or another weight. This approximation, however, cannot be used directly if such parameters as local pressure or its derivatives are to be evaluated near the body's surface. In this case the step-wise constant $\Theta_0(\xi)$ computed for large n should be replaced by an appropriate continuous or sufficiently smooth interpolation, evaluation of higher derivatives requiring higher orders of interpolation.

Concluding remarks

In the above consideration the flow was normalised for convenience so that the draft was always equal to π. To return to the original flow with a draft D we have to perform a similarity transformation of the computed flow with a scale factor D/π. Such parameters as the coordinates of the separation point, or related quantities, can be obtained for different values of the draft by choosing the body's height in the normalised flow in the required proportion to π.

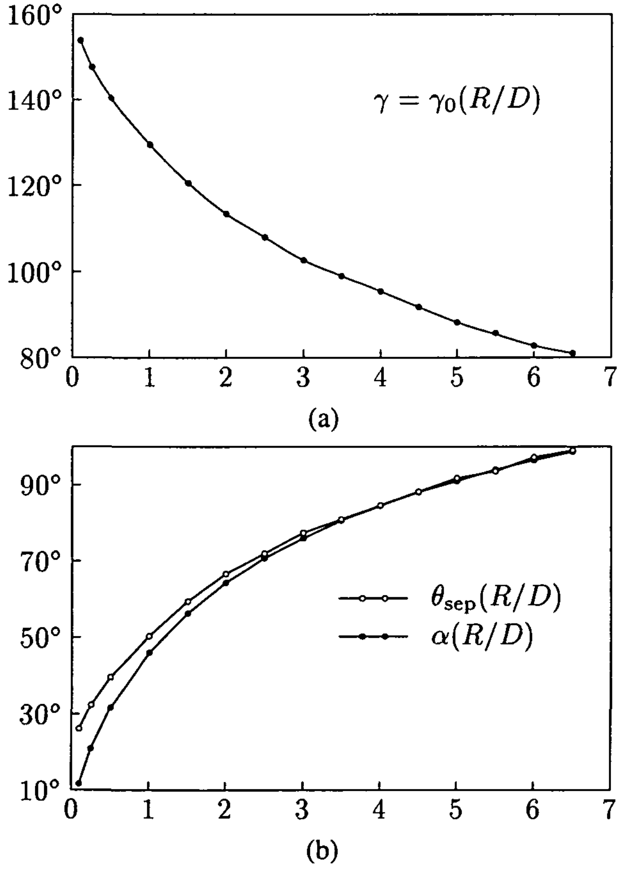

FIGURE 3. Graphs $\gamma_0(R/D)$, $\theta_{sep}(R/D)$ and $\alpha(R/D)$

The plot in Figure 3(a) shows how the separation angle γ_0 from the example with a round front face depends on the ratio of the radius R to the draft D. We can draw from this plot an intuitively obvious conclusion that the smaller the draft is, the earlier the separation occurs.

Figure 3(b) presents the plots for the velocity angle at separation θ_{sep} and the angle

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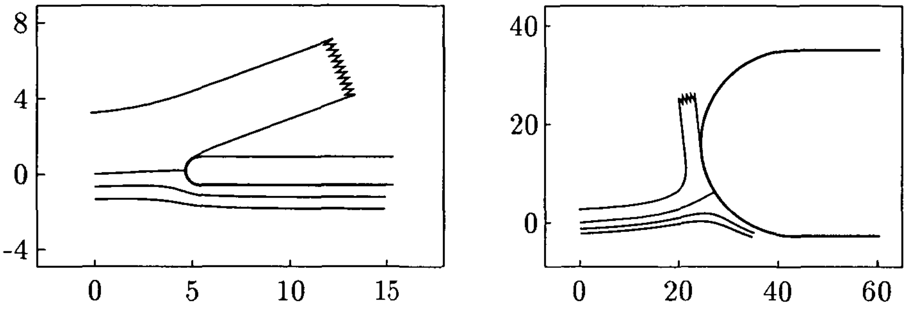

(a) (b)

FIGURE 4. Computed flows for $R/D = 1/4$ (a) and $R/D = 6$ (b)

of the jet α against the same parameter $r = R/D$. Due to concavity of the free streamline the graph $\theta_{\text{sep}}(r)$ lies slightly higher than the graph $\alpha(r)$. The difference between them is insignificant at large values of r but becomes considerable when r is small. If the ratio r is not big enough, the jet makes an acute angle with the main stream, whereas, if r is sufficiently large, the jet is diverted backward. We can see from Figure 3(b) that the critical ratio, which separates these two cases and for which the direction of the jet becomes vertical, is approximately $r = 4.8$. Flows computed for $r = 1/4$ and $r = 6$ and shown in Figure 4 illustrate each of these cases.

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Appendix: Curvature and asymptotics of some integrals

Given a curve set in a parametric form, we can calculate its curvature κ with the aid of the formula:

$$\kappa = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

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Here dots denote differentiation with respect to the curve parameter. The free streamlines satisfy the equation:

$$\frac{dz}{d\xi} = (1 - \xi^{-1})e^{i\theta(\xi)}.$$

Hence, $\kappa = \theta'(\xi)/(1 - \xi^{-1})$. The expression for the angle is given by (7) and (10). Differentiating (10) we find

$$\theta'(\xi) = \frac{1}{2\pi(a-\xi)^{1/2}} \int_a^b \frac{\Theta(s)\Big(a - \xi - (s-a)\Big)}{(s-a)^{1/2}(s-\xi)^2} ds.$$

Passing to a new variable $t \equiv (s-a)^{1/2}$ and introducing the notation $\delta \equiv (a-\xi)^{1/2}$ we rewrite this expression in the form:

$$\theta'(\xi) = \frac{2\delta}{\pi} \int\limits_0^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{(t^2+\delta^2)^2} dt - \frac{1}{\pi\delta} \int\limits_0^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{(t^2+\delta^2)} dt. \tag{23}$$

Our task is to determine the $\delta \to 0+0$ behaviour of the curvature. We start with estimation of the first integral.

$$\begin{aligned} \frac{2\delta}{\pi} \int\limits_0^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{(t^2+\delta^2)^2} dt &= \frac{2\delta}{\pi} \int\limits_0^{\sqrt{b-a}} \frac{\Theta(a) + \Theta'(a)t^2 + \gamma(t)t^4}{(t^2+\delta^2)^2} dt \\ &= \frac{2\Theta(a)}{\pi\delta^2} \int\limits_0^{\infty} \frac{d\omega}{(\omega^2+1)^2} + \frac{2\Theta'(a)}{\pi} \int\limits_0^{\infty} \frac{\omega^2 d\omega}{(\omega^2+1)^2} + O(\delta) \\ &= \frac{\Theta(a)}{2\delta^2} + \frac{1}{2}\Theta'(a) + O(\delta). \end{aligned} \tag{24}$$

In the expressions above, $\gamma(t)$ is a bounded function. Throughout, $\Theta(a) \equiv \Theta(a+0)$ and $\Theta'(a) \equiv \Theta'(a+0)$. While making transformations we took advantage of the fact that

$$\frac{2\delta}{\pi} \int\limits_{\sqrt{b-a}}^{\infty} \frac{\Theta(a) + \Theta'(a)t^2}{(t^2+\delta^2)^2} dt = O(\delta)$$

and passed to the new variable $\omega = t/\delta$.

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Now let us process the second integral in (23). Let σ be a positive number that is less than $\sqrt{1-a}$. Consider the chain of transformations:

$$
\begin{aligned}
-\frac{1}{\pi\delta}\int_0^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{(t^2+\delta^2)}dt &= -\frac{1}{\pi\delta}\int_0^{\sigma} \frac{\Theta(a+t^2)}{(t^2+\delta^2)}dt - \frac{1}{\pi\delta}\int_\sigma^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{t^2+\delta^2}dt \\
&= -\frac{1}{\pi\delta}\int_0^{\sigma} \frac{\Theta(a+t^2)-r(\delta)}{(t^2+\delta^2)}dt - \frac{r(\delta)}{\pi\delta}\int_0^{\sigma} \frac{dt}{(t^2+\delta^2)} \\
&\quad - \frac{1}{\pi\delta}\int_\sigma^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{t^2}dt + O(\delta),
\end{aligned}
$$

where $r(\delta) \equiv \Theta(a) - \Theta'(a)\delta^2$. Let us denote the first and the second terms in the expression above by $A(\delta)$ and $B(\delta)$ respectively. First we estimate $A(\delta)$. For this purpose we represent it in the form:

$$
A(\delta) = -\frac{1}{\pi\delta}\int_0^{\sigma} \Theta'(a)dt - \frac{1}{\pi\delta}\int_0^{\sigma} \frac{r_2(t)dt}{(t^2+\delta^2)},
$$

where $r_2(t) \equiv \Theta(a+t^2) - \Theta(a) - \Theta'(a)t^2$. Further,

$$
A(\delta) = -\frac{\sigma\Theta'(a)}{\pi\delta} - \frac{1}{\pi\delta}\int_0^{\sigma} \frac{r_2(t)}{t^2}dt + \frac{1}{\pi\delta}\int_0^{\sigma} \frac{r_2(t)\delta^2/t^2}{t^2(1+\delta^2/t^2)}dt.
$$

Clearly, $|r_2(t)| < Ct^4$ for some positive constant C, so the last term is less than $(\pi\delta)^{-1}\sigma C\delta^2 = O(\delta)$. Hence,

$$
A(\delta) = -\frac{\sigma}{\pi\delta}\Theta'(a) - \frac{1}{\pi\delta}\int_0^{\sigma} \frac{r_2(t)}{t^2}dt + O(\delta),
$$

or,

$$
A(\delta) = -\frac{1}{\pi\delta}\int_0^{\sigma} \frac{\Theta(a+t^2)-\Theta(a)}{t^2}dt + O(\delta). \tag{25}
$$

Now let us evaluate $B(\delta)$:

$$
B(\delta) \equiv -\frac{r(\delta)}{\pi\delta}\int_0^{\sigma} \frac{dt}{t^2+\delta^2} = -\frac{r(\delta)}{\pi\delta^2}\arctan(\sigma/\delta).
$$

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Since, as $\delta \to 0+0$,

$$\arctan(\sigma/\delta) = \frac{\pi}{2} - \frac{\delta}{\sigma} + O(\delta^3),$$

we obtain

$$B(\delta) = -\frac{r(\delta)}{\pi\delta^2}\left(\frac{\pi}{2} - \frac{\delta}{\sigma} + O(\delta^3)\right) = -\frac{\left(\Theta(a) - \Theta'(a)\delta^2\right)}{\pi\delta^2}\left(\frac{\pi}{2} - \frac{\delta}{\sigma}\right) + O(\delta),$$

or,

$$B(\delta) = -\frac{\Theta(a)}{2\delta^2} + \frac{\Theta(a)}{\delta\pi\sigma} + \frac{1}{2}\Theta'(a) + O(\delta). \tag{26}$$

Now combining (23), (24), (25) and (26) we find:

$$\Theta'(\xi) = K_2\delta^{-2} + K_1\delta^{-1} + K_0 + O(\delta),$$

where

$$K_2 = \frac{\Theta(a)}{2} - \frac{\Theta(a)}{2} = 0, \quad K_0 = \frac{\Theta'(a)}{2} + \frac{\Theta'(a)}{2} = \Theta'(a)$$

and

$$K_1 = -\frac{1}{\pi}\int_0^{\sigma} \frac{\Theta(a+t^2) - \Theta(a)}{t^2}dt - \frac{1}{\pi}\int_{\sigma}^{\sqrt{b-a}} \frac{\Theta(a+t^2)}{t^2}dt + \frac{\Theta(a)}{\pi\sigma}.$$

It is easy to see that

$$\frac{\Theta(a)}{\pi\sigma} = \frac{1}{\pi}\int_{\sigma}^{\sqrt{b-a}} \frac{\Theta(a)}{t^2}dt + \frac{\Theta(a)}{\pi(b-a)^{1/2}},$$

so

$$K_1 = \pi^{-1}\left(\Theta(a)(b-a)^{-1/2} - \int_0^{\sqrt{b-a}} (\Theta(a+t^2) - \Theta(a))t^{-2}dt\right).$$

The final formula then can be written as

$$\theta'(\xi) = K_1(a-\xi)^{-1/2} + \theta'(a+0) + O((a-\xi)^{1/2}).$$

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References

[1] G. Birkhoff and E. Zarantonello, *Jets, Wakes and Cavities* (Academic Press, New York, 1957).

[2] S. Brodetsky, "Discontinuous fluid motion past circular and elliptic cylinders", *Proc. R. Soc. Lond.* **A102** (1923) 542.

[3] R. Courant, *Partial Differential Equations* (Interscience Publishers, New York, 1962).

[4] G. Dagan and M. P. Tulin, "Two-dimensional free-surface gravity flow past blunt bodies", *J. Fluid Mech.* **51** (1972) 529.

[5] E. O. Tuck and J.-M. Vanden-Broeck, "Ploughing flows", *European J. Appl. Math.* (1998) in press.

[6] J.-M. Vanden-Broeck, "The influence of surface tension on cavitating flow past a curved obstacle", *J. Fluid Mech.* **133** (1983) 255.